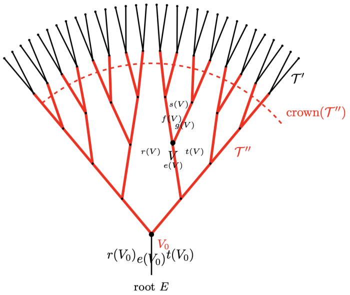
$$4\sum_{\substack{(x,y)\in\mathbb{Z}^2_{\geq 0}\\ \det(x,y)=1}} \frac{1}{\|x\|^2\,\|y\|^2\,\|x+y\|^2} = \pi,$$

$$h(D) = \frac{|D|^{3/2}}{12\pi}\sum_{\substack{a>0\\ b^2-4ac=D}} \frac{1}{a(a+b+c)c}, \text{ for } D < -4$$

$$D_{1,1,1}(z) = \sum_{\substack{\omega_1+\omega_2+\omega_3=0\\ \omega_i\in\mathbb{Z}z+\mathbb{Z}}}' \frac{\operatorname{Im}(z)^3}{|\omega_1\omega_2\omega_3|^2} = 2E_3(z) + \pi^3\zeta(3)$$



Evaluation of lattice sums via telescoping over topographs

Nikita Kalinin, Guangdong Technion Israel Institute of Technology

Special Functions and Number Theory seminar

https://arxiv.org/abs/2510.02082

$$\arctan(x_1) + \arctan(x_2) + \arctan(x_3) = \arctan\left(\frac{x_1+x_2+x_3-x_1x_2x_3}{1-x_1x_2-x_2x_3-x_3x_1}\right)$$

$$\operatorname{arctanh}(x_1) + \operatorname{arctanh}(x_2) + \operatorname{arctanh}(x_3) = \operatorname{arctanh}\left(\frac{x_1+x_2+x_3+x_1x_2x_3}{1+x_1x_2+x_2x_3+x_3x_1}\right)$$

Corollary: "A half-shift reflection identity for the digamma function", N.K., https://arxiv.org/abs/2510.00012.

$$2W_1(x) + \log 4 + \psi\left(\tfrac{1}{2} + x\right) + \psi\left(\tfrac{3}{2} - x\right) = 0,$$

where $\psi(z) = \frac{d}{dz} \ln \Gamma(z) = -\gamma + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+z}\right)$ is the digamma and

$$W_1(x) = 2 \int_0^\infty \Re\left(\frac{y}{(y^2+1)\left(e^{\pi(y+2ix)} - 1\right)} \right) dy.$$

Proof. Define the cosine integral

$$\mathrm{Ci}(z) := -\int_z^\infty \frac{\cos t}{t}\, dt.$$

Because the function is even about $x = \frac{1}{2}$, all sine Fourier coefficients vanish. Collecting the constant and cosine terms yields, for $0 < x < 1$,

$$\psi\left(\tfrac{1}{2} + x\right) + \psi\left(\tfrac{3}{2} - x\right) = -2\log 2 + 4 \sum_{k=1}^{\infty} (-1)^k\ \mathrm{Ci}(\pi k) \cos(2\pi k x).$$

Plan

1. Sums in number theory
2. Class number for binary quadratic forms
3. Conway's topographs
4. Telescoping identities
5. Zagier's identity
6. Geometric interpretation

Sums in number theory (I): zeta values

Basel problem 1650: Find $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$
Leonhard Euler 1734: Define $\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots$, then

$$\zeta(2) = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots = \frac{\pi^2}{6}, \zeta(2n) = (-1)^{n+1}\frac{(2\pi)^{2n}B_{2n}}{2(2n)!}.$$

Roger Apéry 1978: $\zeta(3)$ is irrational.

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Don Zagier 1993: Define $f(m,n) = \frac{1}{mn^3} + \frac{1}{2m^2n^2} + \frac{1}{m^3n}$. Then one checks directly that

$$f(m,n) - f(m+n,n) - f(m,m+n) = \frac{1}{m^2n^2}$$

and hence, summing over all $m, n > 0$, that

$$\zeta(2)^2 = \left(\sum_{m,n>0} - \sum_{m>n>0} - \sum_{n>m>0}\right) f(m,n) = \sum_{n>0} f(n,n) = \frac{5}{2}\zeta(4).$$

Sums in number theory (II): lattice sums

J. P. G. L. Dirichlet 1839: Dedekind zeta function for $\mathbb{Z}[i]$. In simple terms, for $s > 1$ one considers the sum over lattice points in $\mathbb{Z}^2$ and proves that

$$\sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(m^2+n^2)^s} = 4\zeta(s)\cdot(1-\frac{1}{3^s}+\frac{1}{5^s}-\frac{1}{7^s}+\dots) = 4\zeta(s)\,L(s,\chi_{-4}).$$

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Leonard Tornheim 1950, Louis J. Mordell 1958: Tornheim introduced

$$T(s_1,s_2,s_3) = \sum_{m,n\geq 1} \frac{1}{m^{s_1} n^{s_2} (m+n)^{s_3}},$$

and Mordell later considered higher-dimensional analogues. For $s_i, s > 1$

$$\zeta_{MT,r}(s_1,\dots,s_r;s) := \sum_{m_1,\dots,m_r\geq 1} \frac{1}{m_1^{s_1}\cdots m_r^{s_r}(m_1+\cdots+m_r)^s}.$$

It interpolates between products of zeta values and multiple zeta values, admits meromorphic continuation and functional relations, and at integer arguments is expressed as $\mathbb{Q}$-linear combinations of multiple zeta values.

A new proof of an old result

Theorem. For $A = \{(x,y) \mid x, y \in \mathbb{Z}^2_{\geq 0}, \det(x,y) = 1\}$, we have

$$2 \sum_A \frac{1}{\|x\|^2 \cdot \|y\|^2 \cdot \|x+y\|^2} = \pi/2.$$

Proof (N.K., 2024, `https://arxiv.org/abs/2410.10884`).

Define $F(x,y) = \frac{x \cdot y}{\|x\|^2 \cdot \|y\|^2}$, $F : (\mathbb{Z}^2)^2 \to \mathbb{R}$. Then,

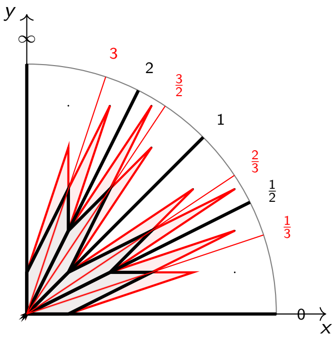
$$F(x,y) - F(x+y,y) - F(x,x+y) = \frac{-2\det(x,y)^2}{\|x\|^2 \cdot \|y\|^2 \cdot \|x+y\|^2}.$$

We telescope

$$F(x,y) - F(x+y,y) - F(x,x+y)$$

over $A \cap [0,n]^4$ obtaining the sum of $-F(x+y,y) - F(x,x+y)$ over $\{(x,y) \in A, x+y \notin [0,n]^2\}$.
The latter sum tends to $-\pi/2$ since the set of angles at the origin of the parallelograms partition the angle $\pi/2$ of the first quadrant. □

Farey sunburst and the boundary term asymptotics



At step 1 there is a single parallelogram spanned by $(1,0)$ and $(0,1)$. At step 2 it is bisected by the mediant ray $(1,1)$, producing two parallelograms, spanned by $(1,0),(1,1)$ and by $(1,1),(0,1)$. The figure above shows the next refinements: step 3 (in black) and step 4 (in red).

Let $x, y \in \mathbb{Z}^2_{\geq 0}$ with $\det(x,y) = 1$, and let $\theta = \angle(x,y)$. Then the area identity gives

$$\sin\theta = \frac{\det(x,y)}{\|x\|\,\|y\|} = \frac{1}{\|x\|\,\|y\|},$$

$$\theta = \frac{1}{\|x\|\,\|y\|} + O\left(\frac{1}{\|x\|^3\|y\|^3}\right).$$

For the telescoping primitive

$$F(x,y) = \frac{x \cdot y}{\|x\|^2\|y\|^2} = \frac{\cos\theta}{\|x\|\,\|y\|}$$

$$= \frac{1}{\|x\|\,\|y\|} + O\left(\frac{1}{\|x\|^3\|y\|^3}\right),$$

so $F(x,y) = \theta + O(\|x\|^{-3}\|y\|^{-3})$.

Class number for binary quadratic forms (I)

Adolf Hurwitz 1905:

Theorem

For $D < 0$, a fundamental discriminant, all the terms in the following sum are positive and this sum converges to $h(D)$:

$$h(D) = \frac{\omega_D}{12\pi}|D|^{3/2} \sum_{\substack{A>0 \\ B^2-4AC=D}} \frac{1}{A(A+B+C)C}, \tag{1}$$

where $A, B, C, D \in \mathbb{Z}$ and

$$\omega_D = \begin{cases} 1 \text{ for } D < -4, \\ 2 \text{ for } D = -4, \\ 3 \text{ for } D = -3. \end{cases} \tag{2}$$

The formula on the previous slide is a particular case of this theorem for $D = -4$.

Class number for binary quadratic forms (II)

For $v = (m, n)$, let $q = [A, B, C]$ denote a quadratic form

$$q(v) = Am^2 + Bmn + Cn^2.$$

Two binary quadratic forms $[A, B, C]$ and $[A', B', C']$ with integer coefficients are called equivalent if there exist $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$ such that

$$A'm^2 + B'mn + C'n^2 = A(am + bn)^2 + B(am + bn)(cm + dn) + C(cm + dn)^2.$$

That is,

$$[A, B, C] \sim [Aa^2 + Bac + Cc^2, 2Aab + B(ad + bc) + 2Ccd, Ab^2 + Bbd + Cd^2].$$

In particular,

$$[1, 0, 1] \sim [a^2 + c^2, 2ab + 2cd, b^2 + d^2] = [A, B, C], \text{ and} \tag{3}$$

$$A + B + C = (a + b)^2 + (c + d)^2 \text{ in this case.}$$

The *discriminant* D of a binary quadratic form $[A, B, C]$ is $B^2 - 4AC$ and the discriminant D is invariant under the above equivalence.

Denote by $h(D)$ the number of equivalence classes of forms with $\gcd(A, B, C) = 1$ and discriminant D. Although formulas for $h(D)$ exist, it is still an open problem to prove that there exist infinitely many values of $D > 0$ for which $h(D) = 1$.

Example

For the quadratic form $q(v) = \|v\|^2 = m^2 + n^2$, the discriminant $D = -4$, and $h(D) = 1$. Since we have

$$\sum_{\substack{a,b,c,d \in \mathbb{Z}_{\geq 0},\\ ad-bc=1}} \frac{1}{(a^2+b^2)(c^2+d^2)\Big((a+c)^2+(b+d)^2\Big)} = \frac{\pi}{4}, \qquad (4)$$

and

$$\frac{1}{A(A+B+C)C} = \frac{1}{(a^2+b^2)\Big((a+c)^2+(b+d)^2\Big)(c^2+d^2)},$$

putting it all together and multiplying by the constant 3 coming from the three cyclic orders, we rewrite Hurwitz's theorem as

$$h(-4) = 1 = \frac{2}{12\pi} \cdot 2^3 \cdot \frac{3\pi}{4} = \frac{\omega_{-4}}{12\pi} \cdot |-4|^{3/2} \sum_{\substack{A>0\\ B^2-4AC=-4}} \frac{1}{A(A+B+C)C}.$$

Class number for binary quadratic forms (III)

Hurwitz's result has been largely unnoticed for nearly a century, with the only exceptions: L.E. Dickson's *History of number theory* in 1952, and Robert Sczech's work on Eisenstein cocycles for $GL_2\mathbb{Q}$ in 1992.
It was revived in 2019, in the Duke-Imamoḡlu-Tóth paper "On a class number formula of Hurwitz", and a formula for the indefinite case ($D > 0$), similar to Hurwitz's, was established:

Theorem (Duke-Imamoḡlu-Tóth)

For $D > 0$, a fundamental discriminant,

$$h(D)\log\varepsilon_D = \sum_{\substack{[A,B,C] \text{ reduced} \\ B^2-4AC=D}} \frac{D^{1/2}}{B} + \sum_{\substack{A,C,A+B+C>0 \\ B^2-4AC=D}} \frac{D^{3/2}}{3(B+2A)B(B+2C)}.$$

Here ε_D, the fundamental unit, is defined as $\varepsilon_D := (t_D + u_D\sqrt{D})/2$, where (t_D, u_D) is the smallest solution to $t^2 - Du^2 = 4$ in positive integers.

Conway's Topographs (I)

In 1997, in his book *The Sensual (Quadratic) Form*, John H. Conway introduced topographs, a graphical tool for visualizing binary quadratic forms and their values over the integers. A topograph provides an intuitive and surprisingly powerful visual framework for understanding, for example, reduction algorithms.

In 2024, Cormac O'Sullivan proposed a unifying approach to the above class number series via topographs.

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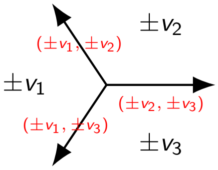
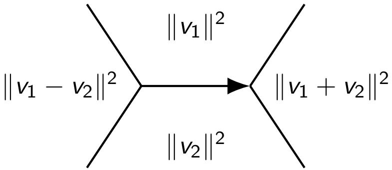
In 2024, Cormac O'Sullivan proposed a unifying approach to the above class number series via topographs.

A topograph for a binary quadratic form q is an infinite trivalent planar tree $\mathcal{T}$ with labels in the connected components (regions) of $\mathbb{R} \setminus \mathcal{T}$. Each region corresponds bijectively to a pair $(v, -v)$ of primitive lattice vectors in $\mathbb{Z}^2$ (i.e., a point in $P\mathbb{Q}^2$), and the label on this region is the value $q(v) = q(-v)$ of the quadratic form q. At each vertex, the three adjacent regions correspond to three primitive vectors $v, w, v+w$, forming a basis of $\mathbb{Z}^2$. Thus, near each vertex of $\mathcal{T}$ the labels r, s, t on regions are exactly $q(v), q(w), q(v+w)$ for a certain basis (v, w) of $\mathbb{Z}^2$.

Conway's Topographs (II): labels via a quadratic form



Local picture near a vertex (a superbase $\{v_1, v_2, v_3\}$).



Local picture near an edge (here $q(v) = \|v\|^2$).

We label each region (corresponding to $\pm v \in P\mathbb{Q}^2$) by $q(v) = q(-v)$. For the norm form $q(v) = \|v\|^2$ with $v = (m, n)$ this is $\|v\|^2 = m^2 + n^2$, and the *parallelogram law* says

$$\|v_1 - v_2\|^2 + \|v_1 + v_2\|^2 \;=\; 2\big(\|v_1\|^2 + \|v_2\|^2\big).$$

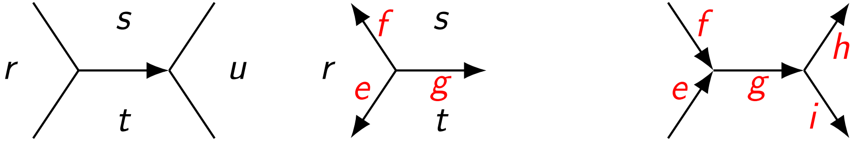
More generally, for any binary quadratic form q one has

$$q(v_1 - v_2) + q(v_1 + v_2) \;=\; 2\big(q(v_1) + q(v_2)\big).$$

This relation propagates labels across the tree: once q is known on a superbase, one can recover q on all of $\mathbb{Z}^2$ by iterating these local moves.

Conway's Topographs (III)

Equivalently, a topograph is a planar connected 3-valent tree with labels on vertices, edges, and regions. Near each edge, labels must satisfy $r + u = 2(s + t)$. So, $r, s + t, u$ form an arithmetic progression with difference $g := s + t - r$. Note: $e + g = 2t$.



$D := -ef - fg - eg$ is the same for all vertices,

(indeed, we have $g - e = 2t = i - g$ and $(g - e)(g - f) = (i - g)(h - g) = g^2 - eg - gf + ef = g^2 + ih - ig - hg$)

and is called the discriminant of the topograph.

Useful identities in a topograph (I)

$$\frac{g}{st}+\frac{f}{rs}+\frac{e}{rt}=\frac{gr+ft+es}{rst}=\frac{g(f+e)+f(e+g)+e(f+g)}{2rst}=\frac{-D}{rst},$$

$$\frac{1}{e}+\frac{1}{f}+\frac{1}{g}=\frac{ef+fg+ge}{efg}=\frac{-D}{efg},$$

$$\frac{s}{fg}+\frac{r}{ef}+\frac{t}{eg}=\frac{gr+ft+es}{efg}=\frac{g(f+e)+f(e+g)+e(f+g)}{2efg}=\frac{-D}{efg}.$$

Higher-order analogues include:

$$\frac{g}{s^2t^2}+\frac{f}{r^2s^2}+\frac{e}{r^2t^2}=-\frac{6}{rst}-\frac{D(r+s+t)}{r^2s^2t^2},$$

$$\frac{s}{f^2g^2}+\frac{r}{e^2f^2}+\frac{t}{e^2g^2}=-\frac{3}{2efg}-\frac{D(e+f+g)}{2e^2f^2g^2},$$

$$\frac{1}{e^3}+\frac{1}{f^3}+\frac{1}{g^3}=-\frac{D^3}{e^3f^3g^3}-\frac{3D(e+f+g)}{e^2f^2g^2}+\frac{3}{efg}.$$

Useful identities in a topograph (II)

Remarkably, the same local combinatorics also encodes trigonometric or hyperbolic *angle-addition* relations, depending on the sign of the discriminant.

Lemma (Local trigonometric/hyperbolic relations)

For $D < 0$, for appropriate branches of arcsin *and* arctan*, we have*

$$\arcsin\Big(\frac{e}{rt}\cdot\frac{\sqrt{-D}}{2}\Big)+\arcsin\Big(\frac{f}{rs}\cdot\frac{\sqrt{-D}}{2}\Big)+\arcsin\Big(\frac{g}{st}\cdot\frac{\sqrt{-D}}{2}\Big)=0, \quad (5)$$

$$\arctan\Big(\frac{\sqrt{-D}}{e}\Big)+\arctan\Big(\frac{\sqrt{-D}}{f}\Big)+\arctan\Big(\frac{\sqrt{-D}}{g}\Big)=0. \quad (6)$$

For $D > 0$ and $|e|, |f|, |g| > \sqrt{D}$, we have

$$\operatorname{arcsinh}\Big(\frac{e}{rt}\cdot\frac{\sqrt{D}}{2}\Big)+\operatorname{arcsinh}\Big(\frac{f}{rs}\cdot\frac{\sqrt{D}}{2}\Big)+\operatorname{arcsinh}\Big(\frac{g}{st}\cdot\frac{\sqrt{D}}{2}\Big)=0, \quad (7)$$

$$\operatorname{arctanh}\Big(\frac{\sqrt{D}}{e}\Big)+\operatorname{arctanh}\Big(\frac{\sqrt{D}}{f}\Big)+\operatorname{arctanh}\Big(\frac{\sqrt{D}}{g}\Big)=0. \quad (8)$$

Proof idea: $-\arcsin A = \arcsin B + \arcsin C$ is equivalent to $-A = B\sqrt{1-C^2} + C\sqrt{1-B^2}$; squaring twice makes all terms cancel.

O'Sullivan interpretation

Theorem (O'Sullivan, 2024, Topographs for binary quadratic forms and class numbers)

Let $\mathcal{T}$ be any topograph of discriminant $D < 0$. Then

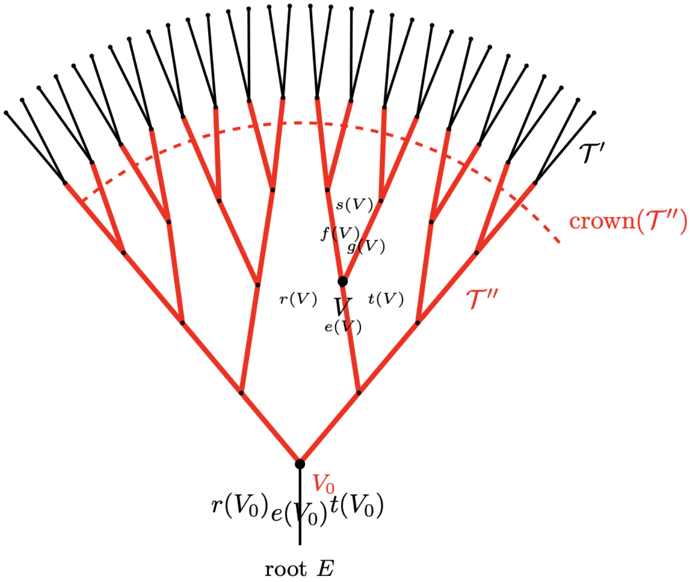
$$|D|^{3/2} \sum_{r \succ \frac{s}{t} \in \mathcal{T}} \frac{1}{|rst|} = 4\pi, \tag{9}$$

where we sum over all vertices of $\mathcal{T}$, each vertex contributing one term; here r, s, t denote the labels on regions adjacent to a given vertex of $\mathcal{T}$.

Then, Hurwitz's theorem follows because the RHS of it is essentially the sum over all vertices of all topographs of discriminant D, and the number of topographs of discriminant D is $h(D)$.

A similar formula can be written for $D > 0$.

Telescoping identities (I)



Telescoping identities (II)

Theorem (N.K. 2025, `https://arxiv.org/abs/2510.02082`)

Let $\mathcal{T}$ be any topograph of discriminant $D < 0$. Let $\mathcal{T}'$ be the upper half of $\mathcal{T}$ with respect to a root E. Suppose $\mathcal{T}'$ is admissible. Then, for a suitable branch of arcsin *and for the principal branch of* arctan,

$$\sum_{r \succ^{s}_{t} \in \mathcal{T}'} \frac{1}{|rst|} = \frac{1}{D}\left(\frac{e_0}{r_0 t_0} - \frac{2\arcsin\left(\frac{e_0}{r_0 t_0}\cdot\frac{\sqrt{-D}}{2}\right)}{\sqrt{-D}}\right), \tag{10}$$

$$\sum_{{}^{f}_{e}\succ^{g} \in \mathcal{T}'} \frac{1}{|efg|} = \frac{1}{D}\left(\frac{\arctan\left(\frac{\sqrt{-D}}{e_0}\right)}{\sqrt{-D}} - \frac{1}{e_0}\right), \tag{11}$$

where the summation is over all vertices of $\mathcal{T}'$, with each contributing one term.

Telescoping identities (III)

The proofs consist of telescoping $F(x,y) - F(x+y,y) - F(x,x+y)$ over the vertices $V = (x, y, x+y)$ of a topograph for $F(V) = \frac{e}{rt}(V)$ due to

$$\frac{1}{rst} = \frac{1}{D}\left(\frac{e}{rt} - \frac{f}{rs} - \frac{g}{st}\right),$$

To control the sum "at infinity" we use

$$\arcsin\left(\frac{e}{rt} \cdot \frac{\sqrt{-D}}{2}\right) + \arcsin\left(\frac{f}{rs} \cdot \frac{\sqrt{-D}}{2}\right) + \arcsin\left(\frac{g}{st} \cdot \frac{\sqrt{-D}}{2}\right) = 0.$$

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It is interesting to note that the above identities are equivalent to

$$\cot(X)\cot(Y) = \cot(X)\cot(X + Y) + \cot(Y)\cot(X + Y) + 1.$$

Similarly for $(D < 0)$ arctan and $(D > 0)$ arctanh, applying the identity below:

$$\arctan(x_1) + \arctan(x_2) + \arctan(x_3) = \arctan\left(\frac{x_1 + x_2 + x_3 - x_1x_2x_3}{1 - x_1x_2 - x_2x_3 - x_3x_1}\right).$$

$$\operatorname{arctanh}(x_1) + \operatorname{arctanh}(x_2) + \operatorname{arctanh}(x_3) = \operatorname{arctanh}\left(\frac{x_1 + x_2 + x_3 + x_1x_2x_3}{1 + x_1x_2 + x_2x_3 + x_3x_1}\right).$$

Zagier's identity (I)

Let $H = (\mathbb{Z} \times \mathbb{Z}_{>0}) \cup (\mathbb{Z}_{\geq 0} \times \{0\})$, the set of lattice vectors in the closed upper half-plane with the negative x-axis removed.

Theorem (N.K., https://arxiv.org/abs/2410.10884)

For every integer $n \geq 1$,

$$\sum_{\substack{x,y \in H \\ \det(x,y)=n}} \frac{n^2}{\|x\|^2 \, \|y\|^2 \, \|x+y\|^2} = \frac{\pi}{2n} \sigma_1(n).$$

Each non-collinear triple yields exactly 12 ordered pairs (x, y) with $x, y \in H$ and $\det(x, y) > 0$: six permutations of $(\omega_1, \omega_2, \omega_3)$ and a factor 2 from central symmetry.

Thus

$$D_{1,1,1}(i) - 2E(i, 3) = 12 \sum_{n \in \mathbb{Z}_{>0}} \sum_{\substack{x,y \in H \\ \det(x,y)=n}} \frac{1}{\|x\|^2 \, \|y\|^2 \, \|x+y\|^2}.$$

Zagier's identity (II): at $z = i$, $s = 3$ and general case

$$\pi^3\zeta(3) = D_{1,1,1}(i) - 2E(i,3) = 12\sum_{n\geq 1}\sum_{\substack{x,y\in H\\ \det(x,y)=n}} \frac{1}{\|x\|^2\,\|y\|^2\,\|x+y\|^2}$$

$$= 12\sum_{n\geq 1}\frac{\pi}{2n^3}\,\sigma_1(n) = 6\pi\sum_{n\geq 1}\frac{\sigma_1(n)}{n^3} = 6\pi\,\zeta(3)\zeta(2) = \pi^3\zeta(3),.$$

because $\sum_{n\geq 1}\sigma_1(n)n^{-s} = \zeta(s)\zeta(s-1)$ and $\zeta(2) = \pi^2/6$. Similarly,

Theorem (N.K., https://arxiv.org/abs/2410.10884)

For $z = x + iy \in \mathcal{H}$ and $s > -1$,

$$\sideset{}{'}\sum_{\substack{\omega_1+\omega_2+\omega_3=0\\ \omega_k\in\mathbb{Z}z+\mathbb{Z}}} \frac{|\det(\omega_1,\omega_2)|^{-s}}{|\omega_1\omega_2\omega_3|^2} = \frac{6\pi}{y^3}\,\zeta(s+3)\,\zeta(s+2).$$

Here $\sum'$ indicates that collinear triples are omitted.

Geometric interpretation

Use the rational parametrization of $x^2 + y^2 = 1$ by

$$f : \binom{a}{b} \to \left(\frac{2ab}{a^2 + b^2}, \frac{a^2 - b^2}{a^2 + b^2} \right).$$

Note that $f(\binom{1}{0}) = (0, 1), f(\binom{0}{1}) = (0, -1)$. By a direct calculation, the area of the triangle with vertices $f(\binom{a}{b}), f(\binom{c}{d}), f(\binom{a+c}{b+d})$ equals to

$$\frac{2|ad - bc|^3}{(a^2 + b^2) \cdot (c^2 + d^2) \cdot ((a + c)^2 + (b + d)^2)} = \frac{2}{rst}. \qquad (12)$$

Here r, s, t denote the values of the quadratic form $q(n, m)$ on a superbase vectors $\left\{ \binom{a}{b}, \binom{c}{d}, -\binom{a+c}{b+d} \right\}$, the notation that we use for the topograph's labels on regions.

Theorem (Hurwitz, 1905)

$$\sum_{\substack{a,b,c,d \in \mathbb{Z}_{\geq 0} \\ ad - bc = 1}} \frac{1}{(a^2 + b^2)(c^2 + d^2)((a + c)^2 + (b + d)^2)} = \frac{\pi}{4}.$$

Thank you very much!