Relation to Modular Graph Functions from String Theory

Let

$$A = \left\{(x, y) \mid x, y \in \mathbb{Z}^2_{\geq 0}, \det(x \;\; y) = 1\right\},$$

i.e., the set of pairs of lattice vectors in the first quadrant that span lattice parallelograms of the oriented area one.
This theorem was proved on the first lecture via telescoping and via geometric arguments with areas of triangles:

$$4 \sum_A \frac{1}{|x|^2 \cdot |y|^2 \cdot |x+y|^2} = \pi.$$

Today we will connect this formula to MGF (Modular Graph Functions) like

$$C_\Gamma(\tau) = \Sigma'_{p_1,\ldots,p_w \in \Lambda} \prod_{\alpha=1}^{w} \frac{y}{\pi |p_\alpha|^2} \prod_{i=1}^{N} \delta(\sum_{\alpha=1}^{w} \Gamma_{i\alpha} p_\alpha), \tau = x + iy$$

“The structure of the low energy expansion of genus-one Type II superstring amplitudes leads one to associate modular functions with Feynman graphs for a conformal scalar field on a torus.”

This talk is based on the following papers:

• D. Zagier, Notes on lattice sums (unpublished) (I have in posession a note called “Evaluation of lattice sums” **(2009)**)

• *E. d’Hoker, M. B. Green, and P. Vanhove*, On the modular structure of the genus-one type II superstring low energy expansion, Journal of High Energy Physics, (8):1–69, **(2015)**,

• *Eric D’Hoker, Michael B. Green*, Identities between modular graph forms, Journal of Number Theory 189 25–80 **(2018)**

String Theory, Feynman graphs

an edge is a particle propagating – external (incoming/outgoing) or internal (virtual)
a vertex corresponds to an interaction point where particles are created or destroyed
a loop is a quantum fluctuation – internal virtual particles whose momenta are integrated over
an external legs are the actual incoming and outgoing particles in a scattering event

Eisenstein series, $C_{a,b,c}$ and $D_4 = C_{1,1,1,1}$, $D_5 = C_{1,1,1,1,1}$

Laplacian

Define $\Delta = 4y^2\partial_\tau\partial_{\bar\tau} = y^2(\partial_x^2 + \partial_y^2)$, and $\nabla = 2iy^2\partial_\tau$

$$\Delta E_w = w(w-1)E_w, \Delta C_{1,1,1} = 6E_3.$$

In fact $C_{1,1,1} = E_3 + \zeta(3)$ due to D. Zagier. $C_{2,2,1} = \frac{2}{5}E_5 + \zeta(5)/30.$

$$(\Delta - 2)C_{2,1,1} = 9E_4 - E_2^2.$$

$$\Delta C_{2,2,1} = 8E_5.$$

$$D_4 - 24C_{2,1,1} - 3E_2^2 + 18E_4$$

$$D_5 = 60C_{3,1,1} + 10E_2C_{1,1,1} + 48E_5 - 16\zeta(5)$$

Proof ideas: check few terms (in q), then have estimates for modular forms. We use

$$\sum \frac{1}{z+n} = -i\pi\frac{1+e^{2\pi iz}}{1-e^{2\pi iz}} = -i\pi\frac{1+q}{1-q}$$

Poincaré series

A Poincaré series is an expression of the type

$$P(\tau; s_1, s_2, s_3) = \sum_{\gamma \in \Gamma} H(\gamma\tau; s_1, s_2, s_3)$$

where $\Gamma = SL(2, \mathbb{Z})$ and $\tau \in \mathbb{H}$ (upper half-plane). In our case, let $\tau = x + yi$ and

$$H(\tau; s_1, s_2, s_3) = \frac{y^{s_1+s_2+s_3}}{|\tau|^{2s_2}|\tau - 1|^{2s_3}}.$$

Let $\tau = \frac{ai+b}{ci+d}$, let us see what is $H(...)$ and compute Δ.

Connection between Poincar'e series and summation over a topograph

$q = [a, b, c]$, $z_q = \frac{-b+\sqrt{D}}{2a}$, then

$$H(z_q; s_1, s_2, s_3) = \left(\frac{\sqrt{|D|}}{2}\right)^{s_1+s_2+s_3} \frac{1}{a^{s_1} c^{s_2} (a+b+c)^{s_3}}.$$

So $P(\tau; s_2, s_2, s_3)$ is just a sum over a topograph. Then

$P(\tau; 1, 1, 1) = 3\pi/2$, $P(\tau; 2, 2, 1) = 3\pi/4$

$$\frac{12\pi}{w_Q} = |D|^{3/2} \sum_{q=[a,b,c]\sim Q} \frac{1}{a(a+b+c)c}$$

so we recover the Hurwitz formula:

Relation to class numbers for quadratic forms

Theorem (Hurwitz, 1905)

$$\frac{1}{\omega_D} h(D) = \frac{1}{12\pi} |D|^{3/2} \sum_{a>0,\, b^2-4ac=D} \frac{1}{a(a+b+c)c}, \omega_d = \begin{cases} 2 & D < -4 \\ 4 & D = -4 \\ 6 & D = -3 \end{cases}$$

where $h(D)$ is the number of equivalence classes of binary quadratic forms with discriminant $D < 0$.

[Cormac O'Sullivan] Let $\mathcal{T}$ be any topograph of discriminant $D < 0$ then

$$|D|^{3/2} \sum_{r \left\rangle \frac{s}{t} \right. \in \mathcal{T}} \frac{1}{|rst|} = 4\pi, \qquad |D|^{5/2} \sum_{r \left\rangle \frac{s}{t} \right. \in \mathcal{T}} \frac{|r+s+t|}{|rst|^2} = 24\pi, \qquad (1)$$

$C^k_{a,b,c}$

$$C^k_{a,b,c}(\tau) = \sum_{(m_r,n_r)\neq 0} \frac{\delta_{m,0}\delta_{n,0}\left(m_1 n_2 - n_1 m_2\right)^{2k}\left(y/\pi\right)^{a+b+c}}{\left|m_1\tau + n_1\right|^{2a}\left|m_2\tau + n_2\right|^{2b}\left|m_3\tau + n_3\right|^{2c}}$$

$$\left(\Delta - w(w-1)\right) C^k_{a,b,c} = -4ab C^{k+1}_{a+1,b+1,c} - 4bc C^{k+1}_{a,b+1,c+1} - 4ca C^{k+1}_{a+1,b,c+1}$$

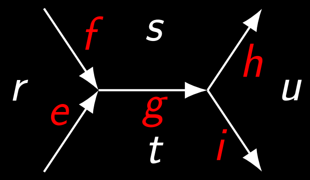
$$-4 C^{k+1}_{a,b,c} = C^k_{a-2,b,c} + C^k_{a,b-2,c} + C^k_{a,b,c-2} - 2C^k_{a,b-1,c-1} - 2C^k_{a-1,b,c-1} - 2C^k_{a-1,b-1,c}$$

$$\sum_{r=1}^{3}\left|m_r\tau + n_r\right|^4 - \sum_{r\neq r'}\left|m_r\tau + n_r\right|^2\left|m_{r'}\tau + n_{r'}\right|^2 = -4\tau_2^2\left(m_1 n_2 - n_1 m_2\right)^2$$

$$\begin{aligned}
&\left(\Delta - a(a-1) - b(b-1) - c(c-1)\right) C^k_{a,b,c} \\
&\quad = + ab\left(C^k_{a-1,b+1,c} + C^k_{a+1,b-1,c} + C^k_{a+1,b+1,c-2} - 2C^k_{a,b+1,c-1} - 2C^k_{a+1,b,c-1}\right) \\
&\qquad + bc\left(C^k_{a-2,b+1,c+1} + C^k_{a,b-1,c+1} + C^k_{a,b+1,c-1} - 2C^k_{a-1,b,c+1} - 2C^k_{a-1,b+1,c}\right) \\
&\qquad + ca\left(C^k_{a-1,b,c+1} + C^k_{a+1,b-2,c+1} + C^k_{a+1,b,c-1} - 2C^k_{a,b-1,c+1} - 2C^k_{a+1,b-1,c}\right).
\end{aligned}$$

(from “On the modular structure of the genus-one type II superstring low energy expansion” E. d’Hoker et al.)

Telescoping over Conway's topographs



(c)

Labels on regions satisfy $r+u=2(s+t)$. So, $r, s+t, u$ form an arithmetic progression with difference $g := s+t-r$. Note: $e+g=2t$.
$D := -ef - fg - eg$ is the same for all vertices (taken with signs as in figure b)), and is called the discriminant of the topograph.
Fundametal identities (abusing the signs):

$$\frac{g}{st}+\frac{-h}{su}+\frac{-i}{tu}=\frac{gu+ht+is}{stu}=\frac{-g(i+h)-h(g+i)-i(h+g)}{2stu}=\frac{2D}{stu}$$

Similar for

$$\frac{t}{eg}+\frac{s}{fg}+\frac{r}{ef}$$

Geometric meaning of telescoping $\frac{1}{egf}$

So, over a branch of a topograph we can telescope $\frac{1}{rst}$ (labels on regions), gives the class number formula for $D < 0$ (and corresponds to the area of the ellipse)
We can also telescope $\frac{1}{efg}$ (labels on egdes), gives the class number formula for $D > 0$.
Telescoping over all topograph gives Modular Graph Functions.
What is the telescoping $\frac{1}{egf}$ for $D < 0$? Well, just the area **outside!**

Geometric meaning of terms at the crown: angles

What is the geometric meaning of $\frac{g}{st} \sim \frac{2x\cdot y}{|x|^2|y|^2} \sim \sin 2\alpha \sim 2\alpha$? if it is small,

$$\frac{g}{st} \sim \arcsin \frac{g}{st}$$

and then (since for $x^2 + y^2$ the discriminant is -4)

$$\arcsin(\frac{g}{st} \cdot \frac{\sqrt{-D}}{4}) = \arcsin(\frac{h}{su} \cdot \frac{\sqrt{-D}}{4}) + \arcsin(\frac{i}{tu} \cdot \frac{\sqrt{-D}}{4})$$

So the sum of the terms over the crown in the $\arcsin(\frac{g}{st} \cdot \frac{\sqrt{-D}}{4})$ at the root.
Thus $\sum \frac{1}{rst} = \frac{1}{2D}\left(\frac{g}{st}(root) - \frac{4}{\sqrt{-D}} \cdot \arcsin(\frac{g}{st} \cdot \frac{\sqrt{-D}}{4})\right)$

For the case $\sum \frac{1}{|x|^2|y|^2|x+y|^2}$ we get $\frac{1}{-8}(0 - \frac{4}{2}\pi) = \pi/4$