EVALUATION OF LATTICE SUMS VIA TELESCOPING OVER TOPOGRAPHS

NIKITA KALININ

Abstract. Topographs, introduced by Conway in 1997, are infinite three-valent planar trees used to visualize the values of binary quadratic forms. In this work, we study series whose terms are indexed by vertices of a topograph and show that they can be evaluated using telescoping sums whose terms correspond to the edges of a topograph.

Our technique provides new arithmetic proofs for modular graph function identities arising in string theory, yields alternative derivations of Hurwitz-style class number formulas in number theory, and serves as a unified framework for well-known Mordell-Tornheim series and Hata's series for the Euler constant γ.

Our theorems are of the following spirit: let us cut a topograph along an edge (called the *root*) in two parts, and then sum $\frac{1}{rst}$ (the reciprocal of the product of labels on regions adjacent to a vertex) over all vertices of one part. Then the sum is equal to an explicit expression depending only on the root and the discriminant of the topograph.

Keywords: modular graph functions, lattice sums, telescoping sums, binary quadratic forms, topographs, class number. AMS classification: 11E16, 11F67, 11M35

Contents

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1. Introduction

The purpose of this article is to introduce and develop the *telescoping over topograph* method and to connect it to the works on class number formulas for quadratic number fields (a classical one [9] of Adolf Hurwitz and two recent ones: [7] of Duke, Imamoḡlu, Tóth and [12] by O'Sullivan) as well as to the modular graph functions in the low energy genus one expansion of type II string amplitudes due to d'Hoker, Green, Gürdoğan, Vanhove [5, 4].

1.1. Formulas related to class number.

For $v = (m, n)$, denote by $q = [A, B, C]$ a quadratic form

$$q(v) = Am^2 + Bmn + Cn^2.$$

Two binary quadratic forms $[A, B, C]$ and $[A', B', C']$ with integer coefficients are called equivalent if there exist $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$ such that

$$A'm^2 + B'mn + C'n^2 = A(am + bn)^2 + B(am + bn)(cm + dn) + C(cm + dn)^2.$$

That is,

$$[A, B, C] \sim [Aa^2 + Bac + Cc^2, 2Aab + B(ad + bc) + 2Ccd, Ab^2 + Bbd + Cd^2],$$

In particular,

$$[1, 0, 1] \sim [a^2 + c^2, 2ab + 2cd, b^2 + d^2] = [A, B, C], \text{ and} \tag{1}$$

$$A + B + C = (a + b)^2 + (c + d)^2 \text{ in this case.}$$

The *discriminant* D of a binary quadratic form $[A, B, C]$ is $B^2 - 4AC$ and it is preserved by the above equivalence. A discriminant D is called a *fundamental* discriminant if either (a) D is square-free and $D \equiv 1 \pmod 4$, or (b) $D \equiv 0 \pmod 4$, $D/4$ is square-free and $D/4 \equiv 2, 3 \pmod 4$.

Denote by $h(D)$ the number of equivalence classes of forms with $\gcd(A, B, C) = 1$ and discriminant D. Although formulas for $h(D)$ exist, it remains difficult to estimate its asymptotic behavior; for example, it is still an open problem to prove that there exist infinitely many values of $D > 0$ for which $h(D) = 1$.

In 1905, Adolf Hurwitz wrote a paper on an infinite series representation of the class number $h(D)$ in the positive-definite case.

Theorem 1 (Hurwitz,[9])**.** For $D < 0$, a fundamental discriminant, all the terms in the following sum are positive and this sum converges to $h(D)$:

$$h(D) = \frac{\omega_D}{12\pi}|D|^{3/2} \sum_{\substack{A>0 \\ B^2-4AC=D}} \frac{1}{A(A + B + C)C}, \tag{2}$$

where

$$\omega_D = \begin{cases} 1 \text{ for } D < -4, \\ 2 \text{ for } D = -4, \\ 3 \text{ for } D = -3. \end{cases} \tag{3}$$

Hurwitz's proof amounts to computing area of a certain domain in two different ways, see Section 3.4 for details.

Example 1. For the quadratic form $q(v) = ||v||^2 = m^2 + n^2$, the discriminant $D = -4$, and $h(D) = 1$. Hurwitz's arguments [9, p. 20] lead to the formula

$$\sum_{\substack{a,b,c,d \in \mathbb{Z}_{\geq 0}, \\ ad-bc=1}} \frac{1}{(a^2 + b^2)(c^2 + d^2)((a + c)^2 + (b + d)^2)} = \frac{\pi}{4}, \tag{4}$$

By the change of variables (1), it follows that

$$\frac{1}{A(A+B+C)C} = \frac{1}{(a^2+b^2)((a+c)^2+(b+d)^2)(c^2+d^2)},$$

thus, putting it all together, and multiplying by constant 3 coming from the three cyclic orders of the denominators of the terms, we get

$$h(-4) = 1 = \frac{2}{12\pi}\cdot 2^3 \cdot \frac{3\pi}{4} = \frac{\omega_D}{12\pi}|D|^{3/2} \sum_{\substack{A>0\\ B^2-4AC=D}} \frac{1}{A(A+B+C)C}.$$

Hurwitz's result has been largely unnoticed for nearly a century with the only exceptions of Dickson's *History of number theory* [6, p.167] in 1952 and Sczech's work [13] on Eisenstein cocycles for $GL_2\mathbb{Q}$ in 1992.

It has been revived in 2019, in the Duke-Imamoḡlu-Tóth paper [7], and a formula for the indefinite case ($D > 0$), similar to (2), was established:

Theorem 2 ([7], Theorem 3, p. 3997)**.** For $D > 0$, a fundamental discriminant,

(5)
$$h(D)\log\varepsilon_D = D^{1/2} \sum_{\substack{[A,B,C]\ \text{reduced}\\ B^2-4AC=D}} \frac{1}{B} + d^{3/2} \sum_{\substack{A,C,A+B+C>0\\ B^2-4AC=D}} \frac{1}{3(B+2A)B(B+2C)}.$$

Here ε_D, the fundamental unit, is defined as $\varepsilon_D := (t_D + u_D\sqrt{D})/2$ where (t_D, u_D) is the smallest solution to $t^2 - Du^2 = 4$ in positive integers.

In 1997, in his book *The Sensual (Quadratic) Form* [3], John H. Conway introduced topographs, a graphical tool for visualizing binary quadratic forms and their behavior over the integers. A topograph provides an intuitive and surprisingly powerful visual framework for understanding, for example, reduction algorithms. A topograph for a binary quadratic form q is an infinite three-valent planar tree $\mathcal{T}$ with labels in the connected components (regions) of $\mathbb{R}\setminus\mathcal{T}$. To each region there correspond bijectively a pair $(v, -v)$ of primitive lattice vectors in $\mathbb{Z}^2$, and the label on this region is $q(v) = q(-v)$. At each vertex the three adjacent regions correspond to three primitive vectors $v, w, v+w$, forming a basis of $\mathbb{Z}^2$. Thus, near each vertex of $\mathcal{T}$ the labels on regions are exactly $q(v), q(w), q(v+w)$ for a certain basis (v, w) of $\mathbb{Z}^2$. Topographs are related to many objects in mathematics [2] and may be used to study a variation of Markov triples, see a popular exposition [15].

Below we will use the notation of topographs which will be explained later in Section 2. We will consider summations over all vertices V of a topograph, and we sum the reciprocal to the product $|rst|$ of labels on the adjacent to V regions or the product $|egf|$ of labels on the adjacent to V edges.

In 2024, O'Sullivan proposed in [12] a unifying approach to these class number series via topographs. In particular, (4) was rewritten in the language of topographs as

Theorem 3 ([12], Theorem 9.1)**.** Let $\mathcal{T}$ be any topograph of discriminant $D < 0$. Then

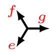
(6)
$$|D|^{3/2} \sum_{r \succ^{s}_{t} \in \mathcal{T}} \frac{1}{|rst|} = 4\pi,$$

where we sum over all vertices of $\mathcal{T}$, each vertex contributing one term; here r, s, t denote the labels on regions adjacent to a given vertex of $\mathcal{T}$, as explained in Section 2.

In the same paper the formula (5) for the class number for indefinite case $D > 0$ was rewritten in terms of topographs as

Theorem 4 ([12], Theorem 9.2)**.** Let $\mathcal{T}$ be any topograph of a non-square discriminant $D > 0$. Define $\mathcal{T}_\star$ to be equal to $\mathcal{T}$ except that all the river edges are relabeled with $\sqrt{D}$. Then

$$D^{3/2} \sum \frac{1}{|efg|} = 2 \log \varepsilon_D,$$



$\in \mathcal{T}_\star$

where we sum over all vertices of $\mathcal{T}_\star$ modulo the river period (each vertex contributing one term, see [12] for details on the river and its period), and e, f, g are labels on the edges.

1.2. **String theory formulas.** There was an independent parallel story. Beyond number theory, similar lattice sums appear in the analysis of modular graph functions in string theory. In 2008, in an unpublished note [16] Zagier considers

$$D_{1,1,1}(z) = \sum_{\omega_1+\omega_2+\omega_3=0}{}' \frac{\mathrm{Im}(z)^3}{|\omega_1\omega_2\omega_3|^2}, \quad \omega_1, \omega_2, \omega_3 \in \mathbb{Z}z + \mathbb{Z},$$

where $\sum'$ denotes the summation over the fractions with non-zero denominators, and proves

$$D_{1,1,1}(z) = 2E(z,3) + \pi^3\zeta(3), \tag{7}$$

Zagier's proof involves analytic manipulations, partial telescoping and reduces the question to sums of $\frac{1}{(z+n)(z+m)}$, then to sums involving the real part of $\frac{1+q}{1-q}$ for $q = e^{2\pi i z}$. Finally, Zagier proves (7) up to certain holomorphic, $SL(2,\mathbb{Z})$-invariant and small at infinity function, hence identically zero.

Let us specialize $z = i$, so $\mathbb{Z}z+\mathbb{Z}$ becomes $\mathbb{Z}^2$. Then, one can rewrite $D_{1,1,1}(i)$ via the sum over ω_1, ω_2 that span parallelograms of area one and hence, up to standard transformations, (7) is equivalent to (4), see [11] for details.

In 2017 in [4] there were defined Modular Graph Functions (MGF) appearing in low energy expansion of genus-one Type II superstring amplitudes. In perturbative type-II superstring theory, the genus-one four-graviton amplitude can be written as an integral over the torus moduli space of products of Green functions; then the integrals turned into lattice sums that are modular invariant by construction. At weight two one obtains classical Eisenstein series. At weight three the unique connected vacuum diagram is a "sunset" graph with two vertices joined by three propagators, and the corresponding lattice sum is precisely $D_{1,1,1}(z)$ defined above. In other words, Zagier's arithmetic identity (7) provides the first closed formula for a non-trivial modular graph function that appears in the low energy expansion of superstring amplitudes.

A modular graph function is constructed from a graph Γ. Assign to each edge $e \in \Gamma$ a variable $\frac{y}{|\omega_e|^2}$ and consider the sum of products $\prod_{e\in G} \frac{y}{|\omega_e|^2}$ over all such tuples of $\omega_e \in \mathbb{Z}z + \mathbb{Z}$ such that the sum of incoming ω_e at each vertex is zero.

If Γ consist of two vertices and w edges between them, then

$$D_\Gamma(z) = \sum_{\substack{\omega_1,\dots,\omega_w \in \Lambda \\ \omega_1+\dots+\omega_w=0}}{}' \frac{y^w}{\prod_{i=1}^{w} |\omega_i|^2}, \qquad \Lambda = \mathbb{Z} + \mathbb{Z}z,\ z = x + iy, \tag{8}$$

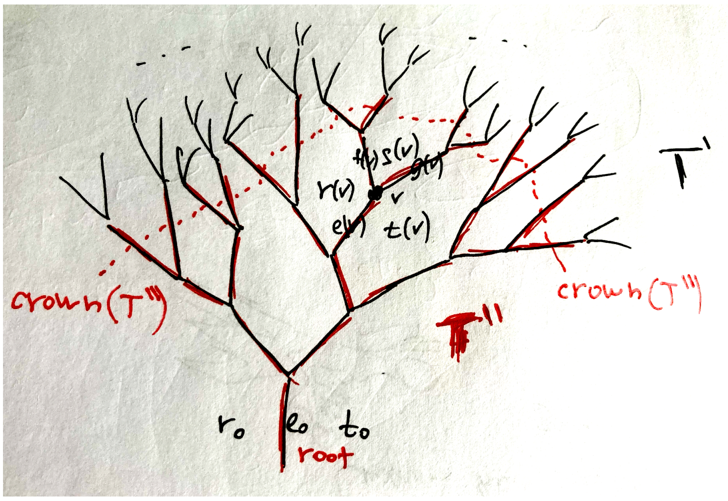


FIGURE 1. Illustration of a tree $\mathcal{T}'$ with a root, a subtree $\mathcal{T}''$ (in red) with a crown, consisting of leaves intersecting the dashed line. For each vertex V we may consider the labels $r(V), s(V), t(V)$ on the adjacent regions and labels $e(V), g(V), f(V)$ on the adjacent edges.

So we obtain the definition of $D_{1,1,1}$ when Γ is a graph with two vertices and three edges. The study of modular graph functions revealed a rich network of differential and algebraic relations between them and resulted in hundreds of articles. Our telescopic viewpoint furnishes a parallel, purely arithmetical derivation of some of those relations.

Note that the arithmetic structure in the modular graph functions and Hurwitz-type series for class number is the same.

1.3. **New results.** In [11] the Zagier's formula (7) was obtained by a telescopic method altogether with (2). Then it became clear that the telescopic method allows to derive the above Hurwitz-type formulas. The telescopic approach allows explicit evaluations of lattice sums by reducing global series to boundary contribution in topographs.

Definition 1. Let $\mathcal{T}$ be a topograph, and let E be an oriented edge of $\mathcal{T}$ labelled e_0, with adjacent regions r_0, t_0. Cutting $\mathcal{T}$ along E separates it into two infinite components; let $\mathcal{T}'$ denote the component containing the target of E. The edge E is called the *root* of the subtree $\mathcal{T}'$. It will be convenient to assume that $\mathcal{T}'$ contains the edge E but not its source vertex. Let $\mathcal{T}''$ be a connected subgraph of $\mathcal{T}'$, containing E, all of whose vertices have degree three or one. An edge $E' \in \mathcal{T}''$ is called a *leaf* if it is not a root and is adjacent to a vertex of $\mathcal{T}''$ of degree one. The set of leaves of $\mathcal{T}''$ is called the *crown* of $\mathcal{T}''$, refer to Figure 2.

There is a natural distance between a leaf in a crown to the root – the length of the shortest path. In (9),(11) we require that $\frac{e}{rt}$ computed at any of the leaves tends to zero as the distance between the crown and the root grows. In (10),(12) we require that $\frac{1}{|e|}$ at leaves tends to zero as the distance between the crown and the root grows.

Theorem 5. Let $\mathcal{T}$ be any topograph of discriminant $D < 0$. Then, for the appropriate branch of arcsin,

$$\sum_{r \succ^{s}_{t} \in \mathcal{T}'} \frac{1}{|rst|} = \frac{1}{D}\left(\frac{e_0}{r_0 t_0} - \frac{2\arcsin\left(\frac{e_0}{r_0 t_0}\cdot\frac{\sqrt{-D}}{2}\right)}{\sqrt{-D}}\right), \tag{9}$$

$$\sum_{e \succ^{f}_{g} \in \mathcal{T}'} \frac{1}{|efg|} = \frac{1}{D}\left(\frac{\arctan\left(\frac{\sqrt{-D}}{e_0}\right)}{\sqrt{-D}} - \frac{1}{e_0}\right), \tag{10}$$

where the summation is over all vertices of $\mathcal{T}'$, each contributing one term, provided the terms $\frac{e}{rt}$ at the edges e of the crown tend to zero as the distance between the crown and the root grows.

Theorem 6. Let $\mathcal{T}$ be any topograph of discriminant $D > 0$. Then

$$\sum_{r \succ^{s}_{t} \in \mathcal{T}'} \frac{1}{|rst|} = \frac{1}{D}\left(\frac{e_0}{r_0 t_0} - \frac{2\,\mathrm{arsinh}\left(\frac{e_0}{r_0 t_0}\cdot\frac{\sqrt{D}}{2}\right)}{\sqrt{D}}\right), \tag{11}$$

$$\sum_{e \succ^{f}_{g} \in \mathcal{T}'} \frac{1}{|efg|} = \frac{1}{D}\left(\frac{\mathrm{arctanh}\left(\frac{\sqrt{D}}{e_0}\right)}{\sqrt{D}} - \frac{1}{e_0}\right), \tag{12}$$

where the summation is over all vertices of $\mathcal{T}'$, each contributing one term, provided the terms $\frac{1}{|e|}$ at the edges of the crown tend to zero as the distance between the crown and the root grows.

In the above theorems one should take the appropriate branches of arcsin, arctan and the arguments of functions must belong to the domain of definition.

We can also pass to the limit when $D \to 0$. We get

Theorem 7. Let $\mathcal{T}$ be any topograph of discriminant $D = 0$. Then

$$\sum_{r \succ^{s}_{t} \in \mathcal{T}'} \frac{1}{|rst|} = \left(\frac{e_0}{r_0 t_0}\right)^3 /24, \tag{13}$$

$$\sum_{e \succ^{f}_{g} \in \mathcal{T}'} \frac{1}{|efg|} = \left(\frac{1}{e_0}\right)^3 /3, \tag{14}$$

where the summation is over all vertices of $\mathcal{T}'$, each contributing one term, provided the terms at the crown tend to zero.

Corollary 1. As a direct corollary of Theorem 9.12 in [12] and formula (12) we get

$$2\log\varepsilon_D = \sum \mathrm{arctanh}\,\frac{\sqrt{D}}{|e|},$$

where the sum involves only edges, adjacent to the vertices on the topograph's river, but not in the river (modulo river period), hence the sum is finite.

1.4. **Comparison with existing results.** Modular graph functions such as (8) are defined up to a change of coordinates for indefinite quadratic forms. Our approach allows to consider forms with any discriminant, but we cannot take the sum over all the vertices of the topograph as this sum diverges for $D \geq 0$. One can consider sums with terms like $\frac{1}{|r^n s^m t^k|}$ for natural n, m, k, in [4] there were deduced relations between such sums for various m, n, k. Naturally, we can write the same relations.

In [12] the sums of summands like $\frac{1}{|rst|}$ are considered for $D < 0$ and the sums with summands like $\frac{1}{|efg|}$ for $D > 0$. We can consider both types of sums for any $D \in \mathbb{R}$, but in case the sum over all the topograph diverges we should take only an appropriate half of topograph, as it is stated in our theorems. Also, in both [7],[12] there appear sums with $\frac{1}{|r^2 s^2 t|}$ and bigger powers, in view of the above discussion about modular graph functions, there is a bunch of relations between them.

To the best of the author's knowledge, the formula for $\log \varepsilon_D$ as in Corollary 1 does not appear in the literature.

1.5. **Plan of the paper.** Section 2 defines topographs and establishes a bunch of useful identities for the labels near vertices of topographs. Section 3.1 illustrates the main idea, how to get (4) via telescoping. In Section 3.2 we show how to obtain Mordell-Torhheim series using our approach, in Section 3.3 we show that Hata's series for the Euler constant is also a particular case of our construction. Section 3.4 highlights the geometric meaning of the summands and illustrates the duality between formulas including region labels and formulas including edge labels. Section 4 presents proofs in general case.

2. Topographs

A topograph is a planar connected 3-valent tree $\mathcal{T}$ with labels on vertices, edges, and regions (connected components of $\mathbb{R}^2 \setminus \mathcal{T}$); all this information encodes the values of a binary quadratic form q and, at the same time, helps to navigate the set of forms $SL(2,\mathbb{Z})$-equivalent to q. Topographs were introduced by J.-H. Conway in [3] in 1997. Topographs provide a powerful geometric tool for visualizing the behavior of binary quadratic forms and understanding their equivalence classes.

Let us start from the graph structure.

Definition 2. A superbase is a triple $v_1, v_2, v_3 \in \mathbb{Z}^2, v_1 + v_2 + v_3 = 0$ and $\{v_1, v_2\}$ forms a basis of $\mathbb{Z}^2$. We consider superbases up to sign, i.e. triples $\{v_1, v_2, v_3\}$ and $\{-v_1, -v_2, -v_3\}$ are equal. Consider a graph $\mathcal{T}$ whose vertices represent all superbases. Let edges connect vertices of the form

$$\{v_1, v_2, -v_1 - v_2\} \text{ and } \{v_1, -v_2, -v_1 + v_2\}.$$

Thus each edge corresponds to four bases $\{\pm v_1, \pm v_2\}$ of $\mathbb{Z}^2$. Each vertex $\{v_1, v_2, v_3\}$ has degree three, with edges $\{\pm v_1, \pm v_2\}, \{\pm v_1, \pm v_3\}, \{\pm v_2, \pm v_2\}$.

The graph $\mathcal{T}$ is connected and can be embedded in $\mathbb{R}^2$ without self-intersections; the resulting planar graph is called a topograph. Each region in the complement to the graph correspond to a primitive vector $\pm v$, see Figure 2 for a local picture near a vertex and an edge.

We label the regions of the topograph with numbers.

Example 2. Let us label the region corresponding to $\pm v$ by $||v||^2 = m^2 + n^2$ where $v = (m, n)$. Recall the *parallelogram law*: for any v_1, v_2,

$$||v_1 - v_2||^2 + ||v_1 + v_2||^2 = 2(||v_1||^2 + ||v_2||^2).$$

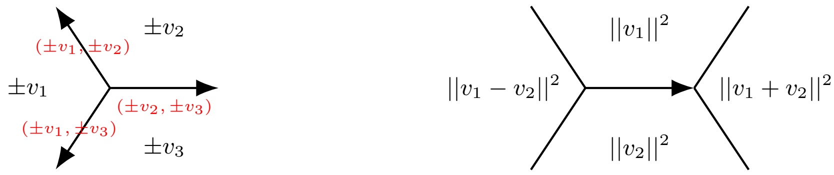


FIGURE 2. Left: local picture near a vertex corresponding to a superbase $\{v_1, v_2, v_3\}$. Right: local picture near an edge with labels corresponding to the quadratic form $q(v) = ||v||^2$.

Given a binary quadratic form q, label the region corresponding to $\pm v$ by $q(v)$.

Note that $q(v_1 - v_2) + q(v_1 + v_2) = 2(q(v_1) + q(v_2))$. Using this identity one can recover all the values of a quadratic form on vectors in $\mathbb{Z}^2$ knowing values of q on vectors in a superbase.

Conversely, if we label all the regions of the topograph, such that near every edge as in Figure 3, a), labels satisfy

$$r + u = 2(s + t), \tag{15}$$

these labels determine a unique quadratic form q. Indeed, if

$$q(v_1, v_2) = av_1^2 + bv_1v_2 + cv_2^2$$

then a, b, c can be found from $q(\binom{1}{0}) = a, q(\binom{0}{1}) = b, q(\binom{-1}{-1}) = a + b + c.$

Note that $r, s+t, u$ form an arithmetic progression with difference $g := s+t-r$, so the oriented edge pointing from r to u receives the label g, see Figure 3, a,b), which changes sign if the orientation of the edge is reversed.

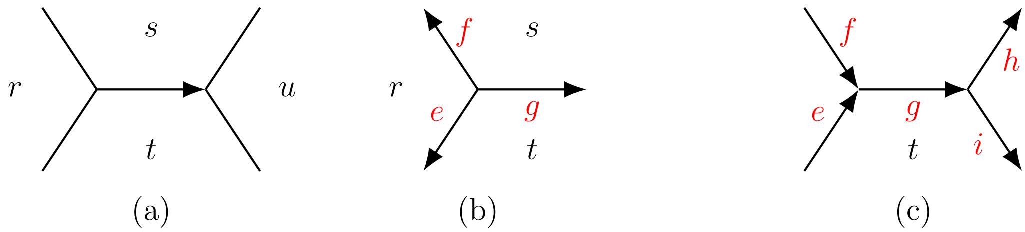


FIGURE 3. Topographs locally.

Similarly define $e = r + t - s, f = r + s - t$, see Figure 3. Then $e + g = 2t$.

Definition 3. The number $D := -ef - fg - eg$, where e, f, g are the oriented edge labels near a vertex(as in Figure 3 b)), is called the discriminant of the topograph.

It is a straightforward computation to verify that D is independent of the choice of vertex. Indeed, on figure c) we have $g - e = 2t = i - g$, and hence

$$(g - e)(g - f) = (i - g)(h - g) = g^2 - eg - gf + ef = g^2 + ih - ig - hg,$$

thus $-eg - gf + ef = ih - ig - hg$, with the orientations as in Figure c).

Given a quadratic form q with an associated bilinear form $B(x, y)$, the label on the edge $\{\pm e_i, \pm e_j\}$ is $|2B(\pm e_i, \pm e_j)|$, with sign depending on the orientation of the edge; for example, for the standard dot product $B(x, y) = x \cdot y$ we have

$$s + t - r = ||x||^2 + ||y||^2 - ||x - y||^2 = 2(x \cdot y).$$

2.1. **Useful identities in a topograph.** In this section we state the identities which are at the base of the telescoping and cancellation argument since they show that the certain first-order approximations are additive, and so their telescoping is trivial. Let us recall the following identity:

$$\frac{g}{st}+\frac{f}{rs}+\frac{e}{rt}=\frac{gr+ft+es}{rst}=\frac{g(f+e)+f(e+g)+e(f+g)}{2rst}=\frac{-D}{rst} \tag{16}$$

$$\frac{1}{e}+\frac{1}{f}+\frac{1}{g}=\frac{ef+fg+ge}{efg}=\frac{-D}{efg}. \tag{17}$$

$$\frac{s}{fg}+\frac{r}{ef}+\frac{t}{eg}=\frac{gr+ft+es}{efg}=\frac{g(f+e)+f(e+g)+e(f+g)}{2efg}=\frac{-D}{efg} \tag{18}$$

Interestingly, the combinatorial structure of a topograph encodes also trigonometric and hyperbolic relations, depending on the sign of the discriminant.

Lemma 1. For $D<0$, we have

$$\arcsin\Big(\frac{e}{rt}\cdot\frac{\sqrt{-D}}{2}\Big)+\arcsin\Big(\frac{f}{rs}\cdot\frac{\sqrt{-D}}{2}\Big)+\arcsin\Big(\frac{g}{st}\cdot\frac{\sqrt{-D}}{2}\Big)=0, \tag{19}$$

$$\arctan\Big(\frac{\sqrt{-D}}{e}\Big)+\arctan\Big(\frac{\sqrt{-D}}{f}\Big)+\arctan\Big(\frac{\sqrt{-D}}{g}\Big)=0 \tag{20}$$

for the appropriate branches of arcsin and arctan. If the sum of any two terms does not exceed π in absolute value, then the formula holds for the principal values of arcsin and arctan (that belong to $(-\pi/2,\pi/2)$).

For $D>0$ we have

$$\operatorname{arsinh}\Big(\frac{e}{rt}\cdot\frac{\sqrt{D}}{2}\Big)+\operatorname{arsinh}\Big(\frac{f}{rs}\cdot\frac{\sqrt{D}}{2}\Big)+\operatorname{arsinh}\Big(\frac{g}{st}\cdot\frac{\sqrt{D}}{2}\Big)=0 \tag{21}$$

$$\operatorname{arctanh}\Big(\frac{\sqrt{D}}{e}\Big)+\operatorname{arctanh}\Big(\frac{\sqrt{D}}{f}\Big)+\operatorname{arctanh}\Big(\frac{\sqrt{D}}{g}\Big)=0. \tag{22}$$

Proof. To prove the first identity, note that $-\arcsin A=\arcsin B+\arcsin C$ (for appropriate branches of arcsin) if and only if $-A=B\sqrt{1-C^2}+C\sqrt{1-B^2}$, then we square, and then square again; then all terms cancel. The same proof works for arsinh. For arctan and arctanh these formulas follow from (17) and the identities

$$\arctan(x_1)+\arctan(x_2)+\arctan(x_3)=\arctan\left(\frac{x_1+x_2+x_3-x_1x_2x_3}{1-x_1x_2-x_2x_3-x_3x_1}\right)$$

$$\operatorname{arctanh}(x_1)+\operatorname{arctanh}(x_2)+\operatorname{arctanh}(x_3)=\operatorname{artanh}\left(\frac{x_1+x_2+x_3+x_1x_2x_3}{1+x_1x_2+x_2x_3+x_3x_1}\right)$$

provided the denominators are not zero and $|x_i|<1$ in the formula for arctanh. □

3. Examples and ideas

This section is actually the core of the paper. In Section 3.1 we discuss a telescopic proof of (4), thus highlighting the main idea in the simplest case. In Section 3.2 we obtain Mordell-Torhheim series $\sum_{n,m\geq 1}\frac{1}{n^2m^2(n+m^2)}$ taking a limit by a parameter. In Section 3.3 we derive Hata's series for the Euler constant, using a form $q(n,m)=nm$. In Section 3.4 we explain the geometric meaning of the summand and illustrate the duality between formulas including region labels and formulas including edge labels.

3.1. **Hurwitz series.** Let $\det(x,y)$ denote the determinant of 2×2 matrix with columns $x, y \in \mathbb{Z}^2$. Let

$$A = \big\{(x,y) \mid x, y \in \mathbb{Z}^2_{\geq 0}, \det(x,y) = 1\big\},$$

i.e., the set of pairs of lattice vectors $x = (a,b), y = (c,d)$ in the first quadrant that span lattice parallelograms of oriented area one, i.e. $ad - bc = 1$.

We prove the Hurwitz result (4) via a telescopic method.

Theorem 8.

$$4\sum_A \frac{1}{|x|^2|y|^2|x+y|^2} = \pi.$$

Proof. Define $F(x,y) = \frac{2x\cdot y}{|x|^2|y|^2}$, $F : (\mathbb{Z}^2)^2 \to \mathbb{R}$. An explicit computation shows that

$$F(x,y) - F(x+y,y) - F(x,x+y) = \frac{-4\det(x,y)^2}{|x|^2|y|^2|x+y|^2}. \tag{23}$$

Consider the sum of the expressions

$$F(x,y) - F(x+y,y) - F(x,x+y)$$

over the set $\{(x,y)|x,y \in \mathbb{Z}^2_{\geq 0} \cap [0,n]^2, \det(x,y) = 1\}$. By cancelling identical terms with opposite signs we get $F(\binom{1}{0},\binom{0}{1})$ together with the sum of $-F(x+y,y) - F(x,x+y)$ over the set

$$\big\{(x,y)|x,y \in \mathbb{Z}^2_{\geq 0} \cap [0,n]^2, \det(x,y) = 1, x+y \notin \mathbb{Z}^2_{\geq 0} \cap [0,n]^2\big\}.$$

Note that each element (x,y) of the above set represents a parallelogram spanned by x and y. All these parallelograms have area one and their angles at the origin partition the angle $\pi/2$ of the first quadrant. Next, $\frac{2x\cdot y}{|x|^2\cdot|y|^2}$ is 2α up to third order terms, where α is the angle between x and y. Indeed, $\sin\alpha\cdot|x||y| = 1$

$$\frac{2x\cdot y}{|x|^2|y|^2} = 2\cos\alpha\sin\alpha = 2\alpha - \frac{1}{6}(2\alpha)^3 + \ldots$$

Also, for $\sum_{i=1}^n \alpha_i = \pi/2$ we have that

$$|2\sum_{i=1}^n \sin\alpha_i\cos\alpha_i - \pi| \leq \frac{8\pi}{3}\max_{i=1..N}|\alpha_i|^2 \to 0 \text{ as } \max_{i=1..N}|\alpha_i| \to 0.$$

Thus, since $F(\binom{1}{0},\binom{0}{1}) = 0$, as $n\to\infty$,

$$F(\binom{1}{0},\binom{0}{1}) + \sum(-F(x+y,y) - F(x,x+y)) \to -\pi.$$

Finally, we multiply by -4 from (23) to get the desired formula. $\square$

In terms of topographs, we consider the formula (9)

$$\sum_{r \succ_{t}^{s} \in \mathcal{T}'} \frac{1}{|rst|} = \frac{1}{D}\left(\frac{e_0}{r_0t_0} - \frac{2\arcsin(\frac{e_0}{r_0t_0}\cdot\frac{\sqrt{-D}}{2})}{\sqrt{-D}}\right),$$

for the edge $E = \{\pm\binom{1}{0}, \pm\binom{0}{1}\}$ and the positive-definite quadratic form $q(n,m) = n^2 + m^2$ with $D = -4$, $e_0 = 0, r_0 = t_0 = 1$.

To prove the formula we telescope

$$\frac{e}{rt} = 2\frac{x\cdot y}{|x|^2|y|^2}.$$

The $\frac{1}{D}$ in the formula is -4 in (23). The only surviving term $\frac{e_0}{r_0 t_0}$ is zero in this case.

Each term at the crown, up to third order, is

$$\frac{2\arcsin(\frac{e_0}{r_0t_0}\cdot\frac{\sqrt{-D}}{2})}{\sqrt{-D}}=\arcsin(\frac{e_0}{r_0t_0}).$$

So the geometric meaning of the terms that we telescope is the angle between vectors. Note that to get the telescoping relation we may consider other quadratic forms, not necessarily $q(v)=||v||^2$. This leads to telescoping identitites over topographs, i.e. formulas 9,10,11,12,13,14.

3.2. **Mordell-Tornheim series.** Let us pick $\mu>0$ and compute the following sum:

$$\sum_{\mu}=\sum_{\substack{a\geq b\\ c\geq d\\ ad-bc=1}}\frac{2\mu^2}{(a^2+\mu^2b^2)(c^2+\mu^2d^2)\left((a+c)^2+\mu^2(b+d)^2\right)}.$$

This is equivalent to computing

$$\sum_{\mu}=\sum\frac{2|\det(x,y)|^2}{|x|^2|y|^2|x+y|^2},$$

where

$$x,y\in\mathbb{Z}_{\geq 0}\cdot\begin{pmatrix}1\\0\end{pmatrix}+\mathbb{Z}_{\geq 0}\cdot\begin{pmatrix}1\\\mu\end{pmatrix},$$

Thus we can use the same method, and the same function F as in the proof of Theorem 8.

Performing telescoping we get

$$\sum_{\mu}=-\left(F\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1\\\mu\end{pmatrix}\right)-\sum_{(u,v)\in\text{crown}}F(u,v)\right),$$

since the determinant of the matrix formed by these basis vectors is μ, refer to (23). Therefore, we proved

Theorem 9.

$$\sum_{\substack{a\geq b\\ c\geq d\\ ad-bc=1}}\frac{1}{(a^2+\mu^2b^2)(c^2+\mu^2d^2)\left((a+c)^2+\mu^2(b+d)^2\right)}=\frac{1}{2\mu^2}\left(\frac{\arctan\mu}{\mu}-\frac{1}{1+\mu^2}\right).$$

Taking the limit as $\mu\to 0$ yields $\sum_{\mu}/2\mu^2\to 1/3$. Indeed,

$$\sum_{\substack{a,c\geq 1\\ \gcd(a,c)=1}}\frac{1}{a^2c^2(a+c)^2}=\frac{1}{3},\tag{24}$$

because for each such a pair (a,c) there exists a unique pair $(b,d)\in\mathbb{Z}^2_{\geq 0}$ such that $ad-bc=1, b\leq a, d\leq c$. This identity (24) is well-known [14, 8].

Since in both sides of equality we have analytic functions, we may substitute $\mu=\frac{i}{2}$. Then,

$$\sum_{\substack{a\geq b\\ c\geq d\\ ad-bc=1}}\frac{1}{(4a^2-b^2)(4c^2-d^2)\left(4(a+c)^2-(b+d)^2\right)}=$$

$$=\frac{1}{64}\cdot\left(\frac{\arctan\frac{i}{2}}{\frac{i}{2}}-\frac{1}{1+(\frac{i}{2})^2}\right)\Big/\left(2\cdot\left(\tfrac{i}{2}\right)^2\right)=\frac{\frac{4}{3}-\ln 3}{32}.$$

Remark 1. Since

$$\sum_{\substack{a\geq b\\ c\geq d\\ ad-bc=1}} \frac{1}{(a^2+\mu^2b^2)(c^2+\mu^2d^2)\left((a+c)^2+\mu^2(b+d)^2\right)} = \frac{1}{2\mu^2}\left(\frac{\arctan\mu}{\mu} - \frac{1}{1+\mu^2}\right) =$$

$$= \sum_{k=0}^{\infty}(-1)^k\frac{k+1}{2k+1}\mu^{2k} = \frac{1}{3} - \frac{2}{5}\mu^2 + \frac{3}{7}\mu^4 - \dots.$$

we can derive the identities looking for other coeffcients, for example

$$\sum_{\substack{a\geq b\\ c\geq d\\ ad-bc=1}} \frac{1}{a^2c^2(a+c)^2}\left(\frac{b^2}{a^2}+\frac{d^2}{c^2}+\frac{(b+d)^2}{(a+c)^2}\right) = \frac{2}{5}\ .$$

3.3. **Hata's series.** Let us look on a series for γ, due to Hata Masayoshi.

Definition 4. Let $\mathcal{F}$ denote the set of ordered pairs of fractions $\left(\frac{a}{b},\frac{c}{d}\right)$ in lowest terms such that: $0\leq\frac{a}{b}<\frac{c}{d}\leq 1$, $ad-bc=-1$. Thus $\mathcal{F}$ is the set of pairs of consecutive Farey fractions. Let $\mathcal{F}^*=\left\{\left(\frac{a}{b},\frac{c}{d}\right)=\left(\frac{0}{1},\frac{1}{n}\right): n\in\mathbb{N}\right\}$.

Masayoshi Hata proved the following theorem.

Theorem 10 ([8])**.** In the above notation

$$\gamma = \frac{1}{2}+\frac{1}{2}\sum_{\left(\frac{a}{b},\frac{c}{d}\right)\in\mathcal{F}\setminus\mathcal{F}^*}\frac{1}{abcd(a+c)(b+d)}.$$

Hata's proof is very nice. He studies presentations of functions in a certain Schauder basis associated with pairs of consecutive Farey fractions. Then, using an identity of Parseval-type for the function $\psi(x)=x\{\frac{1}{x}\}(1-\{\frac{1}{x}\})$ he arrives to the above theorem.

Note that

$$\frac{1}{abcd(a+c)(b+d)} = \frac{1}{q(x)q(y)q(x+y)} = \frac{1}{rst}$$

for the quadratic form $q(v)=mn$, for $v=(m,n)$. Thus, this formula can be deduced using our telescopic method.

Lemma 2. For $a,b,c,d\geq 0$ with $ad-bc=-1$ one has

$$\operatorname{arsinh}\left(\frac{ad+bc}{2abcd}\right) = \log\left(\frac{ad}{bc}\right).$$

Proof. By a direct check we see that

$$\sqrt{16+\left(\frac{2(ad+bc)}{abcd}\right)^2} = \frac{(ad+bc)^2+1}{abcd},$$

then,

$$\operatorname{arsinh}\left(\frac{ad+bc}{2abcd}\right) = \log\left(\frac{-(ad+bc}{2abcd}+\sqrt{1+\left(\frac{(ad+bc)}{2abcd}\right)^2}\right) =$$

$$\log\left(\frac{(ad+bc)^2-2(ad+bc)+1}{4abcd}\right) = \log\left(\frac{(ad)^2}{abcd}\right) = \log\left(\frac{ad}{bc}\right).$$

∎

Thus we see that

$$\operatorname{arsinh}\left(\frac{ad+bc}{2abcd}\right)=\operatorname{arsinh}\left(\frac{a(b+d)+b(a+c}{2ab(a+c)(b+d)}\right)+\operatorname{arsinh}\left(\frac{(a+c)d+(b+d)c}{2(a+c)(b+d)cd}\right).$$

In order to find $\sum\frac{1}{rst}$ one take a telescopic sum of $\frac{e}{rt}-\frac{g}{st}-\frac{h}{rs}$ which, in our case, becomes (since $e=q(x+y)-q(x)-q(y)$)

$$F(\binom{a}{b},\binom{c}{d})=\frac{ad+bc}{abcd}\sim 2\operatorname{arsinh}\left(\frac{ad+bc}{2abcd}\right),$$

if it is small.

Therefore

$$\sum_{(\frac{a}{b},\frac{c}{d})\in\mathcal{F}\setminus\mathcal{F}^*}\frac{1}{abcd(a+c)(b+d)}=\sum_{n=1}^{\infty}(F(\binom{1}{n},\binom{1}{n+1})-2\operatorname{arsinh}\left(\frac{1\cdot(n+1)+n\cdot 1}{2n(n+1)}\right))=$$

$$=2\lim_{n\to\infty}(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n)-1=2\gamma-1.$$

3.4. **Duality: inside and outside the circle.** The telescopic identities have a natural geometric interpretation, revealing a duality between sums of the products of reciprocals to region labels and sums of the products of reciprocals to edge labels of the topograph. The product of the region labels at a vertex correspond to the area of the inscribed triangle, while the product of the edge labels gives the area of the triangle formed by the tangent lines at the corresponding points, cf. Legendre duality in [10].

To relate the Farey tessellation to points on a unit circle, we use the rational parametrization of $x^2+y^2=1$ by

$$f:\binom{a}{b}\to\left(\frac{2ab}{a^2+b^2},\frac{a^2-b^2}{a^2+b^2}\right).$$

Note that $f(\binom{1}{0})=(0,1), f(\binom{0}{1})=(0,-1)$. By a direct calculation, the area of the triangle with vertices $f(\binom{a}{b}), f(\binom{c}{d}), f(\binom{a+c}{b+d})$ is equal to

$$\frac{2|ad-bc|^3}{(a^2+b^2)\cdot(c^2+d^2)\cdot((a+c)^2+(b+d)^2)}=\frac{2}{rst}. \tag{25}$$

Here r,s,t denote the values of the quadratic form $q(n,m)$ on a superbase vectors $\left\{\binom{a}{b},\binom{c}{d},-\binom{a+c}{b+d}\right\}$, the notation that we use for the topograph's labels on regions.

Theorem 11 ([9]).

$$\sum_{\substack{a,b,c,d\in\mathbb{Z}_{\geq 0}\\ ad-bc=1}}\frac{1}{(a^2+b^2)(c^2+d^2)((a+c)^2+(b+d)^2)}=\frac{\pi}{4}.$$

Proof. The area of the right half of the unit disc is $\pi/2$ and the triangles with vertices

$$f(\binom{a}{b}), f(\binom{c}{d}), f(\binom{a+c}{b+d}), a,b,c,d\geq 0, ad-bc-1$$

tile it completely. So we divide (25) by two and sum over all tuples (a,b,c,d) with $a,b,c,d\geq 0, ad-bc-1$. □

This reasoning is due to the original article of A. Hurwitz [9]. Hurwitz's proof works for any positive-definite binary quadratic form q (the above case corresponds to $q(v) = ||v||^2, v \in \mathbb{Z}^2$) and consists of using a rational parametrization of a quadric curve to cut its interior into triangles corresponding to consecutive Farey fractions $(\frac{a}{b}, \frac{a+c}{b+d}, \frac{c}{d})$, and then the areas of triangles are proportional to

$$(q(a,b) \cdot q(c,d) \cdot q(a+c, b+d))^{-2}.$$

Alternatively, consider the tangent lines $l_{a,b}$ to the unit circle at points $f(\binom{a}{b})$. The area of the triangle formed by these lines $l_{a,b}, l_{c,d}, l_{a+c,b+d}$ is

$$\frac{|ad-bc|^3}{(ac+bd)(a(a+c)+b(b+d))((a+c)c+(b+d)d)} = \frac{8}{efg}.$$

Note that the dot products in the denominators are exactly e, f, g on the edges in the topograph. Since these triangles tile the domain in between of the circle and the tangents at $(0,1)$ and $(1,0)$ we can evaluate the sum $\sum \frac{1}{efg}$, namely:

Theorem 12.

$$\sum_{\substack{a \geq b \\ c \geq d \\ ad-bc=1}} \frac{1}{(ac+bd)(a(a+c)+b(b+d))((a+c)c+(b+d)d)} = \frac{1-\pi/4}{8}.$$

Remark 2. This identity may be generalized by deforming the lattice as in Section 3.2. Consider the vectors $a\binom{1}{0} + b\binom{1}{\mu}$ and draw the tangent lines at points $f(\binom{a}{\mu b})$. Then the following sum is equal to the area of a part in between of the unit circle and tangents to it at $(0,1)$ and $\frac{2\mu}{\mu^2+1}, \frac{\mu^2-1}{\mu^2+1}$, namely

$$\sum_{\substack{a \geq b \\ c \geq d \\ ad-bc=1}} \frac{\mu^3}{(ac+\mu^2 bd)(a(a+c)+\mu^2 b(b+d))((a+c)c+\mu^2(b+d)d)} = \mu - \arctan \mu.$$

Dividing by μ^3 and substituting $\mu = 0$ we get (24) one more time:

$$\sum_{\gcd(a,c)=1} \frac{1}{(ac)(a(a+c))((a+c)c)} = \frac{1}{3}.$$

Investigating the next coefficient in the Taylor series of arctan we get

$$\sum_{\substack{a \geq b \\ c \geq d \\ ad-bc=1}} \frac{1}{a^2c^2(a+c)^2} \left(\frac{bd}{ac} + \frac{b(b+d)}{a(a+c)} + \frac{(b+d)d}{(a+c)c} \right) = \frac{1}{5}.$$

We may also plug $\mu = i/2$ getting

$$\sum_{\substack{a \geq b \\ c \geq d \\ ad-bc=1}} \frac{1}{(4ac-bd)(4a(a+c)-b(b+d))(4(a+c)c-(b+d)d)} =$$

$$= \frac{1}{64(i/2)^3}(i/2 - \arctan i/2) = \frac{\ln 3 - 1}{16}.$$

4. Proofs

We now generalize the telescoping argument of Section 3.1 to prove Theorems 5,6 for arbitrary discriminant D.

Our arguments rely on the trigonometric identities derived in Section 2.1. To proceed we need the following lemma, in the notation presented in Figure 4.

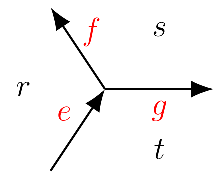


Figure 4. Topographs locally

Lemma 3. Consider a topograph with discriminant $D < 0$ and its vertex with labels as in Figure 4. If the right-hand side of the following formula lies in $(-\pi/2, \pi/2)$, we have

$$\arcsin(\frac{e}{rt} \cdot \frac{\sqrt{-D}}{2}) = \arcsin(\frac{f}{rs} \cdot \frac{\sqrt{-D}}{2}) + \arcsin(\frac{g}{st} \cdot \frac{\sqrt{-D}}{2})$$

$$\arctan(\frac{\sqrt{-D}}{e}) = \arctan(\frac{\sqrt{-D}}{f}) + \arctan(\frac{\sqrt{-D}}{g})$$

For a topograph with discriminant $D > 0$ we have

$$\operatorname{arsinh}(\frac{e}{rt} \cdot \frac{\sqrt{D}}{2}) = \operatorname{arsinh}(\frac{f}{rs} \cdot \frac{\sqrt{D}}{2}) + \operatorname{arsinh}(\frac{g}{st} \cdot \frac{\sqrt{D}}{2})$$

$$\operatorname{arctanh}(\frac{\sqrt{D}}{e}) = \operatorname{arctanh}(\frac{\sqrt{D}}{f}) + \operatorname{arctanh}(\frac{\sqrt{D}}{g})$$

Proof. Follows immediately from Lemma 1 since we only change the orientation of the edge with label e. □

Proof of Theorem 5, part (9). In the notation of Figure 4, the identity (16) becomes

$$\frac{1}{D}(\frac{e}{rt} - \frac{f}{rs} - \frac{g}{st}) = \frac{1}{rst}. \tag{26}$$

This is immediate generalisation of (23) for $q(v) = ||v||^2$, because $F(x, y)$ in (23) is $\frac{e}{rt}$ and $D = -4$. Summing (26) over the vertices of $V \in \mathcal{T}''$ of degree three we see that all the intermediate terms cancel, and only the terms corresponding to the root and the edges in the crown survive, so we have

$$\sum_{V \in \mathcal{T}''} \frac{1}{rst} = \frac{1}{D} \left(\frac{e}{rt}(\text{root}) - \sum_{V \in \text{crown}(\mathcal{T}'')} \frac{e}{rt}(V) \right).$$

Now, if $\frac{e}{rt}(V) \to 0$ as the distance from V to the root grows, then

$$\sum_{V \in \text{crown}(\mathcal{T}'')} \frac{e}{rt}(V) - \sum_{V \in \text{crown}(\mathcal{T}'')} \frac{\arcsin(\frac{e}{rt}(V) \cdot \frac{\sqrt{-D}}{2})}{\sqrt{-D}/2} = 0$$

when the distance between the crown and the root tends to infinity. Now, it follows from Lemma 1 that

$$\sum_{V\in \text{crown}(\mathcal{T}'')} \frac{\arcsin(\frac{e}{rt}(V)\cdot\frac{\sqrt{-D}}{2})}{\sqrt{-D}/2} = \frac{\arcsin(\frac{e}{rt}(\text{root})\cdot\frac{\sqrt{-D}}{2})}{\sqrt{-D}/2}$$

and this finishes the proof.

□

The proof of Theorem 6 part (11) is identical, the only difference is that we use formulas for arsinh.

So, when we telescope $\frac{1}{rst}$ over both branches $\mathcal{T}', \mathcal{T}\setminus\mathcal{T}'$ of a topograph, we get

$$\frac{1}{D}\left(\frac{e_0}{r_0t_0} - \frac{2\arcsin(\frac{e_0}{r_0t_0}\cdot\frac{\sqrt{-D}}{2})}{\sqrt{-D}}\right) + \frac{1}{D}\left(\frac{-e_0}{r_0t_0} - \frac{2\arcsin(\frac{-e_0}{r_0t_0}\cdot\frac{\sqrt{-D}}{2})}{\sqrt{-D}}\right) = \frac{2\cdot 2\pi}{(-D)^{3/2}},$$

i.e. exactly (9), and then in implies the Hurwitz class number formula.

Telescoping $\frac{1}{efg}$ (labels on egdes) over a topograph gives the class number formula for $D > 0$.

Proof of Theorem 5, part (10). Here we will telescope $\frac{s}{fg}$ and use the telescoping identity (18).

Denote the edges of the region s by $f = f_0, f_1, \dots$ on the left and $g = g_0, g_1, \dots$ on the right. Note that $f_{k+1} = f_k + 2s, g_{k+1} = g_k + 2s$ and

$$\sum_{k=0}^{\infty} \frac{s}{f_k f_{k+1}} = \frac{s}{2s}\sum_{k=0}^{\infty}(\frac{1}{f_k} - \frac{1}{f_{k+1}}) = \frac{1}{2f_0}.$$

Therefore in the sum of $\frac{s}{fg}$ we have $\frac{1}{2e_0}$ twice, all intermediate terms cancel because $\frac{1}{2f_0} + \frac{1}{2g_0} - \frac{s}{f_0g_0} = 0$ for $s = \frac{f_0+g_0}{2}$. The sum of the terms at the crown is $-\sum\frac{1}{e}$ over all e in the crown. Then we use the approximation

$$\frac{\sqrt{-D}}{e} \sim \arctan(\frac{\sqrt{-D}}{e})$$

at the edges of the crown and Lemma 3 to conclude the proof of the theorem.

□

The second part of Theorem 6 is proven similarly.

Theorem 7 is proven by taking the limit $D \to 0$.

5. Acknowledgements

In 2024 I discovered a simple telescopic proof [11] of (2) without knowing all the above story about topographs that I learned recently thanks to users of MathOverflow [1]. I thank Wadim Zudilin, Mikhail Shkolnikov, and Ernesto Lupercio for discussions.

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