How to teach mathematics to sociologists?

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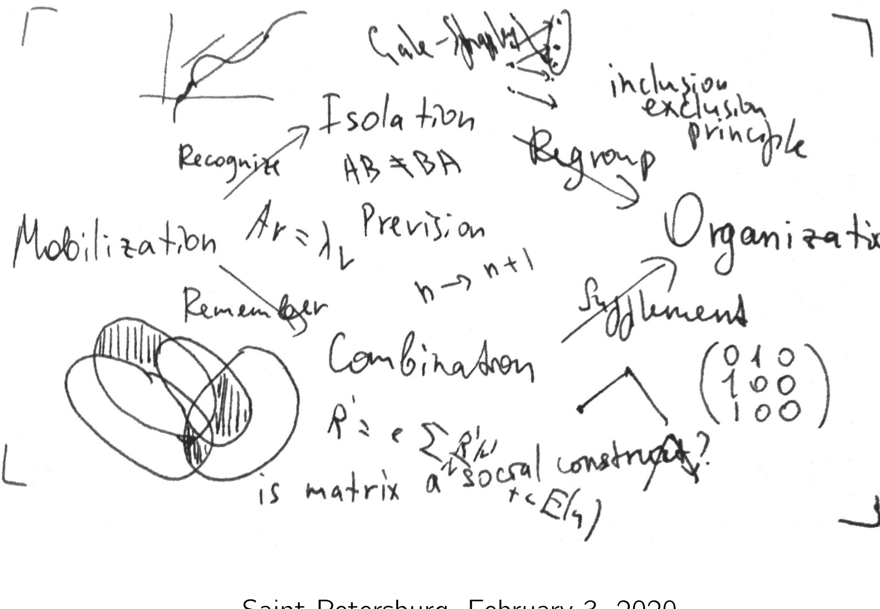
My advice for teachers of a math course for freshman sociologists. Aligned with references to relevant sociological literature, teaching goals, and objectives, and how this worked in practice. Enjoy, engage, and write to me what do you think.
http://mathcenter.spb.ru/nikaan/teach.html



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0 Preface

0.1 :: why a sociologist should care about math?

It is not uncommon that a new mathematical tool contributes to applications not by answering a pressing question–of–the–day but by revealing a different (and perhaps more significant) underlying principle.

Robert W Ghrist. Elementary applied topology

Linear algebra. Imagine a huge network such as facebook. Who are the most influential figures? The first idea might be to look at the number of friends or subscribers: who has more friends is more influential. Unfortunately, this can be easily manipulated by creating a lot of fake accounts-friends. A more robust approach is to say that the influence of a person is proportional to the sum of his friends' influences. To make this idea precise, we need *eigenvectors* and *eigenvalues*, important concepts of linear algebra. Nowadays, looking into networks you can say a lot about migration, education, culture, beliefs, etc. Hence sociologists should learn big data analysis which, in turn, massively uses linear algebra.

Often sociological theory is verbally stated, and ambiguously relates uninterpreted concepts in terms of far too general functional relationships, such as 'varies positively with', without specification of the form of such a relationship. Consequently, there is often no clear way to validate (or, in some cases, to falsify) such theories, and the structure is sometimes so obscure that almost any testable deductions are possible. To some extent this state of affairs is a consequence of the sheer complexity of social phenomena, and more rigorous attempts often come unstuck simply because the genuinely sociological content disappears in the process of making a theory tractable.

Anthony P.M. Coxon. Mathematical applications in sociology: measurement and relations.

Logic, the habit of making statements formal and precise. Studying linear algebra or statistics is very hard without an acquaintance with the mathematical concept of proof. Mathematics is a language, way more formal and precise than any spoken language; it is an advantage to speak and understand it at least on the survival level. Though it is difficult to imagine that sociologists will use induction or will prove any theorem about graphs in their professional life, I am convinced that these topics help a lot in developing skills which are **obligatory prerequisites** in order to study linear algebra or statistics. In short, I think that omitting these (or functionally similar) topics in the courses of mathematics for non-mathematicians is the reason why those non-mathematicians know mathematics so poorly and suffer on math courses so miserably.

> *In the beginning, we should ask: What is the causal relationship of interest? Although purely descriptive research has an important role to play, we believe that the most interesting research in social science is about cause and effect, like the effect of class size on children's test scores... A causal relationship is useful for making predictions about the consequences of changing circumstances or policies; it tells us what would happen in alternative (or 'counterfactual') worlds. For example, as part of a research agenda investigating human productive capacity – what labor economists call human capital – we have both investigated the causal effect of schooling on wages.*
>
> J.D. Angrist, and J.-S.Pischke. Mostly harmless econometrics: An empiricist's companion.

Sociologists are concerned with causal effects. So, if they want not only to run linear regression but also understand what does the result mean, they need to study econometrics. No surprise, this requires a good mastery of analysis, linear algebra, and statistics.

I think that these three reasons (networks, clarifying the language, econometrics) are already enough to convince a freshman that math is worth to study hard and thoroughly.

Last but not least: sociologists compete with economists (who study more math) on the labor market. Knowing math (as well as programming skills) is an advantage. Soft skills are much easy to learn (when you care) than math.

0.2 :: foreword for teachers

Dear teacher,

Probably, you are a mathematician assigned to teach mathematics to sociologists. Beware! Sociologists are not mathematicians, and they are not stupid. The worst thing you can do is to teach the standard course of linear algebra or analysis omitting proofs: there is no chance for even the brightest students to make sense of it (try to teach mathematicians in the same way).

In writing notes for teachers, I owe my inspiration to the lecture of Prof. Rokhlin[1] where he argued that the structure of a math course for non-mathematicians should be **different**. Convinced by his ideas, I start with integration, then go to differentiation, and never touch the limits, because limits require at least 2-3 lectures to thoroughly

[1] V.A. Rokhlin. "Teaching mathematics to non-mathematicians". In: *http://www.math.stonybrook.edu/~oleg/Rokhlin/LectLMO-eng.pdf* (1981).

comprehend them. Since I have only five lectures for analysis, something must be sacrificed and let it be not derivatives and integrals.

While preparing this course I have read a lot about mathematics in sociology and this determined the choice of material. References to sociological articles are provided in abundance, showing applications of concepts that we study and future directions to self-study. I recommend you to read these articles.

Here you find the materials given to students, my explanation of the logic behind the problems and lectures, my pedagogical tricks and rough estimates of the pedagogical results. [bonus] topic were covered occasionally if time permitted.

0.3 :: foreword for students

Dear student,

> *You know, you think sociologically, and then you think mathematically. But these are often hard to fit together. The mathematics enforces a discipline that the other discipline doesn't really value in the same way. It has its own forms of rigor but they're not the same. To bring those two into conjunction has always been the sort of thing that I thought of as important.* Thomas Fararo.

This text is for teachers but you may also benefit from it: check that you know all the material sketched, and forecast your final note for the course by answering the questions and solving the problems at the end of each topic.

We are here to **do** mathematics, not to hear how others did it. Mathematics is a powerful **language**, to learn it you must practice speaking (which means: solving problems). Learn the alphabet: notation (greek letters, matrices, indices, symbols in mathematical logic, basic definitions). Learn the grammar: ideas of proofs, different types of reasoning (logical, geometrical, algebraical). Read papers[2] which advertise mathematics in sociology and list a huge bunch of applications varying from dynamical model of segregation[3] to population ecology in organization study[4] and dissemination

[2]Christofer R Edling. "Mathematics in sociology". In: *Annual review of sociology* 28.1 (2002), pp. 197–220,Olof Backman and Christofer Edling. "Review Essay: Mathematics Matters: On the Absence of Mathematical Models in Quantitative Sociology". In: *Acta sociologica* 42.1 (1999), pp. 69–78,Christofer Edling. *Interviews with mathematical sociologists*. 2007

[3]Thomas C Schelling. "Dynamic models of segregation". In: *Journal of mathematical sociology* 1.2 (1971), pp. 143–186.

[4]Michael T Hannan and John Freeman. "The population ecology of organizations". In: *American journal of sociology* 82.5 (1977), pp. 929–964.

of culture[5]. Many links can be found in an old review[6]. Check a great review[7] on agent-based models and a [probably, not] freshman level nice book[8] on using game theory and agent-based model; a book[9] with advice on modelling for a social scientist. Here I stop: I mentioned the best texts to read, and if you apprehend them, it will not be a problem to find something else for further reading. True learning comprises self-education and speaking with clever people. Nothing less, nothing more.

Where to go to get more solid knowledge in mathematics? Try a book[10] about math for PhD students in sociology.

0.4 :: for administrators: which mathematics do sociologists need?

For a one-semester course (15 lectures, 15 practices by 80 minutes each) I propose the following list, where each subtopic leads to a meaningful rather complicated result requiring efforts to conceive :

- Venn diagrams, induction, logic puzzles, proof of Newton's binomial formula, two proofs of the exclusion-inclusion formula
- graphs, number of paths between vertices as a power of the adjacency matrix, Gale-Shapley algorithm for stable marriages, random walking on graphs
- linear algebra, geometric meaning of a determinant, change of coordinates, linear independence, basis
- relation between the integral and the derivative, plot analysis, finding maxima and minima, a proof of linear regression formula.
- non-obligatory bonus lecture (if time permits): eigenvalue centrality and PageRank, SVD decomposition for recommendation systems.

If you want to add something, you must remove something from the program. Removing fundamentals (induction, proofs for graphs, and logic puzzles) implies impossibility to understand more abstract mathematics: you may not jump over first ten steps while ascending stairs. Squeezing the material is similar to running up to the stairs instead of walking. Not all students will agree to follow you.

[5] Robert Axelrod. "The dissemination of culture: A model with local convergence and global polarization". In: *Journal of conflict resolution* 41.2 (1997), pp. 203–226.

[6] Thomas J Fararo. "Reflections on mathematical sociology". In: *Sociological Forum*. Vol. 12. 1. Springer. 1997, pp. 73–102.

[7] Michael W Macy and Robert Willer. "From factors to factors: computational sociology and agent-based modeling". In: *Annual review of sociology* 28.1 (2002), pp. 143–166.

[8] Robert Axelrod. *The complexity of cooperation*. 1997.

[9] Charles A Lave and James G March. *An introduction to models in the social sciences*. University Press of America, 1993.

[10] John Fox. *A mathematical primer for social statistics*. 159. Sage, 2009.

0.5 :: handwriting vs slides vs scans

It is very convenient to write on whiteboard, then the speed of the lecture is just right. Slides are very bad for teaching math, except if you write by hand and then scan. Since the audience is about ninety persons, I provide in advance the scans of everything that I intend to write on whiteboard. It is optimised for reading on smartphones.

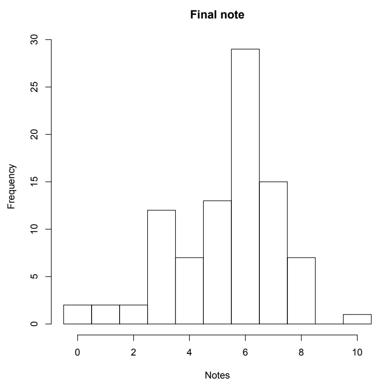
http://mathcenter.spb.ru/nikaan/teach/socmathbooknotes.pdf.

0.6 :: home reading

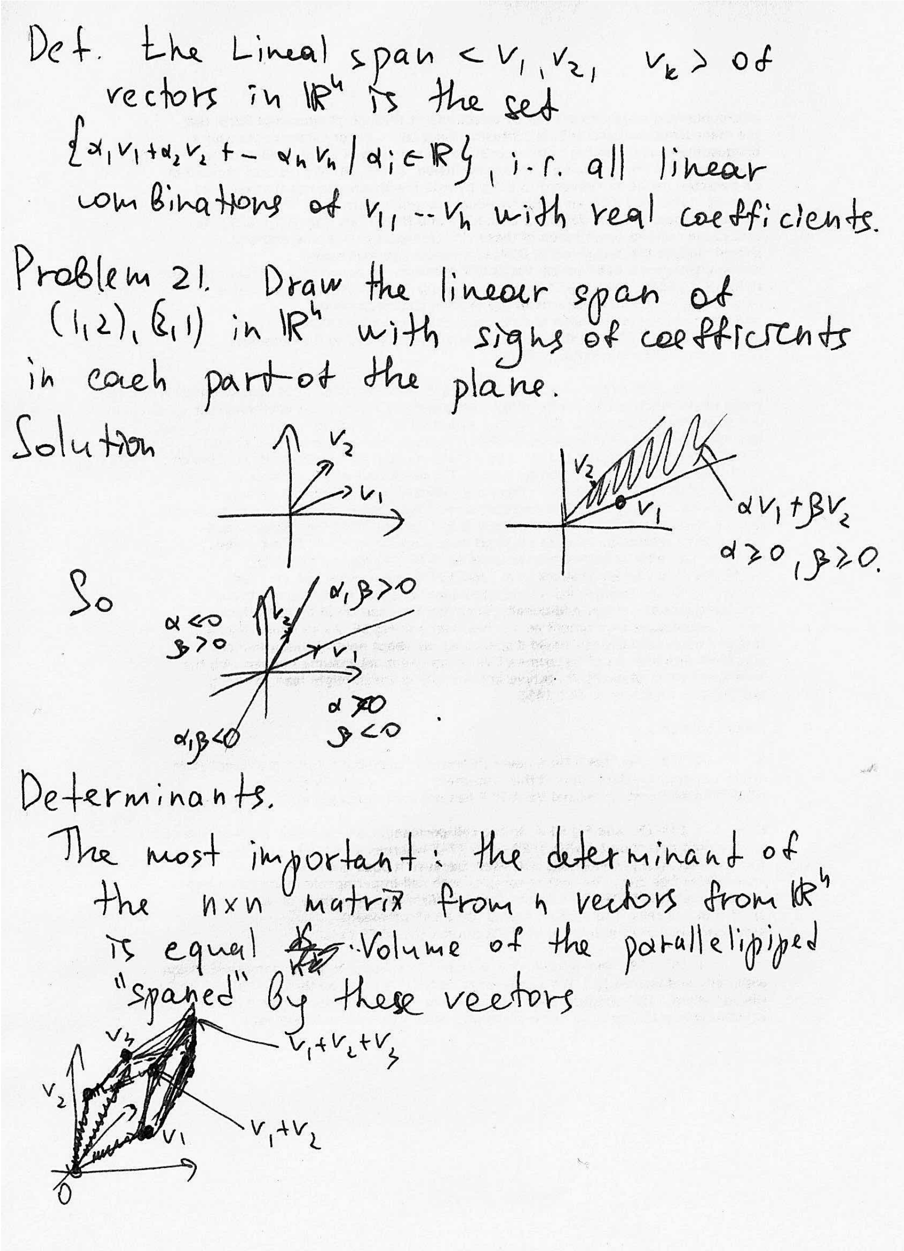
One of the most important activities in learning mathematics is speaking about it, explaining solutions, answering questions. In general, to doubt your own reasoning it is useful to explain your thoughts to another person. So, each student was assigned with a chapter ($20 - 30$ pages) from a book about mathematical sociology and then had to recite the main ideas to an assistant, and then spread the word about these ideas further. The reading was: Chapters 1,4-11 from *Introduction to mathematical sociology*, Chapters 3-7 from *Matrices and society*, Chapters 1,3,7 from *Complexity and cooperation* books.

0.7 :: final note

The passing grade was 4. This histogram of grades in 2019 was as follows:



Examples of the three midterms, final exams, and histograms for the grades can be found further in this text (see the table of contents).


Def. the Lineal span $\langle v_1, v_2, \; v_k \rangle$ of vectors in $\mathbb{R}^n$ is the set $\{\alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_n v_n \,/\, \alpha_i \in \mathbb{R}\}$, i.e. all linear combinations of $v_1, \dots v_n$ with real coefficients.

Problem 21. Draw the linear span of (1,2), (3,1) in $\mathbb{R}^n$ with signs of coefficients in each part of the plane.

Solution

So

Determinants.

The most important: the determinant of the $n \times n$ matrix from n vectors from $\mathbb{R}^n$ is equal Volume of the parallelipiped "spaned" by these vectors

Figure 1: A page from notes available prior to a lecture

0.8 :: aims and objectives

A list of potential aims for the course[11]:

- lose a fear of math
- learn how to find everything about math and computations on youtube, use internet to perform computations
- understand SVD decomposition and the meaning of eigenvectors
- learn how to calculate without mistakes
- analyse plots of functions
- learn derivation and integration
- learn a bit of PDE
- learn a concept of proof
- learn how to read mathematical texts
- translate real world problems into mathematical language
- study several interesting topics, but non-systematically

Some of them are *mutually exclusive*: it is not possible to loose fear and learn how to calculate at the same time, because for the latter there should be a lot of graded homework and tests with a punishment for the mistakes, since students initially make mistakes in every line of computation.

Also it should be clear to a reader, that it is not possible to achieve all these aims in one semester course. First, because it takes several courses in math departments to study analysis, linear algebra, graph theory (and not all students are able to accomplish that), here we have less time and not all students are mathematically inclined. And if you take a bit of everything, nobody learns nothing. So I recommend to choose one main aim and 2-3 subaims: I prefer stressing on "how to find everything in the Internet", "proofs", "loose a fear", "collection of nice examples".

[11] I highly recommend a teacher to read LRB Elton. "Aims and Objectives in the Teaching of Mathematics to non-Mathematicians". In: *International Journal of Mathamatical Education in Science and Technology* 2.1 (1971), pp. 75–81

1 Set theory, logic, combinatorics, graphs, matrices

goal: acquire the idea of proof, basic mathematical, basics of graphs, combinatorics, and matrices

1.1 :: matrices

goal: learn matrix multiplication, solving linear equations, formulae for matrix multiplication, elementary transformation

We will see how recommendation systems work and how to rank web-pages by popularity, but we need matrices. The rule of multiplication of two-by-two matrices:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} ax_1 + by_1 & ax_2 + by_2 \\ cx_1 + dy_1 & cx_2 + dy_2 \end{pmatrix}$$

Just multiply the rows of the first matrix by the columns of the second matrix, do it for all rows of the first matrix and all columns of the second matrix.

Problem 1. Compute (and the result is called **the identity** 2×2 **matrix**)

$$\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \cdot \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}.$$

Solve a system of linear equations by Gauss elimination method.

$$\begin{cases} 2x_1 + 1x_2 = 3 \\ 5x_1 + 3x_2 = 4 \end{cases}$$

Then solve

$$\begin{cases} 2x_1 + 1x_2 = p \\ 5x_1 + 3x_2 = q. \end{cases}$$

Here we get $x_1 = 3p - q, x_2 = -5p + 2q$. So,

$$\begin{cases} x_1 = 3p - q \\ x_2 = -5p + 2q \end{cases}$$

Now, a new notation: let us write this system as

$$\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix}, Ax = b, \text{where } A = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}, x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, b = \begin{pmatrix} p \\ q \end{pmatrix}.$$

By analogy with one-dimensional case if $ax = b$, $a, b \in \mathbb{R}$ (this notation means that a, b are real numbers), then $x = a^{-1}b$. We expect to have $x = A^{-1}b$, this is true: as we already found out

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} \cdot \begin{pmatrix} p \\ q \end{pmatrix}.$$

And $\begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$ is called the **inverse** of A and is denoted by A^{-1} because

$$\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \cdot \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This should convince you that, $x = A^{-1}b$.

Definition 1. A matrix is a rectangular table with numbers, for example

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}, \begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{pmatrix}, \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$$

Definition 2. For $A = [a_{ij}]_{i=1..m,j=1..n}$, $B = [b_{ij}]_{i=1..n,j=1..k}$, i.e.

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}, B = \begin{pmatrix} b_{11} & \dots & b_{1k} \\ \vdots & \ddots & \vdots \\ b_{n1} & \dots & b_{nk} \end{pmatrix},$$

define $C = AB$ by

$$C = [c_{ij}]_{i=1..m,j=1..k} \text{ where } c_{ij} = \sum_{s=1..n} a_{is}b_{sj}. \tag{1}$$

Indeed, the set of n numbers $a_{is}, s = 1..n$ is exactly the i-th row of A, and the set of numbers $b_{sj}, s = 1..n$ is the j-th column of B.

Given a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ its **determinant**, $\det(A)$, is the number $ad - bc$.

:: reading

Ian Bradley and Ronald L Meek. *Matrices and society: matrix algebra and its applications in the social sciences*. Vol. 501. Princeton University Press, 2014, the first two chapters. Some sections of this book (e.g. about dominance in a group, work mobility), the first important book for our course, will be given to students to read at home and then recite to one of the tutors – six students one year older who agreed to help with teaching.

Matrices

A matrix by itself has no sense. It is just a table, so who cares? Then you say that it is used to solve linear equations, the adjacency matrix of a graph is used to represent a random working on a graph, its powers represent the number of paths between vertices in a graph, a matrix may represent a transformation on a plane ..., only then a notion of a matrix makes sense. Math is about understanding, so when there is nothing to understand (and everybody tries to remember the algorithms for computing the rank, determinant, and eigenvalues) it is not math. You need to include in the program something which is difficult to understand, but possible. On the way to understanding a difficult stuff, students learn basic things and remember them better.

:: self-control, satisfactory level (for the grade 4-5):

Exercise 1. Let

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, C = \begin{pmatrix} 2 & -1 & 3 \\ -3 & 4 & -2 \end{pmatrix}.$$

Find the products AB, BA, AC of matrices. stress that the matrix product is not commutative

Exercise 2. Choose two arbitrary 2×2 matrices A, B, compute AB, BA, $\det(A)$, $\det(B)$, $\det(AB), \det(BA)$.

Exercise 3. Let A be a 2×3 matrix and B be 3×2 matrix. What are the sizes of the matrices AB, BA?

Exercise 4. Find the area of the parallelogram on the plane whose four vertices are

$$(0, 0), (1, 2), (3, 3), (2, 1).$$

Teacher: compute the corresponding determinant $\det \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Tell that the determinant is the area. Ask to remember this.

:: self-control, good level (for the grade 6-7):

Exercise 5. We consider the following map $L:\mathbb{R}^2\to\mathbb{R}^2$, it sends each point (x,y) to the point $(-y,x)$, i.e. $L(x,y)=(-y,x)$ by definition. For example $L(1,0)=(0,1)$, $L(3,2)=(-2,3)$. Find a simple geometric description of L. Suggestion: take some points at random and investigate where L sends these points.

Exercise 6. Let A,B by any 2×2 matrices. Verify that $\det(AB)=\det(BA)=\det(A)\det(B)$. tell that the same is true for determinants of bigger matrices too, we will use this without a proof.

Exercise 7. Using Gaussian elimination method, find real numbers x,y,z such that $x+2y+3z=1, 2x+3y+4z=2, x-3y-5z=7$.

Exercise 8. Let α,β be some real numbers. Recall formulae for $\sin(\alpha+\beta), \cos(\alpha+\beta)$. Find AB, BA, $\det(A), \det(B)$ for

$$A=\begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix},\quad B=\begin{pmatrix}\cos\beta & \sin\beta\\ -\sin\beta & \cos\beta\end{pmatrix}.$$

say that these are rotation matrices, so it is expectable that a rotation by α following by a rotation by β is a rotation by $\alpha+\beta$.

Exercise 9. Find a 2×2 matrix B such that

$$B\cdot\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}a-c & b-d\\ c & d\end{pmatrix}$$

Let A be any matrix with at least two rows. Find a matrix B such that $BA=A'$ where A' is obtained from A by subtraction of the second row of A from the first row of A.

Exercise 10. Find the 5×5 matrix B such that for each 5×5 matrix A, the matrix AB has five columns as A, but in a different order: the first column of A is the second column of AB, the second column of A is the third column of AB, the third column of A is the first column of AB, the fourth column of A is the fifth column of AB, the fifth column of A is the fourth column of AB.

Exercise 11. Find the $n\times n$ matrix, multiplication by which on the left adds i-th row to j-th row, for given i,j, so changing only j-th row.

Exercise 12. Find the $n\times n$ matrix, multiplication by which on the left exchanges rows with numbers i,j.

Exercise 13. Check that the general formula for matrix multiplication (that with indices, (1) on page 9) works for 3×3 matrices.

:: self-control, excellent level (for the grade 8-10):

Exercise 14. Let A be an $n \times n$ matrix. Let E be the $n \times n$ identity matrix, i.e. $E = [a_{ij}]_{1 \leq i,j \leq n}$, where $a_{ii} = 1$ for each $i = 1, \ldots, n$ and $a_{ij} = 0$ if $i \neq j$. Prove that $AE = EA = A$. exercise to overcome the fear of notation. We prove it honestly and force students to get used to indices and summation formalism

Exercise 15. Consider a population of whales with a young persons, b mature and c old. In a terms of time young person become mature, b mature persons make $2b$ young and become old, and old persons die. How does matrices help to count the population in, say, 10 terms?

Exercise 16. Note that we can add or subtract matrices of the same size:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} a+x & b+y \\ c+z & d+w \end{pmatrix}.$$

Find a formula (using a, b, c) for $\lambda \in \mathbb{R}$ such that

$$\det\left(\begin{pmatrix} a & b \\ b & c \end{pmatrix} - \lambda \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0.$$

Exercise 17. Let, A, B, C be 2×2 matrices. Prove that $(AB)C = A(BC)$.

Exercise 18. Let, A, B, C be $n \times n$ matrices. Prove that $(AB)C = A(BC)$.

1.2 :: set theory and the mathematical language

goal: basic set theory, first proofs, comprehend math as a language

Definition of a set, injection, surjection, bijection, image of a map, preimage, Euler-Venn diagrams, symbols $\cap, \cup, \forall, \exists, \wedge, \vee, \subset, \in, \notin, \Rightarrow, \Leftrightarrow, \varnothing, A \setminus B, \neg$ (negation), constructions such as

$$\mathbb{Q}_+ = \{q \in \mathbb{Q} | q > 0\}, \sum_{i=1..n} f(i), \sum_{a \in A} f(a).$$

The composition $g \circ f$ of two functions $f : A \to B, g : B \to C$ is the function from A to C. It sends each element $a \in A$ to $g(f(a)) \in C$.

:: reading

Chapter 2 in Phillip Bonacich and Philip Lu. *Introduction to mathematical sociology*. Princeton University Press, 2012, the second important book for our course.

Problem 2. Translate into the plain English the following.
Def. $f : \mathbb{R} \to \mathbb{R}$ is continuous $\Leftrightarrow$

$$\Leftrightarrow \forall x \in \mathbb{R}, \forall \varepsilon > 0, \exists \delta > 0, \text{ such that } \forall x' \in [x-\delta, x+\delta] \text{ we have } |f(x') - f(x)| < \varepsilon.$$

At this point students are **not** supposed to understand the meaning of this definition. Here we train the ability to translate from mathematical language to English only.

The final exam for the course happens in written form, so it is not easy to design it. Interesting problems are hard to solve, easy problems will not motivate to study during the semester. My approach is following: we discuss about **one hundred** of concrete problems during lectures and practice lessons, for the exam students will get the **same** problems with minimal variations (preventing them from copying). This allows to cover a huge variety of topics (since essentially we solve each type of problems only once) and, I believe, this tactic lays much less stress on students: they expect nothing new on exam, there is nothing to fear if you understood and remembered all the material.

The problems for practice are known to students in advance, they are supposed to solve them at home. Each student should present two solutions at the whiteboard during the semester. Many problems are too difficult to solve, so students may try to work together or find solutions in the Internet. If, still, nobody solved a problem, hints are provided. The idea is that students solve some problems during the class with the hints of the teacher, and they feel that they solved problems by themselves and remember solution much better than just being told the solution *ex cathedra*.

:: self-control, satisfactory:

Read Polya about how to solve problems http://community.middlebury.edu/~gmelvin/past/math110su12/polya.pdf

Exercise 19. Is is possible that the sets A, B, C have pairwise non-empty intersections but $A \cap B \cap C$ is empty?

Exercise 20. Let A, B be finite sets, $|A| > |B|$. Is it possible to have a surjection $f : A \to B$? An injection $f : A \to B$? A bijection $f : A \to B$?

Exercise 21. Is it possible that $|A \cap B| > |A|$ for finite sets A, B?

Exercise 22. Find $|A|$ (the number of elements in the set A), where A is the set of natural numbers from a)1 to 30, b) 1 to 100 which are divisible by at least one number from the set $\{3, 5, 7, 9\}$. Ask to guess what is a generalisations for 4 sets.

Exercise 23. Draw the Euler-Venn diagrams for $(A \cap B) \setminus C$ and $(A \setminus C) \cap (A \setminus B)$.

Exercise 24. Let $A = \{1, 2, 3, 4\}, B = \{1, 2, 3, 4, 5\}$. Let f be the map from A to B, given by

$$\begin{aligned} 1 &\to 3 \\ 2 &\to 4 \\ 3 &\to 2 \\ 4 &\to 5 \end{aligned}$$

Let $g : B \to A$ be given by

$$\begin{aligned} 1 &\to 2 \\ 2 &\to 1 \\ 3 &\to 2 \\ 4 &\to 3 \\ 5 &\to 4 \end{aligned}$$

Find the compositions $f \circ g : B \to B, g \circ f : A \to A$. Are they injections, surjections, or bijections?

Exercise 25. In cafe, you can order green, red, or black soup and green, red, or black tea. How many different orders you can make if it is prohibited to order soup and tea of the same color?

Exercise 26. Five persons A, B, C, D, E visit a cafe every day and always seat on the same table with five chairs (chairs are numbered: $1, 2, 3, 4, 5$.). In how many different ways they can seat on these five chairs? Recall the notation $n!$

Exercise 27. In cafe, the cook has a) 4, b) 8 ingredients. A dish is made of three different ingredients (their order is not important). How many different dishes the cook can cook? For the case a) one can list all dishes, then explain that it is $\frac{4 \cdot 3 \cdot 2}{3 \cdot 2 \cdot 1}$. Introduce the notation $\binom{n}{k}$, provide the formula for it, tell that the proof is exactly the same as in b).

Exercise 28. How many four-digit numbers contain 7?

:: self-control, good:

Exercise 29. Write a definition of a prime number without using words and using only mathematical notation.

Exercise 30. Is it possible in a group of 15 people that each person knows exactly one other person? explain that one can draw a graph. then the vertices should be paired which is impossible

Exercise 31. Is it possible in a group of five people that the 1st person knows one other person, 2nd – three persons, 3rd – two, 4th – two, 5th – three? the sum of valencies is even, because make a string for each friendship and cut it in between, the sum of half-strings is twice the number of edges and is the sum of valencies.

Exercise 32. Draw all different (up to renumbering the vertices) graphs with four vertices and draw their adjacency matrices. define the adjacency matrix here

Exercise 33. Is it possible that a knight (in chess) starts from the left bottom corner and finishes in the right top corner visiting each other cell in the chess-board exactly once? a knight changes its color each jump. this problem is an associative hint for the next

:: self-control, excellent:

Exercise 34. Show that there is no bijection between natural numbers and real numbers from the interval $[0,1]$. Hint: suppose that there exists such a bijection, $n \to a_n$. Choose a number $X = 0, x_1x_2x_3\ldots$ in $[0,1]$ in the following way: its first digit x_1 is different from the first digit of a_1, its second digit x_2 is different from the second digit of a_2, etc. Note that by construction there does not exist a natural number k such that $X = a_k$. Contradiction.

Exercise 35. Let A, B, C, D be four finite sets such that $A \cap B = C \cap D = \varnothing$. How to compute $|A \cup B \cup C \cup D|$ using numbers $|A|, |B|, |C|, |D|, |A \cap C|, |A \cap D|, |B \cap C|, |B \cap D|$?

1.3 :: power set, Cartesian product, equivalence relation, binomial formula

goal: understand some proofs, learn methods to prove: induction, bijection, combinatorial reasoning

Problem 3. Draw the Euler-Venn diagram for $\big((A \setminus B) \cap (C \setminus D)\big) \cup (A \setminus C)$.

Def. Cartesian product, $A \times B$, of sets A, B is defined as

$$A \times B = \{(a, b) | a \in A, b \in B\}.$$

Def. Let A be a set. By 2^A we denote the set of all subsets of A.

Problem 4. If $|A| < \infty$, then $|2^A| = 2^{|A|}$.

Problem 5. For sets A, B with $A \cap B = \varnothing$, prove that $2^A \times 2^B = 2^{A \cup B}$.

Proof. We construct a **bijection** between $2^A \times 2^B$ and $2^{A \cup B}$. The first set is the set of pairs (a, b) where $a \subset A, b \subset B$. Given such a pair, consider the set $a \cup b \subset A \cup B$, note that $a \cup b$ is an element of $2^{A \cup B}$, so we constructed a map $f : 2^A \times 2^B \to 2^{A \cup B}$. The map in other direction is constructed as follows: let $c \in 2^{A \cup B}$, construct a pair $(c \cap A, c \cap B) \in 2^A \times 2^B$, this gives a map $g : 2^{A \cup B} \to 2^A \times 2^B$. It is easy to see that g is the inverse of f, and both are bijections.

In networks, it is useful to look at the people who are similar with respect to others, for example, two persons are similar if the sets of their friends coincide. So, to gather more information it is more efficient to friend different people; doctors in similar positions are more likely to adopt a new drug simultaneously comparing with doctors-friends, for more details read a book about structural holes[12]. In general, many advances in mathematics eventuated from a change of perspective. The same here: a trivial idea, once you heard it, may come from a change of view suggested by a mathematical theory. So to study mathematics means to get to know different places of gazing at objects. More[13] on mathematics used to study equivalence relations, this is also needed for community detection methods[14].

The Cartesian product $A \times B$ of two sets A, B is the set of pairs where the first element is from A and the second is from B. In other words $A \times B = \{(a, b) | a \in A, b \in B\}$. For example, if $A = \{1, 2\}$ and $B = \{a, b, c\}$ then

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}.$$

If A, B are finite sets, then $|A \times B| = |A| \times |B|$. We denote $A^2 = A \times A$. So $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. $\mathbb{R}^n$ is the space of dimension n, it consists of elements $(x_1, x_2, \ldots, x_n), x_i \in \mathbb{R}$ for $i = 1..n$.

Example 1. We call two integer numbers x, y equivalent if $x - y$ is divisible by 3. We denote this $x \sim y$. Note that if $x \sim y$ and $y \sim z$, then $x \sim z$. Also $x \sim x$. If

[12] Ronald S Burt. *Structural holes: The social structure of competition*. Harvard university press, 2009.

[13] Martin G Everett and Stephen P Borgatti. "Regular equivalence: General theory". In: *Journal of mathematical sociology* 19.1 (1994), pp. 29–52.

[14] Santo Fortunato. "Community detection in graphs". In: *Physics reports* 486.3-5 (2010), pp. 75–174.

$x \sim y$, then $y \sim x$. Then, all the integer numbers are divided in three groups: with remainder 0, remainder 1, and remainder 2, when we divide by 3. Similar things are called equivalence relations.

Definition 3. An equivalence relation on a set X is $R \subset X^2$ with the following properties.

- $\forall x \in X$, $(x, x) \in R$,
- if $(x, y) \in R$, then $(y, x) \in R$,
- if $(x, y) \in R, (y, z) \in R$, then $(x, z) \in R$.

We declare some pairs of elements of a set X to be equivalent (and put all such pairs to R) and write $x \sim y$ if $(x, y) \in R$. So we can reformulate: this is an equivalence relation if a) $x \sim x, \forall x \in X$ b) $x \sim y \Rightarrow y \sim x$, c) $x \sim y, y \sim z \Rightarrow x \sim z$.

Problem 6. Equivalence classes do not intersect or coincide.

Problem 7. Find the coefficient behind x^3 in $(1+x)^{10}(1+2x)^7$.

Solution: we can choose x from three parenthesis of the first type $(1+x)$ and 1 from all parenthesis of the second type $(1+2x)$. We can do this in $10 \times 9 \times 8$ cases, but since it is not important in which order we choose the first type parenthesis, we divide by 6. So, this gives $\binom{10}{3}$. We also can choose two parenthesis of the first type and one of the second type, this gives $2 \times \binom{10}{2}\binom{7}{1}$, etc. The answer:

$$\binom{10}{3} + 2 \cdot \binom{10}{2}\binom{7}{1} + 4 \cdot \binom{10}{1}\binom{7}{2} + 8 \cdot \binom{10}{1}\binom{7}{3}.$$

Problem 8. Prove the **Newton binomial formula:** $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$, for example $(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$.

Problem 9. Show that you can pay any integer sum bigger than 8 by coins in 3, 5.

Problem 10. Show that you can cut a square in n squares for each $n > 6$.

Problem 11. Show that $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

Explanation of mathematical induction method. I find the standard metaphor of "domino" quite misleading, because it neglects the main object: the induction step. I rather prefer the metaphor of a crocodile eating a line of meatballs. The crocodile thinks: (*)*if I have just eaten this meatball, why not to eat the next?* If it convinced itself once and forever that this is great, then if it has eaten the first meatball, it will eat all (infinite number) of them. The internal reasoning of (*) may be very complicated, e.g. in the Problem 9 the crocodile was thinking: *well, if I have* 5*, I can replace it by* $3+3$*. And if I have* $3+3+3$ *I can replace it by* $5+5$*, and if I have something, which is not* $0, 3, 3+3$*, then I can reason as above.* So, $8 = 5+3$ and starting from 8 the crocodile can always perform its induction step.

:: reading

Introduction to mathematical sociology, chapter 4.

:: self-control, satisfactory:

Exercise 36. Find the coefficient behind x^4 in $(1+x)^{10}(1+2x)^7$.

Exercise 37. In cafe, the cook has 9 ingredients. A dish is made of three different ingredients (their order is important). How many different dishes the cook can cook if the 3rd and 4th ingredients are not allowed to use together?

Exercise 38. There is an island populated by knights and liars. Knights always tell truth, liars always lie. A stranger meets three local persons and asks everybody: “how many knights are among you three?”. The first answered: no one. The second answered: one. What is the answer of the third person?

Exercise 39. Automate can split a piece of paper on 4 or 6 pieces. What number of pieces can be reached from one sheet?

:: self-control, good:

Read the best book[15] in the world about solving problems in mathematics, Chapters 10-12.

[15] George Polya. “Mathematical discovery: On understanding, learning, and teaching problem solving”. In: (1981).

Exercise 40. $x + \frac{1}{x}$ is an integer number. Show that for each natural n the number $x^n + \frac{1}{x^n}$ is also integer.

Exercise 41. We draw several lines on the plane. These lines dissect the plane into parts. Show that we can color the parts in black and white such that each two parts with a common interval boundary are colored in different colors.

Exercise 42. On the boundary of a convex polygon there grow hairs, in outside direction. Someone draws several non-intersection diagonals in the polygon, each such diagonal has hairs on one side. These diagonals dissect polygon into parts. Show that among these parts there exists a polygon with hairs outside.

Exercise 43. We draw an arrow on each side and diagonal of a polygon. Show that there exists a vertex Z of this polygon, such that any other vertex is reachable from Z if we allow moving according to arrows.

Exercise 44. Show that $1^3 + 2^3 + \cdots + n^3 = (\frac{n(n+1)}{2})^2$.

Exercise 45. Find all values of n such that a) $2^{n-1} > n$; b) $2^n > n^2$; c) $2^n < n!$ holds.

Exercise 46. Prove that every positive integer can be represented as a sum of different powers of two.

:: self-control, excellent:

Exercise 47. There is an island upon which a tribe resides. The tribe consists of 100 people, with various eye colours. Yet, their religion forbids them to know their own eye color, or even to discuss the topic; thus, each resident can (and does) see the eye colors of all other residents, but has no way of discovering his or her own (there are no reflective surfaces). If a tribesperson does discover his or her own eye color, then their religion compels them to commit ritual suicide at noon the following day in the village square for all to witness. All the tribespeople are highly logical and devout, and they all know that each other is also highly logical and devout (and they all know that they all know that each other is highly logical and devout, and so forth).

Of the 100 islanders, it turns out that n of them have blue eyes and $100 - n$ of them have brown eyes, although the islanders are not initially aware of these statistics (each of them can of course only see 99 of the 100 tribespeople).

One day, a blue-eyed foreigner visits to the island and wins the complete trust of the tribe.

One evening, he addresses the entire tribe to thank them for their hospitality.

However, not knowing the customs, the foreigner makes the mistake of mentioning eye color in his address, remarking "how unusual it is to see another blue-eyed person like myself in this region of the world".

What effect, if anything, does this *faux pas* have on the tribe? a) Consider the case $n = 1$. b) Consider the case $n = 2$. c) Consider the general case.

Exercise 48. Reed about the Tower of Hanoi. We consider an abstract tower with n disks. a) Prove that a solution exists. b) Determine the minimal number of turns, one need to solve the puzzle.

1.4 :: inclusion-exclusion formula

goal: loos fear of notation, indices, summation signs, understand a quite elaborate proof which will increase your capacity for abstract thinking

Inclusion-exclusion formula: let $A_1, A_2, \ldots, A_n$ be finite sets. Then

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n}(-1)^{k-1} \sum_{1\le i_1<i_2<\cdots<i_k\le n} \Big|\bigcap_{j=1..k} A_{i_j}\Big|.$$

Problem 12. Prove the inclusion-exclusion formula directly, by counting.

Proof. Consider any element a which belong to k sets among $A_1, A_2, \ldots A_n$, then a is counted once on the left. Then, it is counted $k = \binom{k}{1}$ times in $\sum |A_i|$, then it is counted $\binom{k}{2}$ times in $\sum |A_i \cap A_j|$ but with "minus" sign etc. Totally it is counted

$$\binom{k}{1} - \binom{k}{2} + \binom{k}{3} - \cdots = 1 - (1-1)^k = 1$$

times, using binomial formula.

Problem 13. Prove the inclusion-exclusion formula using maps from $\bigcup A_i$ to $\{0, 1\}$.

Proof. Consider functions from $A = \bigcup\limits_{i=1}^{n} A_i$ to $\{0, 1\}$. Denote by $\mathbb{1}_{A_k}$ the function which is one on A_k and zero otherwise. Then, on A we have

$$f = (\mathbb{1}_A - \mathbb{1}_{A_1})(\mathbb{1}_A - \mathbb{1}_{A_2})\ldots(\mathbb{1}_A - \mathbb{1}_{A_n}) \equiv 0,$$

because if $a \in A$ belongs to A_i, then $(\mathbb{1}_A - \mathbb{1}_{A_i})$ is zero on a, therefore $f(a) = 0$ for each $a \in A$. Note that $\mathbb{1}_{A_1} \cdot \mathbb{1}_{A_2} = \mathbb{1}_{A_1 \cap A_2}$, the same for multiple intersection. Finally,

$$0 = \sum_{a\in A} f(a) = \sum \mathbb{1}_A - \sum \mathbb{1}_{A_i} + \sum \mathbb{1}_{A_i\cap A_j} - \cdots =$$

$$= |A| - \sum |A_i| + \sum |A_i \cap A_j| - \sum |A_i \cap A_j \cap A_p| + \dots,$$

what we intended to prove. Then we draw these formulae on the Euler-Venn diagram for three sets.

Students are encouraged to read wikipedia and other sources trying to grasp this material. **Reasons of giving proofs[16] here:** 1) in analysis and linear algebra we cannot see completely rigorous proofs, but I never agree to teach a math course without proofs whatsoever; 2) to understand these proofs a student should work hard, and this is good; 3) students who succeed will never have problems with mathematical notation, indices etc, the fear will be gone – and this is one of the main goals of this course.

:: self-control, good:

Exercise 49. Let $A = [0, 1]$, $B = [1/2, 5]$, $A, B \subset \mathbb{R}$. Draw the graphs of the functions $\mathbb{1}_A, \mathbb{1}_B, \mathbb{1}_{A\cap B}$.

:: self-control, excellent:

Exercise 50. Follow both proofs of the inclusion-exclusion formula in the case of 4 sets

1.5 :: Gale-Shapley algorithm, number of paths, random walking on graphs

goal: discuss a nice everyday problem, get to know a proof about an algorithm, see an application of matrices

Stable matchings. Consider two sets B, G of boys and girls respectively. Each boy has a preference (complete linear order) on girls, each girl has a preference for boys. A matching (pairing of some boys with some girls) is stable if there is no boys b_1, b_2 and girls g_1, g_2 such that b_1 is paired with g_1, but b_1 prefers g_2 to g_1, g_1 prefers b_2 to b_1 and b_2, g_2 prefer to be paired with g_1, b_1 respectively, to their current states.

[16] There is a few texts discussing sociology of proof and math, e.g. Bernard Zarca. *The Professional Ethos of Mathematicians*. 10.3917/rfs.525.0153. Paris, 2011. URL: https://www.cairn.info/revue-francaise-de-sociologie-1-2011-5-page-153.htm, and Reuben Hersh and Ivar Ekeland. *What is mathematics, really?* Vol. 18. Oxford University Press Oxford, 1997, but William P Thurston. "On proof and progress in mathematics". In: *For the learning of mathematics* 15.1 (1995), pp. 29–37 seems to be the best.

Problem 14. Consider n boys and n girls with arbitrary preferences (complete orders). Prove that a stable matching exists.

Proof. Algorithm: 1) each boy proposes to the girl he likes most; 2) each girl chooses the best among proposers and keep him; 3) each boy without a pair proposes to the next girl in his list; 4) go to step 2). Indeed, each girl can only improve her situation when time passes. As for boys, if a boy b is paired with somebody, then all the girls whom he likes more are paired with boys whom they like more than b.

Give a correct definition of a stable matching.

Problem 15. Prove that the sum of the valencies of the vertices of a graph is twice the number of the edges.

Problem 16. Let A be the adjacency matrix of a graph G. Prove that the entry (i, j) of A^k is the number of paths of length k between the vertices i and j.

Proof: the case $k = 1$ is the definition of the adjacency matrix, then by induction.

Bonus lecture (not obligatory) about eigenvector centrality and PageRank: **Centrality.** The influence of a person in a network is proportional to the sum of influences of his friends. This gives $\frac{1}{\lambda}Av = v$ for the vector v of influences[17].

Markov chain. Consider the random walking on an oriented graph[18]. If $\tilde{A}$ is the transition matrix and v is the vector of initial probabilities, then the probability vector after k steps is given by $\tilde{A}^k \cdot v$.

Definition of eigenvector and eigenvalue. PageRank formula.

:: reading

Chapter 10 (centrality), 14 (Markov chains) in Introduction to mathematical sociology, read also Wikipedia on PageRank, lurk into youtube to listen about these topics.

:: self-control, good:

Exercise 51. Let A be the adjacency matrix of the graph with four vertices a, b, c, d and edges ab, bc, cd, ad, ac. Find A^2.

[17] See more details in Britta Ruhnau. "Eigenvector-centrality—a node-centrality?" In: *Social networks* 22.4 (2000), pp. 357–365. Now you can read one of the most cited works on centrality and power in networks, e.g. Phillip Bonacich. "Power and centrality: A family of measures". In: *American journal of sociology* 92.5 (1987), pp. 1170–1182

[18] Using random walking, it is possible to describe social mobility, see Anatol Rapoport. *Mathematical models in the social and behavioral sciences*. Wiley New York, 1983, Chapter 10

Exercise 52. Find a graph with the following multiset of degrees or show that such graph does not exist.

- 2, 2, 2, 2, 3, 3;
- 2, 2, 2, 3, 3, 3;
- 1, 1, 3, 3, 5, 5, 7, 7;
- 1, 1, 2, 2, 3, 3, 4, 4

Exercise 53. If a graph G from the previous problem exists, then find it adjacency matrix $A(G)$. Check that the second and the third powers of $A(G)$ count 2- and 3-paths in G.

Exercise 54. If a graph G from the previous problem exists, find the probabilities to be at a vertex after two and three steps of random walking, starting from the first vertex.

A *cycle* in a graph is a sequence $v_1, v_2, \dots, v_k$ of at least three vertices, such that v_1 is connected to v_2 by an edge, v_2 is connected to v_3 by an edge, etc., v_k is connected to v_1 by an edge. A graph is called *connected* if between each of its two vertices there is a path[19].

Exercise 55. Show (hint: by induction) that a connected graph[20] with n vertices and without cycles has exactly $n-1$ edge.

:: self-control, excellent:

Exercise 56. Prove that if a graph with n vertices has only $n-2$ edges, then it is disconnected.

1.6 :: example of the first midterm

Exercise 57. Compute $\begin{pmatrix} 2 & -1 & 3 \\ -3 & 4 & -2 \\ -3 & 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} -2 & 1 & 3 \\ -3 & -4 & -2 \\ 8 & 4 & -3 \end{pmatrix}$.

[19] There is a centrality measure which counts number of paths through a vertex

[20] To those who want to know everything a sociologist should know about graphs I recommend to read Chapter 4 in Stanley Wasserman and Katherine Faust. *Social network analysis: Methods and applications*. Vol. 8. Cambridge university press, 1994.

Exercise 58. Compute $\begin{pmatrix} 2 & -1 & 3 \\ -3 & 4 & -2 \\ -3 & 1 & 3 \end{pmatrix}^{-1}$ and find x, y, z such that $2x - y + 3z = 1$, $-3x + 4y - 2z = -3$, $-3x + y + 3z = 2$.

Exercise 59. Draw $((A \setminus B) \cap (C \cup (D \setminus A))) \cup (D \cap B \cap A)$.

Exercise 60. In cafe, the cook has 9 ingredients. A dish is made of three different ingredients (their order is important). How many different dishes the cook can cook if the 3rd and 4th ingredients are not allowed to use together?

Exercise 61. Find the 5×5 matrix B such that for each 5×5 matrix A, the matrix AB has five columns as A, but in a different order: the first column of A is the second column of AB, the second column of A is the third column of AB, the third column of A is the first column of AB, the fourth column of A is the fifth column of AB, the fifth column of A is the fourth column of AB.

Exercise 62. Using the principle of mathematical induction, prove that

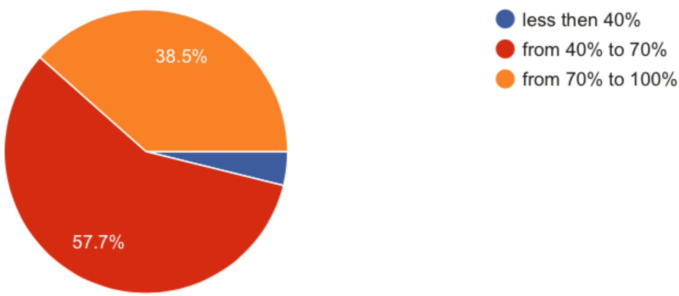
$$1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \cdots + (2n-1) \cdot (2n+1) = \frac{n(4n^2 + 6n - 1)}{3}.$$

1.7 :: feedback

Here I provide the feedback of students for this chapter.

From the first chapter "Set theory, logic, combinatorics, graphs, matrices" I understand ...

26 responses



Some answers to the question "What are the most interesting parts in Chapter 1"

- *"Matrices were great and very interesting - in fact, the most interesting topic of the first chapter, Gale-Shapley algorithm was also really cool, but a bit too short of a topic to really dive in, although I liked the videos on it a lot (thanks for them), sets were easy, as well as inclusion-exclusion formula, Markov chain was surprisingly easy to grasp the idea of - and the explanation in a book suggested for reading was brilliant."*

- *"I really enjoyed induction. I've spent lots of time preparing for classes on the topic, but once I got a grasp of it, it got quite fun. Stable matchings were fascinating, especially in respect to the example with dating proposals (I'm quite sure it was dating, but I do apologize if I'm wrong). It was really nice to learn how to describe such a process via mathematical model. I also liked the tasks on models of population. Oh, and random walking! Before that I could only count the paths by hand."*

- *"The most interesting topic in my point of view was graphs though I did not quite catch it."*

The most difficult parts were induction, inclusion-exclusion formula, Newton's binomial formula. One more answer: *"It was hard for me to understand mathematical language at first (but now it is actually much easier). Also, the proof of inclusion-exclusion formula was rather hard."*

2 Linear algebra, planar and space geometry

goal: learn the geometric meaning of eigenvectors, eigenvalues, determinant, rank, basis

For a comprehensive introduction to linear algebra with all the details read a book[21] for economists or a book[22] for engineers.

2.1 :: metrics, geometry in $\mathbb{R}^n$, scalar product, Cauchy-Schwartz inequality

goal: know basic geometry of space and different examples of metrics

When you perform data analysis, you have some chunks of data, corresponding, for example, to people. It is natural to compare them and find how similar they are, and similarity should be thought of a number. Hence naturally comes the notion of a distance: each person corresponds to a point in some space and similar persons are close points in a space and different persons are points on a big distance.

Definition of a metric space. Examples of metrics in $\mathbb{R}^n$: Euclidean, Chebyshev, Manhattan metric. Similarity between sets. Balls of given radius.

:: self-control, satisfactory level:

Exercise 63. Draw the unit ball for $\mathbb{R}^2$ for Euclidean, Chebyshev, Manhattan metric.

:: self-control, good level:

Exercise 64. Give an example of a metric space X with only three points, with such distances that a ball of bigger radius is contained in a ball of smaller radius.

Exercise 65. Draw the unit ball for $\mathbb{R}^3$ for Euclidean, Chebyshev, Manhattan metric.

Exercise 66. Compute the similarity between sets a) $A = \{a, b, c\}$ and $B = \{1, 2, 3, a\}$, b) A and $A \cap B$, c) A and $A \cup B$.

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[21] Fuad Aleskerov, Hasan Ersel, and Dmitri Piontkovski. *Linear algebra for economists.* Springer Science & Business Media, 2011.

[22] Gilbert Strang. "Introduction to Linear Algebra. 4 edition." In: *Wellesley, MA: Wellesley Cambridge Press.* (2009).

2.2 :: linear span, determinants

goal: understand geometric meaning of determinant and linear span

Distance between points, linear combination of vectors, scalar product, Cauchy-Schwarz inequality.

Problem 17. Find the area of a planar triangle with vertices in $(0,0),(a,b),(c,d)$ with $0 \leq c \leq a, 0 \leq b \leq d$.

Problem 18. Prove Cauchy-Schwarz inequality for $n=2$.

Problem 19. Prove that the area of a triangle with vertices $(0,0),(a,b),(c,d)$ is equal to $\frac{1}{2}\det(\begin{pmatrix} a & b \\ c & d \end{pmatrix})$

Definition of a linear combination of vectors, definition of the linear span of vectors.

Problem 20. Draw the linear span (i.e. all vectors $\alpha v_1 + \beta v_2$) of vectors $v_1 = (2,1), v_2 = (1,2)$ in $\mathbb{R}^2$ and signs of coefficients α, β in each part of the plane.

Definition 4. The determinant of the $n \times n$ matrix made from n vectors in $\mathbb{R}^n$ is equal to the n-dimensional volume of the parallelepiped spanned by these vectors.

Axiom 1. The most important thing to visualise: n vectors in $\mathbb{R}^n$ span the whole $\mathbb{R}^n$ if and only if the determinant of the corresponding $n \times n$ matrix is not zero. The dimension of the span of k vectors is at most k.

Problem 21. Show that the point $(-1,2)$ belongs to the linear span of the vectors $(1,2),(2,1)$.

A recipe to compute $n \times n$ determinant and its properties under multiplication by a number and exchanging rows.

Problem 22. Find

$$\det \begin{pmatrix} a & 0 & b & x \\ 0 & c & 0 & 0 \\ 0 & 0 & d & y \\ f & 0 & 0 & z \end{pmatrix}$$

Problem 23. Let A be an $n \times n$ matrix. Prove that each summand of $\det(A)$ is a product of n entries of A, neither two of them from one row or column.

Problem 24. Calculate

$$\det\begin{pmatrix}1 & 2 & 3\\ 3 & 4 & 1\\ 2 & 3 & 2\end{pmatrix}, \det\begin{pmatrix}1 & 3 & 2\\ 2 & 4 & 3\\ 3 & 1 & 2\end{pmatrix}$$

:: self-control, satisfactory level:

Exercise 67. Let A be a 5×5 matrix. Prove that $\det(\lambda A) = \lambda^5 \det(A)$ where λ is a number, and multiplication of a matrix by a number means that we multiply all the entries of the matrix by this number.

Exercise 68. Suppose that $n \times n$ matrices A, B have inverses. Prove that AB also has an inverse matrix and $(AB)^{-1} = B^{-1}A^{-1}$. just check that the product of $(AB)^{-1}$ and AB is E for both orders of multiplication. Another solution could be that $\det(A) \neq 0, \det B \neq 0$, therefore $\det(AB) \neq 0$, therefore it has an inverse matrix.

Exercise 69. Find the values of μ such that the system

$$\begin{cases} x + \mu y &= 1 \\ \mu x + y &= 1 \end{cases}$$

has a) no solutions, b) exactly one solution, c) infinite number of solutions. Compute the determinant $\det(\begin{pmatrix}1 & \mu\\ \mu & 1\end{pmatrix})$ in all these cases.

:: self-control, good level:

Definition: a vector v is linearly dependent on vectors u, w if there exist numbers μ, ν such that $v = \mu u + \nu w$.

Exercise 70. Prove that if three vectors in $\mathbb{R}^3$ are linearly dependent (one of them is linearly dependent on two others), then the determinant of the corresponding 3×3 matrix is zero. tell that det is the volume of the parallelepiped in $\mathbb{R}^3$ which is obviously zero if three vectors belong to one plane.

2.3 :: new point of view on solving linear systems, rank, eigenvalues

goal: understand: there is a solution $Ax = b$ if and only if b belongs to the linear span of the columns v_i of A. If it is the case, the coordinates of x are the coefficients x_i such that $\sum x_i v_i = b$. eigenvectors are such directions that A makes dilatation in these directions. the rank of a matrix is the dimension of the linear span of its columns (or rows)

To remember: solving a linear system $Ax = b$ is the same as verifying that b belongs to the linear span of the vectors-columns of A.

Problem 25. Prove that a linear system $Ax = b$ with $n \times n$ matrix A has a solution x for each choice of b if $\det(A) \neq 0$.

Proof. Consider n vectors-columns $v_1, v_2, \ldots, v_n$ of A. Since $\det(A) \neq 0$, these vectors $v_1, v_2, \ldots, v_n$ span a parallelepiped of non-zero n-dimensional volume (Axiom 1), therefore $v_1, v_2, \ldots, v_n$ span the whole $\mathbb{R}^n$, so each $b \in \mathbb{R}^n$ can be represented as $b = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$, therefore the vector $x = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ (written vertically) is the solution of $Ax = b$.

Problem 26. Prove that if $Ax = 0$ for an $n \times n$ matrix A has a non-zero solution x, then $\det(A) = 0$.

Proof. Consider vectors-columns $v_1, v_2, \ldots, v_n$ of A. Since $Ax = 0$, therefore, using the coordinates x_i of x we get a linear combination $\sum x_i v_i = 0$, with x_i which are not all zeros. Therefore the dimension of the linear span of $v_1, v_2, \ldots, v_n$ is strictly less than n, therefore the n-dimensional volume of the parallelepiped spanned by $v_1, v_2, \ldots, v_n$ is zero, therefore $\det(A) = 0$.

Problem 27. Prove that if $\det A \neq 0$ then $Ax = b$ has only one solution for any given b.

Proof. Suppose that there are two solutions, x, x', i.e. $Ax = b, Ax' = b$. Then $Ax - Ax' = 0$ (0 is the vector with zeroes). Then $A(x - x') = 0$, but this implies that if $x - x' \neq 0$, then $\det A = 0$. Contradiction. Therefore $x - x' = 0$.

Problem 28. Prove that if $Av = \lambda v, v \neq 0$, then $\det(A - \lambda E) = 0$ where E is the identity matrix.

Proof. Indeed, $Av = \lambda v$, so $Av - \lambda E v = 0$, so $(A - \lambda E)v = 0$. Since $v \neq 0$, we have $\det(A - \lambda E) = 0$, using the previous problem.

Problem 29. Find the eigenvalues and the corresponding eigenvectors for

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$

Problem 30. ((Anti-)example with three points in a space while belong to the same line)Consider the following system of linear equations.

$$\begin{cases} 1x + 1y + 1z & = & p \\ 1x + 1y + 0z & = & q \\ 3x + 3y + 1z & = & r \end{cases}$$

Show that for $p = 1, q = 0, r = 2$ there is no solutions. Find a necessary condition on p, q, r such that the system has solutions. Find all the solutions (x, y, z) in the latter case. Solve by elimination Gauss method. Then note that we consider all vectors $\begin{pmatrix} p \\ q \\ r \end{pmatrix}$ which can be presented as a linear combination

$$x \cdot \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + y \cdot \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + z \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

which is the plane generated by $(1, 1, 3), (1, 0, 1)$, passing through $(0, 0, 0)$. How to find this plane? It is given by an equation $ax_1 + bx_2 + cx_3 = 0$ and $a + b + 3c = 0 = a + c$. So $a = 1, b = 2, c = -1$. Note that (a, b, c) is orthogonal to $(1, 1, 3), (1, 0, 1)$.

Note that the determinant of the considered system is zero. Note that the linear span of three vectors is the same as a matrix (made of these three vectors) multiplied by all possible vectors (whose coordinates now serve as coefficients in the linear span).

Definition of the rank. Linearly dependent set of vectors.

Axiom 2. Geometric meaning of the rank of a matrix A. The rank is equal to the dimension of the linear span of the vectors-columns of A.

Problem 31. Explain why $rk(A)$ is equal to the maximal number of linearly independent vector columns of A.

Follows from geometric meaning of the rank.

:: self-control, satisfactory level:

Exercise 71. Find the rank of the following matrix: $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \\ 2 & 3 & 2 \end{pmatrix}$

A vector v_0 is linearly dependent on a set $v_1, v_2, \ldots, v_n$ of vectors if there exist real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $v_0 = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$. A set $\{v_1, v_2, \ldots, v_n\}$ of vectors is linearly dependent if one of these vectors is linearly dependent on the set of others.

Exercise 72. Explain why any vector v_0 from the linear span of v_1, v_2 is linearly dependent on v_1, v_2.

Exercise 73. Prove that the following sets of vectors are linearly dependent sets: a) $\{v, 0\}$, b) $\{v, w, v + w\}$, c) $\{v, w, u, v - w + u\}$, d) $\{u - v, u + v, u + 2v\}$ where u, v, w – some vectors.

Exercise 74. Find the inverse matrix for $A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}$

Exercise 75. Write the vector $(2,3)$ as a linear combination of two vectors $(1,1)$ and $(1,-1)$.

Note that $(2,3) = 2\cdot(1,0)+3\cdot(0,1)$. So, $(2,3)$ are the coordinates of this point $(2,3)$ in the basis $(1,0),(0,1)$. If $v_1, v_2 \in \mathbb{R}^2$ are two linearly independent vectors, then you can write any vector v as $v = \alpha v_1 + \beta v_2$ with $\alpha, \beta \in \mathbb{R}$. In this case α, β are called the coordinates of v in the basis v_1, v_2.

Exercise 76. Find the condition on p, q, r such that the following system has solutions. Describe all the solutions in this case.

$$\begin{cases} \mu + \nu &= p \\ \mu + 2\nu &= q \\ \mu + 3\nu &= r \end{cases}$$

Exercise 77. Find the condition on p, q, r such that the following system has solutions. Describe all the solutions in this case. Find the determinant of the matrix of the system.

$$\begin{cases} \mu + \nu + \rho &= p \\ \mu + 2\nu + \rho &= q \\ \mu + 3\nu + \rho &= r \end{cases}$$

:: self-control, good level:

Exercise 78. Find an equation of the line through the points $(1,2),(2,4) \in \mathbb{R}^2$. Find the distance from the point $(3,2)$ to this line. (this distance is equal to the length of the perpendicular from the point to the line). ask how to draw a perpendicular line to a given line through a given point, prove the formulae that the students are using

2.4 :: linear algebra, basis, linear maps

goal: understand the notion of linear independence, a basis, a change of a basis

Definition of a linear space, linear maps.

Definition: a set of linearly independent vectors $v_1, v_2, \ldots, v_k$ is a basis in a linear space V if any vectors $v \in V$ can be written as

$$v = \alpha_1 v_1 + \cdots + \alpha_n v_n$$

for some $\alpha_i \in \mathbb{R}$. α_i are called the coordinates of v in the basis $v_, v_2, \ldots, v_n$.

Remark. Any vector have unique coordinates in a basis. Existence follows from the definition of a basis. Uniquence is because if

$$v = \sum \alpha_i v_i = \sum \alpha_i' v_i,$$

then

$$0 = \sum (\alpha_i - \alpha_i') v_i,$$

and since the vectors $v_1, v_2, \ldots, v_n$ are linearly independent, we have $\alpha_i = \alpha_i'$ for all i.

If we make a matrix A using the coordinates of $v_1, v_2, \ldots, v_n$ in a standard basis then $A \cdot \alpha = v$ where $\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \ldots \\ \alpha_n \end{pmatrix}$.

Example: express $(5, 3)$ in the basis $(1, 1), (1, -1)$. We have $(5, 3) = 4(1, 1) + 1(1, -1)$. Note that

$$\begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

Therefore,

$$\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

Thus, if A is the matrix with vectors $v_1, \ldots v_n$ (new basis) as columns and v is a point (with coordinates in old basis), then the coordinates of v in the new basis are given by $A^{-1} \cdot v$.

Theorem about eigendecomposition. If a 2×2-matrix A has different eigenvalues λ_1, λ_2 and eigenvectors v_1, v_2 and Q is the matrix with the first column v_1 and the second column v_2, then

$$A = Q \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cdot Q^{-1}$$

Indeed, if we have any vector $v = \begin{pmatrix} p \\ q \end{pmatrix}$ we can write in as $v = \alpha_1 v_1 + \alpha_2 v_2$. And then $Av = \alpha_1\lambda_1 v_1 + \alpha_2\lambda_2 v_2$. On the other hand, $Q^{-1}v = \begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}$ and

$$\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} \cdot Q^{-1}v = \begin{pmatrix}\lambda_1\alpha_1\\ \lambda_2\alpha_2\end{pmatrix}$$

and since Q is composed out of v_1, v_2, we have

$$Q\begin{pmatrix}\lambda_1\alpha_1\\ \lambda_2\alpha_2\end{pmatrix} = \alpha_1\lambda_1 v_1 + \alpha_2\lambda_2 v_2 = Av.$$

An example of this theorem is Exercise 86.

Definition. The dimension of a linear space is the number of elements in its basis (we will not prove neither that a basis exists, nor that two bases have the same number of elements).

Theorem. Any linear map $f : \mathbb{R}^m \to \mathbb{R}^n$ can be presented by an $n \times m$ matrix A such that $f(v) = Av$ for all $v \in \mathbb{R}^m$.

Definitions of kernel and image of a linear map.

Theorem. For a linear map $f : \mathbb{R}^m \to \mathbb{R}^n$ we have $\dim ker(f) + \dim Im(f) = m$.

:: self-control, good level:

Exercise 79. Prove the triangle inequality, i.e. $|x| + |y| \geq |x + y|$ for any $x, y \in \mathbb{R}^2$.

Exercise 80. The Fibonacci numbers $F_0, F_1, F_2, \ldots$ are defined as follows: $F_0 = F_1 = 1$, $F_2 = F_1 + F_0$, $F_3 = F_2 + F_1$, etc, $F_{n+2} = F_{n+1} + F_n$. Compute F_7. Verify for $n = 1, 2, 3$ and prove (hint: by induction) that $F_0 + F_1 + F_2 + \cdots + F_n = F_{n+2} - 1$.

Exercise 81. Let $v_k = \begin{pmatrix}F_{k+1}\\ F_k\end{pmatrix}$, the vector with Fibonacci numbers. Show that $v_{k+1} = Av_k$ where $A = \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$.

Exercise 82. Find eigenvalues and eigenvectors of $A = \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$.

Exercise 83. Present v_0 as a linear combination of eigenvectors of A. Compute $A^n v_0$ using this presentation. note that this gives an explicit formula for the Fibonacci number F_n.

Exercise 84. Prove that if $\{v_1, v_2, \ldots, v_k\}$ is a linearly independent set of vectors, then $\{v_1, v_2, \ldots, v_{k-1}\}$ is also a linearly independent set of vectors.

Exercise 85. Let $v_1 = (1, 2), v_2 = (2, 1)$. Find a general formula for the coordinates of a point (x, y) in the basis v_1, v_2.

Exercise 86. Consider $A = \begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix}$. Find the eigenvectors v_1, v_2 of A. What is $A(\alpha v_1 + \beta v_2)$? What is A in the basis v_1, v_2 (i.e. what are Av_1, Av_2)? Note that if $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then the columns of A are exactly vectors Ae_1, Ae_2.

Consider the map $f : \mathbb{R} \to \mathbb{R}^2, t \to (1 + t, 3 + 2t)$. I.e. $f(t) = (1 + t, 3 + 2t)$.

Exercise 87. Explain geometrically why $f(t)$ is a line and find it equation.

Exercise 88. Explain geometrically that the image of

$$f : \mathbb{R}^2 \to \mathbb{R}^3, (t_1, t_2) \to (1 + t_1, 2 + 2t_1 - t_2, 3 - t_2)$$

is a plane and find its equation. Recall that an equation of a plane in $\mathbb{R}^3$ in xyz-coordinates is $ax + by + cz + d = 0$.

The presentation of a line or a plane as in the previous two problems are called the parametric presentations.

Exercise 89. Find the parametric presentation of a line in $\mathbb{R}^3$ which passes through two points $(1, 2, 3), (2, 4, 4)$.

Exercise 90. Present the line passing through $(1, 2, 3), (2, 4, 4)$ as an intersection of two planes in $\mathbb{R}^3$, so present it as a solution of a system of two equations on three variables.

Exercise 91. Solve (find a condition for p, q, r such that there exist solutions, and then find solutions in this case)

$$\begin{cases} \mu - \nu & = & p \\ \nu - \rho & = & q \\ -\mu + \rho & = & r \end{cases}$$

Exercise 92. Present the line passing through points $(1, 2, 3), (2, 4, -1) \in \mathbb{R}^3$ as an intersection of two planes and parametrically.

:: self-control, excellent level:

Exercise 93. Reduced row echelon form of a matrix looks like that. (see also wiki)

$$\begin{pmatrix} 1 & 0 & * & * & 0 & * \\ 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * \\ \dots \end{pmatrix}$$

Each row starts from zeroes, then 1, then some numbers which we do not care about, we denote them by $*$. Each next row has 1 on some position on the right from the previous row. On top of each starting 1 we have zeros. Prove that each matrix can be transformed by the elementary transformations (subtraction rows from each other with some coefficients, as in the Gauss elimination method) a matrix of the reduced row echelon form. probably explain what is required to prove, give hints and leave for the next time.

Exercise 94. Let x^k (it is just a notation, k indicates the position of 1, not the power) be the vector $n \times 1$ whose all coordinates are zeros except the coordinate number k which is 1. Let $A = \{a_{ij}\}_{i,j=1}^n$ be any $n \times n$ matrix. Show that $(x^k)^T x^m = \delta_{km}$. By definition δ_{km} is 1 if $k = m$ and 0 in all the other cases. Find $(x^k)^T A(x^m)$.

2.5 :: [bonus] complex numbers

goal: learn that there exist complex numbers and they are useful in a variety of situations

Exercise 95. i is such a letter that $i^2 = i \cdot i = -1$. Using this rule compute $(a + bi) \cdot (c + di)$.

Why a sociologist needs complex numbers? Because complex numbers appear as eigenvalues for matrices that you use to study migration and birth-death processes in populationWolfgang Weidlich and Günter Haag. *Concepts and models of a quantitative sociology: the dynamics of interacting populations*. Vol. 14. Springer Science & Business Media, 2012, p.102.

Exercise 96. Compute $(\cos\alpha + i\sin\alpha) \cdot (\cos\beta + i\sin\beta)$.

Exercise 97. Note that $i = \cos\pi/2 + i\sin\pi/2$. Note that $-1 = i^2 = \cos\pi + i\sin\pi$. Find $\sqrt{i}$. (In other words, find two roots of $x^2 - i = 0$ in the form $x = \cos\alpha + i\sin\alpha$).

Exercise 98. Solve $x^2 + 4x + 5 = 0$.

Exercise 99. Find the eigenvalues and eigenvectors of the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

2.6 :: [bonus] elementary transformations, Gauss elimination method

goal: why Gauss elimination method works

Elementary transformations are

- reorder the rows of an equation,
- multiply a row by a number,
- add a row, multiplied by a number, to another row,
- reorder the columns (by reordering the variables).

Theorem 1. Using elementary transformations we can always make our system as follows:

$$\begin{pmatrix} 1 & 0 & 0 & \dots & * & & \\ 0 & 1 & 0 & \dots & * & & \\ \vdots & & & & & & \\ 0 & \dots & \dots & 1 & * & \dots & * \\ 0 & \dots & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & \dots & 0 & 0 & \dots & 0 \end{pmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ . \\ p_m \end{bmatrix} \quad (2)$$

So, we suppose that we have a system of m equations on n variables $x_1, \dots, x_n$ So we have a matrix A, it is $m \times n$. After application of elementary transformations, A becomes a matrix B which has inside the identity matrix $k \times k$ and then $m - k$ lines with zeros. The last $n - k$ columns have some numbers which we mark by $*$. Last $n - k$ variables are called *independent* variables. First k variables are called dependent variables. This system has solutions only if $p_{k+1} = p_{k+2} = \dots = p_n = 0$. If the latter is satisfied, we choose any values for the independent variables. Then dependent variables can be found uniquely.

This is called Gauss–Jordan elimination. see wiki.

Example

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$$

Normally we use [] for vectors. It does not matter a lot, it is again a questions of notation: when we study matrices we write (), when we solve equations we write [].

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{bmatrix} p \\ q \end{bmatrix}$$

$$\begin{pmatrix}1 & -1\\0 & 1\end{pmatrix}\begin{pmatrix}2 & 1\\1 & 1\end{pmatrix}\cdot\begin{bmatrix}x\\y\end{bmatrix}=\begin{pmatrix}1 & -1\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\-1 & 1\end{pmatrix}\begin{bmatrix}p\\q\end{bmatrix}$$

$$\begin{pmatrix}1 & 0\\-1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}\cdot\begin{bmatrix}x\\y\end{bmatrix}=\begin{pmatrix}1 & 0\\-1 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\-1 & 1\end{pmatrix}\begin{bmatrix}p\\q\end{bmatrix}$$

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\cdot\begin{bmatrix}x\\y\end{bmatrix}=\begin{pmatrix}1 & 0\\-1 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\-1 & 1\end{pmatrix}\begin{bmatrix}p\\q\end{bmatrix}$$

so we have $E \cdot X = A_1 A_2 \ldots A_n B$ is equivalent to $AX = B$. Therefore $A = A_n^{-1} \ldots A_2^{-1} A_1^{-1} E$ In our case we do not need to reorder columns. So $C = E$ and $B = E$ and $T = A_n^{-1} \ldots A_2^{-1} A_1^{-1}$ in the notation of Corollary.

$$\begin{pmatrix}1 & 0\\-\mu & 1\end{pmatrix}^{-1}=\begin{pmatrix}1 & 0\\+\mu & 1\end{pmatrix}$$

2.7 :: [bonus] SVD decomposition, recommendation systems.

Complex numbers, fundamental theorem of algebra, so the number of eigenvalues of an $n \times n$ matrix is n (with multiplicities). Geometric meaning of a matrix.

SVD-decomposition and recommendation systems (next year to another secret class, together with PageRank).

2.8 :: homework and the second midterm

watch video playlist, especially the video about basis and eigendecomposition.

You may think that during the day zero somebody brings you a fresh new bacterium. Every day every bacterium of **age one day** *produces* **one** *bacterium and every bacterium of* **age two days** *produces* **two bacteria and dies**. *How many bacteria are there on day number n?*

If we denote by G_n the number of bacteria on day n, we get the following sequence: $G_0 = 1, G_1 = 1, G_{n+1} = G_n + 2G_{n-1}$ *for $n \geq 1$.*

We may compute the first few terms

$$G_0 = 1, G_1 = 1, G_2 = 3, G_3 = 5, G_4 = 11, G_5 = 21, \ldots$$

Your goal is to find a **short and explicit** *formula for G_n which depends only on n. This can be done in many ways, in particular, using matrices and eigendecomposition.*

1. Explain that if $A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$ then $A \cdot \begin{pmatrix} G_n \\ G_{n-1} \end{pmatrix} = \begin{pmatrix} G_{n+1} \\ G_n \end{pmatrix}$.

2. Explain why

$$\begin{pmatrix} G_{n+1} \\ G_n \end{pmatrix} = A^n \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore we need to find A^n. How to do that? So far we just reformulated the problem of finding G_n using matrices, seemingly making it harder. But no.

3. Let $B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. Explain why $B^n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}$.

4. Let $C = DBD^{-1}$ where D is a matrix, and D^{-1} is its inverse matrix. Explain why $C^n = DB^nD^{-1}$.

Thus, if we manage to present our $A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$ as $A = DBD^{-1}$ and B has only non-zero entries on diagonal, then we can easily compute A^n.

5. Find the eigenvalues λ_1, λ_2 and corresponding eigenvectors v_1, v_2 of A.

6. Present vectors $(1, 0)$ and $(0, 1)$ as linear combinations of v_1, v_2.

7. Take the eigenvectors of A as a basis and write A in this basis. In other words, find a two-by-two matrix D such that

$$A = D \cdot \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cdot D^{-1},$$

where λ_1, λ_2 are the eigenvalues of A.

8. Note that the matrix D in the above formula is not unique: let D' be the matrix $c \cdot D$, $c \in \mathbb{R}$, i.e. we multiplied all entries of D by c. Then $A = DBD^{-1} = D'BD'^{-1}$. (Why?) E.g. one can take D as the matrix with vector columns v_1, v_2.

9. Compute

$$\begin{pmatrix} G_{n+1} \\ G_n \end{pmatrix} = A^n \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = D \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} D^{-1} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

(you know the matrices D, D^{-1} from the previous exercise.) This gives you an explicit formula for G_n.

10. Another way: find $\alpha, \beta \in \mathbb{R}$ such that $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \alpha \cdot v_1 + \beta \cdot v_2$, then $A^n(\alpha \cdot v_1 + \beta \cdot v_2) = \alpha\lambda_1^n \cdot v_1 + \beta\lambda_2^n \cdot v_2$ (explain why).

11. Write the formula for G_n, verify that it works for $G_1 = 1, G_2 = 3, G_4 = 11$.

2.9 :: an example of the second midterm

you may take a cheatlist A4 with you, with your own handwriting. Printed lists are not allowed

(2 points) Simple question about linear algebra (e.g. definition of a linear span. Or geometric meaning of the determinant of a matrix. Or geometric meaning of the rank of a matrix. Or check that some vectors are linearly dependent or independent. Or give an example of a matrix of rank 2. Or an example of non-commuting matrices of the same size $AB \neq BA$)

(4 points) For a given sequence (e.g. $G_0 = 1, G_1 = 2, G_{n+2} = G_n + G_{n+1}$) find the formula for G_n.

(3 points) For the same sequence write the matrix A such that

$$A \cdot \begin{pmatrix} G_n \\ G_{n-1} \end{pmatrix} = \begin{pmatrix} G_{n+1} \\ G_n \end{pmatrix},$$

and find the rank of A, determinant of A, eigenvalues λ_1, λ_2 and eigenvectors v_1, v_2 of A

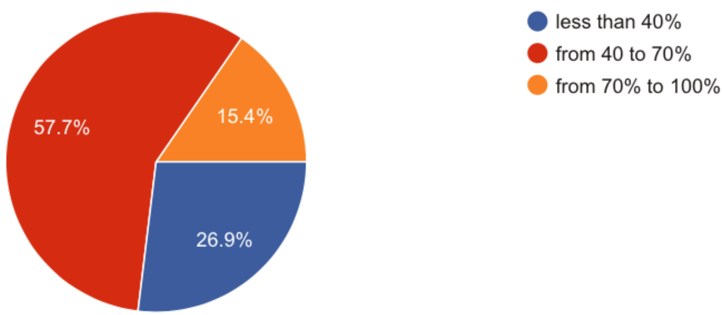
(1 point) find the coefficients of the vector $\begin{pmatrix} G_1 \\ G_0 \end{pmatrix}$ in the basis v_1, v_2

2.10 :: feedback

Here I provide the feedback of students for this chapter.

In the second chapter "Linear algebra, planar and space geometry" I understand...

26 responses



I have rather diverse answers to the question "What are the most interesting parts in Chapter 2"

- *"I want to mention: 3blue1brown is a spectacular teacher, this helped a lot, so thank you for the suggestion to watch the playlist of his. Everything was actually well with this chapter except for a couple of points I would like to comment on in the next section. I also really like complex numbers."*
- *"Linear span and transformations"*
- *"Linear systems, I liked all the basis vectors, determinants, spans."*
- *"Metrics! solving linear system."*
- *"Eigenvectors and their geometric interpretation."*

Students were asked to watch the youtube channel of 3blue1brown about linear algebra, most of them liked this channel very much.

The most difficult parts were:

- *"Kernel, image, dimension."*
- *"I found the eigenbasis concept a bit challenging, as well as the necessity to randomly introduce Fibonacci numbers. The echelon matrix is a mystery to me. SVD decomposition was great, even though I had to spent an ungodly amount of time figuring it out (that is why it is in the difficulties, thought it truly was interesting)."*
- *"Eigenvalues, determinant, everything."*
- *"It happened somehow, that I understand almost every topic in this chapter at the median level, but I am not confident in any of them (except for metrics and solving liner system, as I have already mentioned). the most difficult for me was to combine those topics, as we have done in the second homework. this chapter was more difficult for me than the previous and the next ones."*
- *"Gauss elimination method and all others."*

3 Mathematical analysis

goal: learn geometric meaning of integration and differentiation, learn how to compute them and analyse graphics Sociologists need mathematical analysis to build and evaluate models. Read "An introduction to models in the social sciences"Charles A Lave and James G March. *An introduction to models in the social sciences*. University Press of America, 1993, about philosophy and heuristics of model building, this book contains a lot of reflexive exercises. BooksJohn G Kemeny. *Mathematical models in the social sciences*. Tech. rep. 1972,Edward Beltrami. *Mathematical models for society and biology*. Academic Press, 2013 are also accessible for you after our course, I recommend to read some chapters in them (not starting from the beginning but choosing one chapter you like the most and reading it). RapoportAnatol Rapoport. *Mathematical models in the social and behavioral sciences*. Wiley New York, 1983 is mush more serious book, but it shows detailed studies of processes in society, heavily using analysis.

3.1 :: integration and differentiation

goal: understand geometric meaning of integration and differentiation, several interpretations Integration as the area.

Problem 32. Formulate the basic properties of integration.

Problem 33. Illustrate the basic properties of integration for $\int_1^8 2dx$.

Definition of an integral using Darboux sums.

Problem 34. Compute $\int_0^1 xdx$ using Darboux sums.

Problem 35. What happens if you try to compute $\int_0^1 f(x)dx$ where $f(x) = 0$ if x is an irrational number and $f(x) = 1$ if x is a rational number.

Derivative as the limit of average speed.

Problem 36. Let $f(t) = t^2$. Compute $f'(1)$.

Not all functions are differentiable. For example $y = |x|$ (absolute value) has a minimum at 0 but is not differentiable.

3.2 :: physical and geometric meaning of the derivative, monotonicity, convexity, local extrema

Physical meaning (approximate calculation of $f(x + \varepsilon)$ where ε is very small.) of derivative.

Problem 37. Using the physical meaning of the derivative, prove that a)$(f+g)' = f' + g'$, b) $(cf)' = cf'$ where $c \in \mathbb{R}$, c) $(fg)' = fg' + f'g$.

Geometric meaning of derivative (the slope of the tangent line to the graph of the function).

Problem 38. Explain why f grows near x_0 if $f'(x_0) > 0$.

Problem 39. Show that if $f''(x_0) > 0$ then f is convex near x_0, and vice versa.

Problem 40. For given f (graphically), draw f', f''.

Problem 41. Show that if f is differentiable and $f(x_0)$ is a local maximum, then $f'(x_0) = 0$.

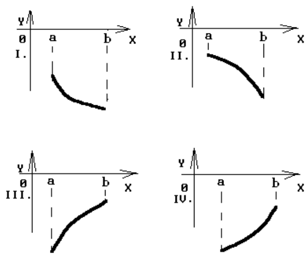
TO REMEMBER: HOW TO FIND MAXIMA AND MINIMA: for a differentiable function $f : [a, b] \to \mathbb{R}$, a maximum of f on $[a, b]$ is achieved at the boundary (points a, b) or at a point where $f' = 0$.

The same for higher dimensions. Let $\Omega \subset \mathbb{R}^n$ be a good set, $f : \Omega \to \mathbb{R}$ is a differentiable map, then the set of local extrema is contained in the boundary and the set of points where all partial derivatives are zeros.

Problem 42. Find the extrema of $f(x, y) = xy - x + y + 2$ on $[-2, 2] \times [-2, 2]$.

:: self-control, satisfactory level:

Exercise 100. Which of the functions in the picture satisfy $f < 0$ on the whole interval? $f' < 0$? $f'' > 0$?



Exercise 101. Draw graphs of $y = \sqrt{x}, y = -\sqrt{x}, y = \sqrt{-x}, y = -\sqrt{-x}$ on the same picture

Exercise 102. Draw $y = \frac{1}{x+2} - 3$. Hint: draw $y = \frac{1}{x}$ and translate it.

Exercise 103. Draw $y = x^3, y = x^3, y = \sqrt{x}, y = x^{1/3}, y = e^x, y = \ln x$ on the same picture.

Exercise 104. Draw the graphs of $y = \sin x$ and $y = \arcsin x$.

Exercise 105. Find $\ln(\sqrt{x^2+1})'$.

Exercise 106. For $F(a, b) = \ln(ab)(a+\sin b)^2 + a^b$ find the partial derivatives F_a, F_b.

:: self-control, good level:

Exercise 107. Using geometric presentation of $\sin\alpha, \tan\alpha$ show that for $\alpha \in (0, \pi/4]$, we have $\sin\alpha < \alpha < \tan\alpha$. here is a discussion about radian and gradus measure of angles

Exercise 108. Draw graphs of $y = e^x, y = x^2$ on log-log scale (watch https://www.khanacademy.org/math/algebra2/exponential-and-logarithmic-functions/log v/logarithmic-scale and https://www.khanacademy.org/math/algebra2/expone logarithmic-scale/v/richter-scale and read https://en.wikipedia.org/wiki/Logarithmic_scale)

Exercise 109. Find a function $y(x)$ such that $y' \sin x = y \cos x$. Hint: what is $(\ln y)'$?

Exercise 110. Find a function $y(x)$ such that $y' = 8 \sin 2x$.

:: self-control, excellent level:

Exercise 111. Draw the graph of $y = \frac{x^3+1}{x^2-2}$ as precisely as possible: 1) where it is defined, 2. roots, 3. where it is positive or negative 4. behavior near $x = \pm\sqrt{2}$) 5) behavior at $\pm\infty$ (asymptotes $y = ax + b$).

Exercise 112. Draw the plot of the function $y(x) = \sin\frac{1}{x}$.

Exercise 113. Draw graphs of $y = \{x\}$ (fractional part of x) and $y = \{1/x\}$.

Exercise 114. Find the line $y = ax + b$ such that the sum of the squares $(ax_i + b - y_i)^2$ of distances from the points $(x_i, y_i) = (1, 2), (2, 4), (3, 3), i = 1, 2, 3$ to this line is minimal among all the lines in the plane. In other words, find the "best" line approximating these three points.

3.3 :: fundamental theorem of calculus, integration by parts,

goal: learn integration by parts

Problem 43. (Fundamental theorem of calculus) Prove that for $F(y) = \int_0^y f(x)dx$ we have $F'(y) = f(y)$.

Remark. If $F' = f$ then $\int_a^b f(x)dx = F|_b^a = F(b) - F(a)$.

Problem 44. Find $\int_2^3 (x^3 + 2x^4)dx$

Problem 45. (Integration by parts) Prove that $\int_a^b f'g = (fg)|_a^b - \int_a^b fg'$.

Problem 46. Compute $\int_2^3 x\cos(2x)dx$.

Problem 47. Prove $(f \circ g)' = g' \cdot (f' \circ g)$.

Problem 48. Compute $(\sin 2x)'$.

Problem 49. Find $(\cos^2 x^2)'$.

Problem 50. Prove that $\sin' = \cos, \cos' = -\sin$.

Taylor series.

Problem 51. Find $\sin\varepsilon$ with accuracy ε^4.

Problem 52. Show that $\sin' x = \cos x, \cos' x = -\sin x$, using Taylor series for them.

:: self-control, satisfactory level:

Exercise 115. Let A be the set of points below the parabola $y(x) = 5 - x^2$ and B be the set above the line passing through points $(-3, -4), (2, 1)$. Find the area of $A \cap B$.

Exercise 116. Find $\int_0^\pi x^2 \cos x dx$

Exercise 117. Calculate the integral $\int\limits_0^1 (x^{3/4} + x\sin x)dx$.

Exercise 118. Two persons are in a prison, incarcerated in the same room, so they communicate. The police propose them the following game: tomorrow they go to different rooms. To each of them it will be proposed a color: white or black. Each of them will must to guess the color of the other person. If at least one of them is right, they both will be freed. If both guess wrong, they will be immediately killed. Should they play this game and, if yes, what should be their strategy?

Exercise 119. Find a such that $\int_0^\pi a\sin x dx = 1$.

:: self-control, good level:

Exercise 120. Find the first three terms in the Taylor series for $y(x) = 1 - \sin^2 x$.

:: self-control, excellent level:

3.4 :: using derivatives: looking for maxima and linear regression.

goal: understand how the formula for linear regression can be easily deduced from what we know The formula for the line $y = ax + b$ which is the best approximation of

$$\{(x_i, y_i)\}_{i=1}^n \subset \mathbb{R}^2$$

with the error term

$$F(a, b) = \sum_{i=1}^{n} (y_i - ax_i - b)^2.$$

Problem 53. Show that $x^T y - n\bar{x}\bar{y} = \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})$.

Problem 54. Show that $x^T x - n\bar{x}^2 = \sum_{i=1}^n (x_i - \bar{x})^2$.

3.5 :: sequences, limits, and continuous functions

goal: understand the definition of a limit

Formally, a sequence of real numbers is a map $x : \mathbb{N} \to \mathbb{R}$. Traditionally, instead of writing $x(1), x(2), \ldots$ we write $x_1, x_2, x_3, \ldots$.

Example. The infinite sequence $0, 0, 0, 0, \ldots$ may be presented as

$$\{x_i\}_{i=1}^{\infty}, \forall i \in \mathbb{N}, x_i = 0.$$

Example. The infinite sequence $1, 1/2, 1/3, 1/4, \ldots$ may be presented as

$$\{x_i\}_{i=1}^{\infty}, \forall n \in \mathbb{N}, x_n = 1/n.$$

Example. The infinite sequence $1, -1, 1, -1, 1, \ldots$ may be presented as

$$\{x_i\}_{i=1}^{\infty}, \forall k \in \mathbb{N}, x_k = (-1)^{k+1}.$$

Definition of limits via traps for sequences. Continuous functions, equivalence of two definitions.

Problem 55. Prove Bolzano-Cauchy theorem.

Problem 56. Prove two policeman theorem.

END OF THE COURSE.

3.6 :: individual homework

goal: learn how to use internet when you need to solve a problem 1. Using the least square method, find the best line approximating the points

$$(1,8),(3,2),(4,7).$$

2. Find SVD decomposition of the matrix $\begin{pmatrix} 2 & 1 \\ 16 & -10 \\ 51 & 5 \\ 1 & 2 \end{pmatrix}$

3. Find the equation of the plane in $\mathbb{R}^3$ passing through $(-6,18,11)$ and the line $(-2\lambda-3, 3\lambda+16, 2\lambda-2)$.

4. Find the derivative $\frac{1}{x^6}\cos(\cos 7x + 2e^{\cos(7x^3-11x+11)})$.

5. Find the integral $\int(2x^{3/8}+6x\sin x+7\frac{1}{1+x^4})dx$.

6. Find the maximum of the function $(x+7)x^2+5x(x-2)+11$ on the interval $[-13,11]$.

7. Find (approximately, two digits after the decimal point) the eigenvalues of $\begin{pmatrix} 2 & 12 & -5 \\ 1 & 2 & 12 \\ 3 & 2 & 5 \end{pmatrix}$

3.7 :: example of 3rd midterm

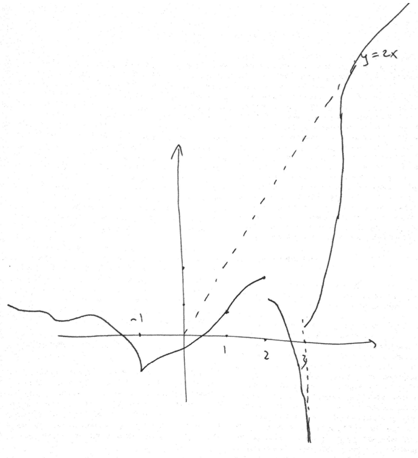
Exercise 121. Calculate the integral $\int\limits_0^1 (x^{3/4}+x\sin x)dx$.

Exercise 122. Draw a graph of a function $f(x)$ such that f has jump discontinuity at point $x=2$, essential discontinuity at $x=3$, monotone (increasing) on $(1,2)$, monotone (decreasing) on $(2,3)$, concave on $(1,2)$, has an asymptote $y=2x$ as $x\to+\infty$, has **no** derivative at $x=-1$. Answer is like on the picture:

Exercise 123. Calculate the derivative of the function

$$f(x)=\ln(\sqrt{x^2-1})+2x\sin(x^2-1)$$

at the point $x=2$.



3.8 :: [bonus] cardinalities of $\mathbb{Q}$ and $\mathbb{R}$

Definition 5. Two sets A, B have the same cardinality if there exists a bijection $A \to B$.

Example, the set of integer numbers, $\mathbb{Z}$ has the same cardinality as the set of pair integers. $2\mathbb{Z} = \{2z | z \in \mathbb{Z}\}$. Indeed, there is a bijection, which is given by $z \to 2z$.

Also the set $\mathbb{Z}$ has the same cardinality with the set of odd integer numbers. Therefore, $\mathbb{Z}$ is decomposed into two parts, each of them has the same cardinality as $\mathbb{Z}$.

Next example, $\mathbb{N}^2$ has the same cardinality as $\mathbb{N}$. Indeed, a bijection between a set A and the set $\mathbb{N}$ is the same as a numbering of the elements of A. So write $\mathbb{N}^2$ as a infinite table, and then number it going by diagonals.

Then, there is injection $\mathbb{Q}_{>0} = \{\frac{p}{q} | p, q \in \mathbb{N}\}$ to $\mathbb{N}^2$. Just $\frac{p}{q} \to (p, q)$.

Theorem 2. Let A, B be two sets. Then if there exists an injection $A \to B$ and an injection $B \to A$, then A, B have the same cardinality.

So it seems that all infinite sets have the same cardinality. It is not so. For example. one can prove that 2^X (the set of subsets of X) has cardinality strictly bigger than X (for each set X, finite or infinite).

Theorem 3. There is no bijection between sets $\mathbb{N}, \mathbb{R}$.

Proof. Suppose that there exists an bijection, $i \to A_i$, $a_{i1}a_{i2}, a_{i3} \ldots, a_{ij} \in \{0, 1, 2, 3, 4$ $\mathbb{Z}$. Construct a new number y, $y = 0.b_1 b_2 b_3 \ldots$ such that $b_1 \neq a_{11}, b_2 \neq a_{22}, \ldots$.

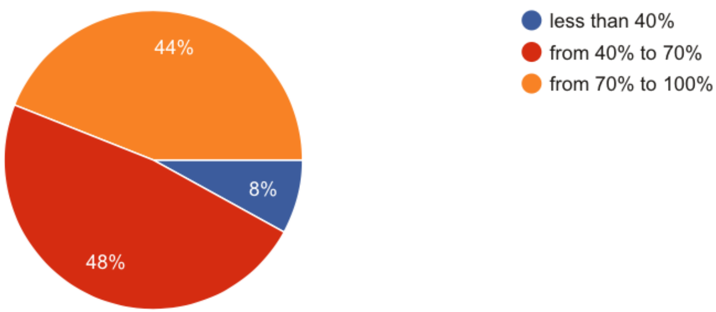
Then y differs from i-th position from i-th number in our numbering, this is true for all $i \in \mathbb{N}$ Therefore y is not numbered, contradiction. Therefore there is no such a bijection.

3.9 :: feedback

Here I provide the feedback of students for this chapter.

In the third chapter "Mathematical analysis" I understand...

25 responses



The most interesting topics were...

- *"Alright, so my take on this chapter is certainly biased, since at my school I had a huge math. analysis course. Therefore to me everything except for linear regressions was kind of boring, but that's just me."*
- *"Derivatives, integrals, integration by parts."*
- *"That must be a really weird one, but I did enjoy all the theory on integration. I also liked the calculation of the best line (linear regression). I finally understood where the trend lines on scatter plots come from."*

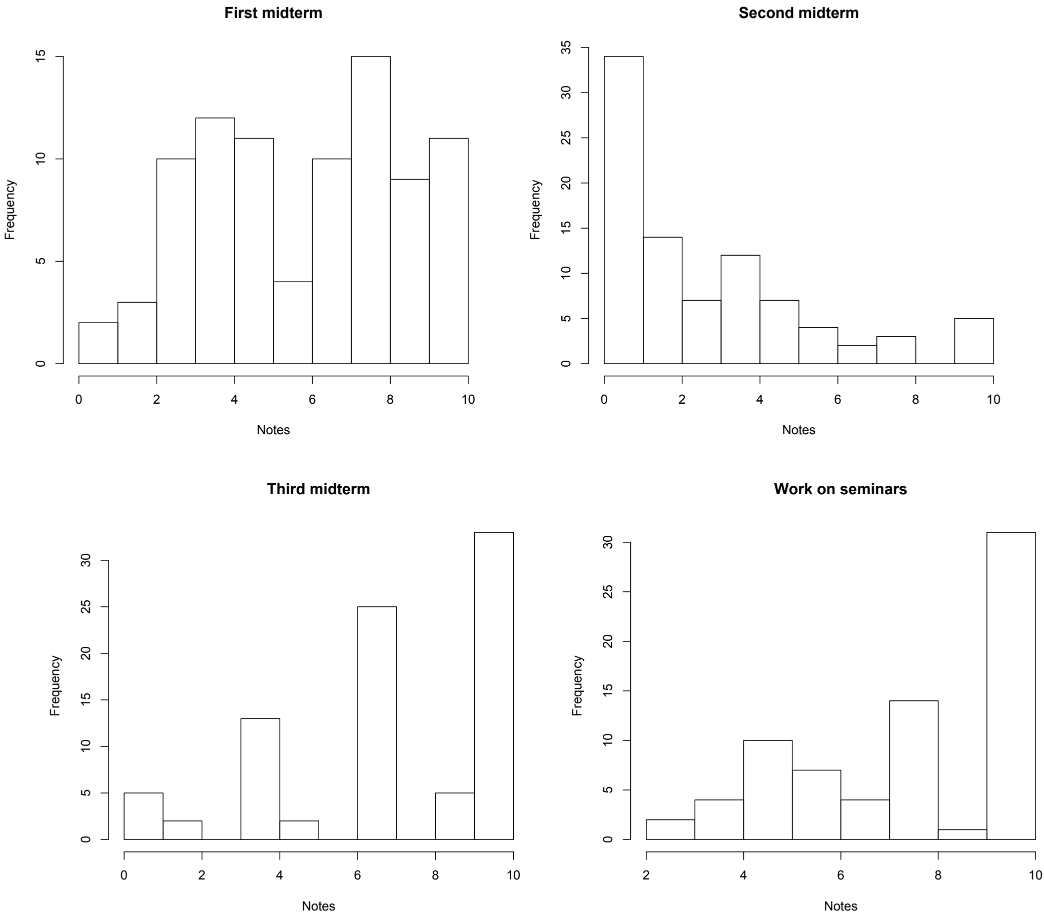
The most difficult parts were...

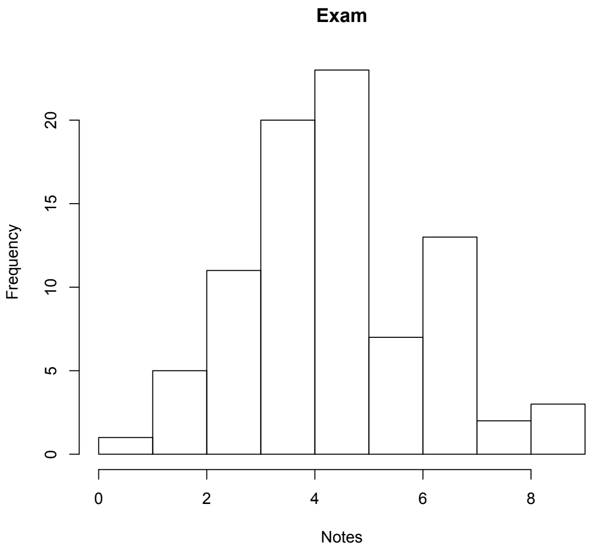
- *"Limits(still don't understand them, please, do not include them in the final exam)"*
- *"Proving that a function is convex or concave?That was a challenge!"*

- *"Limits. We didn't study them at school and had only one lecture to understand them. Though, we used them on several lectures what made no sense to me"*

4 Additional materials

4.1 :: notes for three midterms, exam, and seminars





4.2 :: bonus reading

Experimental techniques, systematic observation in natural settings, and the whole apparatus of research methodology is necessary if empirical study is to carry the science beyond the distillation of personal experience. Similarly, if conceptual elaboration is to progress beyond the proverbs of the ancients, special tools are necessary. The most remarkable of these is mathematics. Mathematics provides a battery of languages which, when carefully fitted to a set of ideas, can lend those ideas great power. The mind falters when faced with a complex system or a long chain of deductions. The crutch that mathematics provides to everyday reasoning becomes essential as sociology moves toward the analysis of complex systems and predictions based on extended chains of deductions.James Samuel Coleman et al. "Introduction to mathematical sociology." In: (1964) Read especially Chapter 18 (different strategies to use mathematics in sociology) of this book.

The data for the choice of pursuing an education and society classes is explained using rational choice theory and a little of mathematicsRichard Breen and John H Goldthorpe. "Explaining educational differentials: Towards a formal rational action theory". In: *Rationality and society* 9.3 (1997), pp. 275–305. Big surveyKim P Corfman and Sunil Gupta. "Mathematical models of group choice and negotiations". In: *Handbooks in operations research and management science* 5 (1993), pp. 83–142 on group choice models: mathematical, economical, marketing etc. Game theory modelClemens Kroneberg and Andreas Wimmer. "Struggling over the boundaries of belonging: a formal model of nation building, ethnic closure, and populism". In: *American Journal of Sociology* 118.1 (2012), pp. 176–230 on cooperation between elites and masses to produce a nation with data on Ottoman Empire. History of mathematical sociology[23] with a lot of conceptual examples.

Mathematics are used in study of of process (discrete and continuous models, James S Coleman. "Social capital in the creation of human capital". In: *American journal of sociology* 94 (1988), S95–S120, Michael T Hannan and John Freeman. *Organizational ecology*. Harvard university press, 1993), structure (social networks, graphs, matrices), and action (rational choice, game theory) Petter Holme, Christofer R Edling, and Fredrik Liljeros. "Structure and time evolution of an Internet dating community". In: *Social Networks* 26.2 (2004), pp. 155–174,Christofer R Edling. "Mathematics in sociology". In: *Annual review of sociology* 28.1 (2002), pp. 197–220.

4.3 :: overall feedback

Most of students love home reading with oral recitation afterwards. I don't know what are the preferred chapters but students read about Markov chains with pleasure.

"[The reading is] the best part of the course really(helped to understand the practical use if mathematics for sociology)"

"It wasn't of particular usefulness for me."

"And thank you for the book about culture, now I see this idea everywhere. Unfortunately, one who should tell me about his book did not do it, but I asked others."

Students say that three midterms were very useful for the study. Also, in order to write a midterm, every student must send a solution of all the problems in the demo-version of this midterm to an assistant. This force them to prepare for the midterm, and the it is not the first time when they see problems when they come to a midterm. In other words, this is just an obligatory homework. Note that it is not graded, so the students have no incentives to copy from others.

"those midterms were useful for me, because I felt that I have to prepare for them, so I devoted time and was trying to complete demo versions and to understand what is happening. It helped me to find the motivation to revise themes that I missed or just don't understand. As far as it was one midterm per theme, it was also useful for me to prepare official cheating papers — it was like extra-revising. Actually, I believe that the practice of cheating papers is very useful too — it helps to understand that math is not (at least not only) about memorizing"

Students were allowed to bring an A4 list with any handwritten information to each midterm (except the third) and the exam.

[23] Chapter 1 in Vittorio Capecchi. "Mathematics and Sociology". In: *Applications of Mathematics in Models, Artificial Neural Networks and Arts*. Springer, 2010, pp. 1–78

Some random answers.

"I think that you are a very professional and understanding teacher and I liked you approach to us very much. But I think half of the course was too complicated, especially the second chapter, and I still remember you saying: you will know as much as a student in a technical university. Yes, HSE is half technical university but some things I believe are not really for first-year students. Preparing for the exam via the Internet, I was convinced that everything can be done by computer programs in less amount of time, and I just questioned myself: for what purpose I try to really understand everything and spend enormous amounts of time on huge computations on the paper, if I study on the first course of non-technical program and later I will use computers for that purpose. Another thing is that except you no one explained the material to us and it was really tough during lectures. The answers to some problems is impossible to find even in the internet and when they are explained after, in the seminar in even more complicated way....it kind of makes you give up."

4.4 :: just problems

Exercise 124. Find three fractions $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}$ (where all $p_i, q_i \in \mathbb{Z}, i = 1, 2, 3$) such that

$$\frac{p_1}{q_1} + \frac{p_2}{q_2} + \frac{p_3}{q_3} = 1 = \frac{q_1}{p_1} + \frac{q_2}{p_2} + \frac{q_3}{p_3}.$$

Exercise 125. Floor $[x]$ of $x \in \mathbb{R}$ is the maximal integer number which is less than or equal to x. The fractional part $\{x\}$ of x is defined as $\{x\} = x - [x]$. Find a positive $\alpha < 1$ such that $\alpha + \{1/\alpha\} = 1$.

Exercise 126. Find x, if you know that

$$\frac{2}{73} = \frac{1}{60} + \frac{1}{219} + \frac{1}{292} + \frac{1}{x}.$$

Exercise 127. We all know that $x^2 \geq 0$ for each real number x.

a) prove that $a^2 + b^2 \geq 2ab$ for real numbers a, b.

b) prove that $a^2 + b^2 + c^2 \geq ab + bc + ac$ for real numbers a, b, c.

c) prove that $(a^2 + b^2)(c^2 + d^2) \geq (ac + bd)^2$.

Exercise 128. Let 50 numbers be placed on a circle, and each number is the average of its neighbors (so, for example, $x_3 = \frac{x_2+x_4}{2}, x_{50} = \frac{x_{49}+x_1}{2}$, etc). Prove that all numbers are equal to each other. ask to solve this problem for three numbers. There are many solutions. 1) find a formula, if you know x_1, x_2, you can find x_3 etc. 2) draw all numbers on a circle. draw the difference between neighbors, note that these differences must be equal. Then if you some all of them, you get zero, so all of them are zeros. 3) just take the maximal of x_i. Note that the neighbors are equal to it, etc. It is good to tell all the solutions 1) 2) 3) each one is simpler than previous

Exercise 129. We deleted from the chess-board left-bottom and right-top cells. Is it possible to cut the rest into domino (2×1 figures)? it is not enough to say that you can not do that. a proof: color cells in two colors, as in a chess-board. a simple idea. you changed the point of view and everything became clear. that is math. formalism, logic, it is to write down and make less mistakes, and unambiguous to read. in essence, math is simple and beautiful ideas, ability to put everything upside down such that answers become transparent and obvious

Exercise 130. Prove (formally, using the formal definition of all operations, and only formal symbols, not words) that

- $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$,
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

solution: $x \in A \setminus (B \cup C) \Leftrightarrow (x \in A \setminus B) \wedge (x \in A \setminus C) \Leftrightarrow x \in (A \setminus B) \cap (A \setminus C)$, etc.

Exercise 131. A bus is called full if it has at least 50 persons. There are several buses on a road, some of them full. What is bigger, the percentage of full buses among buses or the percentage of people traveling in full buses among all people in all buses?

Exercise 132. In an Apple Republic there was an election to the parliament where all people voted. Each person voting for the party "Vegetables" likes vegetables. Among people voting for the others parties, 90% does not like vegetables. What is the result of the party "Vegetables" on elections if exactly 46% of population likes vegetables?

Exercise 133. Let X be a set and $A, B \subset X$ be its subset. We denote by $\overline{A} = A^C$ the complement for A in X. Show using Venn diagrams, that $A^C \cap B^C = (A \cup B)^C$, $A^C \cup B^C = (A \cap B)^C$. tell that this will be needed in probability theory

Exercise 134. Calculate with 4 digits precision,

$$1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}}} \qquad 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2}}} \qquad 1 + \cfrac{1}{2 + \cfrac{1}{2}}$$

write the infinite fraction x, and conclude that $x = \frac{1}{x-1} - 1$ so $x = \sqrt{2}$.

Exercise 135. Let l_1 be the set of points $(x, y) \in \mathbb{R}^2$ such that $2x + 3y = 1$. It is a line. Let l_2 be the set of points $(x, y) \in \mathbb{R}^2$ such that $x + 2y = 3$. Find the intersection point of two lines l_1, l_2.

Exercise 136. Find an equation of the line through the points $(1, 2), (2, 4) \in \mathbb{R}^2$. Find the distance from the point $(3, 2)$ to this line. (this distance is equal to the length of the perpendicular from the point to the line).

Watch all the videos videos in chapters "Analyzing limits graphically", then, "Continuity", and "Optional videos" (the last one is in the bottom part of the page). Solve quiz 1 (Analyzing limits graphically) and quiz 2 (Continuity).

Pass Unit test (in the bottom of the page). Make "practice" parts if you want. Then, I assume that on 28 September you watched these videos and we will solve exercises based on this.

Note that you can turn on the subtitles and change the speed of the videos.

Exercise 137. Check that for $f(x) = (\sin x)^n$, we have $f'(x) = n(\sin x)^{n-1} \cos x$. Find

$$\int_0^{\pi/2} \frac{1}{((\sin x)^n)} n(\sin x)^{n-1} \cos x dx.$$

Hint: use that $(g \circ f)' = f' \cdot (g' \circ f)$

Exercise 138. Find the line $y = ax + b$ such that the sum of the squares of distances from the points $(1, 2), (2, 4), (3, 3)$ to this line is minimal among all the lines in the plane. In other words, find the "best" line approximating these three points.

Exercise 139. Find $\int_1^2 \frac{\ln x}{x} dx$. Hint: change variables $u = \ln x$.

Exercise 140. Find $\int_1^2 \frac{dx}{x \ln^2 x}$. Hint: change variables $u = \ln x$.

Exercise 141. Find $\int_0^{1/2} \frac{x dx}{\sqrt[3]{x^2 - 1}}$.

Exercise 142. Find $\int_0^1 (x + 3)(4x - 7)^{50} dx$.

Exercise 143. Find $\int_0^1 e^x \sin x dx$. Hint: integrate by parts two times

Exercise 144. Let a sequence $\{x_n\}_{n=1}^{\infty}$ has a limit ($A = \lim_{n\to\infty} x_n$). Explain why $\{x_n\}_{n=1}^{\infty}$ is bounded (i.e. there exists a number C such that $x_n < C$ for all $n = 1, 2, \ldots$).

Exercise 145. Find $\lim_{n\to\infty} \frac{n^2-1}{n^2+1}$.

Exercise 146. Using Taylor series for $\ln x$, please, calculate approximately $\ln(\frac{x^2-1}{x^2+1})$ if x is very big.

Exercise 147. Compute $\lim_{x\to\infty}(\frac{x^2-1}{x^2+1})^{2x+3}$ using the previous problem.

Exercise 148. Find the L'Hôpital's rule in the Internet. Using it, compute $\lim_{x\to\infty}(\frac{e^x}{x^2})$.

Exercise 149. Does there exist 1000 consecuitive natural numbers, such that there is exactly five prime numbers among them?

Exercise 150. Let $f(x) = |x|$ (absolute value of x.) In what follows, Δx is a number. Find

$$\lim_{\Delta x\to 0, \Delta x>0} \frac{f(0+\Delta x) - f(0)}{\Delta x}$$

Find

$$\lim_{\Delta x\to 0, \Delta x<0} \frac{f(0+\Delta x) - f(0)}{\Delta x}$$

Exercise 151. Let $f(x) = x^2 + x + 1$. Find

$$\lim_{\Delta x\to 0} \frac{f(1+\Delta x) - f(1)}{\Delta x}$$

Here Δx is any sequence of numbers which tends to 0

Exercise 152. Ecologists protest against a forest cutting. The mayor told them: "The percentage of pines in the forest is 99% now. We will cut only pines, and after the cutting there will be 98% pines in the forest." How much of the forest will be cut?

Exercise 153. Two persons are in a prison, incarcerated in the same room, so they communicate. The police propose them the following game: tomorrow they go to different rooms. To each of them it will be proposed a color: white or black. Each of them will must to guess the color of the other person. If at least one of them is right, they both will be freed. If both guess wrong, they will be immediately killed. Should they play this game and, if yes, what should be their strategy?

4.5 :: use R

in statistics is R. https://www.r-project.org

Some examples: https://www.rdocumentation.org/packages/graphics/versi
3.4.0/topics/plot

https://alstatr.blogspot.ru/2013/06/matrix-operations.html

About linear regression https://www.tutorialspoint.com/r/r_linear_regress
htm.

https://www.cut-the-knot.org/arithmetic/combinatorics/Ramsey44.shtml

4.6 :: variants of the second midterm

Variants of the 1st problem (2 points):

- Show that the vectors $(1,2,3),(3,4,5),(2,3,4)$ are linearly dependent.
- Show that the vectors $(-1,2,-3),(-3,4,-5),(-2,3,-4)$ are linearly dependent.
- Give an example of a matrix 4×5 of rank 3. This matrix should not contain zeros as entries.
- Give an example of a matrix 5×3 of rank 2. This matrix should not contain zeros as entries.
- Give an example of two 3×3 matrices A, B such that $AB\neq BA$, and at least six entries of A are zeroes.
- Give an example of two 3×3 matrices A, B such that $AB\neq BA$, and at least six entries of A are zeroes.
- What is the volume of the parallelepiped spanned by the vectors $(1,2,3),(2,3,1),(1$
- What is the volume of the parallelepiped spanned by the vectors $(1,2,3),(2,3,1),(1$

Variants of the 2nd problem (4 points):
For the sequence

- $G_0=1, G_1=2, G_{n+2}=3G_n+2G_{n+1}$
- $G_0=1, G_1=2, G_{n+2}=3G_n-2G_{n+1}$

- $G_0 = 2, G_1 = 1, G_{n+2} = 3G_n + 2G_{n+1}$
- $G_0 = 2, G_1 = 1, G_{n+2} = 3G_n - 2G_{n+1}$
- $G_0 = 1, G_1 = 2, G_{n+2} = 3G_{n+1} - 2G_n$
- $G_0 = 1, G_1 = 2, G_{n+2} = -2G_n - 3G_{n+1}$
- $G_0 = 2, G_1 = 1, G_{n+2} = 3G_{n+1} - 2G_n$
- $G_0 = 2, G_1 = 1, G_{n+2} = -2G_n - 3G_{n+1}$)

find the formula for G_n.

The 3rd and 4th problems were the same for all 8 variants.

3(3 points) For the same sequence write the matrix A such that

$$A \cdot \begin{pmatrix} G_n \\ G_{n-1} \end{pmatrix} = \begin{pmatrix} G_{n+1} \\ G_n \end{pmatrix},$$

and find the rank of A, determinant of A, eigenvalues λ_1, λ_2 and eigenvectors v_1, v_2 of A.

4 (1 point) find the coefficients of the vector $\begin{pmatrix} G_1 \\ G_0 \end{pmatrix}$ in the basis v_1, v_2.

4.7 :: variants for the exam

Variant 1

1. Give an example of a 3×3 matrix which contains no zeros, but has no inverse matrix.

2. Write 3×3 matrix with determinant 2.

3. Let X, Y be finite sets, $|X| > |Y|$. Is it possible to have a surjection $f : X \to Y$? An injection $f : X \to Y$? A bijection $f : X \to Y$? Why?

4. We draw an arrow on each side and diagonal of a polygon. Show that there exists a vertex P of this polygon, such that any other vertex is reachable from P if we allow moving according to arrows.

5. Find the values of α such that the system

$$\begin{cases} x + 2\alpha y &= 1 \\ 2\alpha x + y &= 1 \end{cases}$$

has a) no solutions, b) exactly one solution, c) infinite number of solutions.

6. For $F(x, y) = \cos(xy)(x + \ln y) + 2x^2y$ find the partial derivatives F_x, F_y.

7. Find a function $z(y)$ such that $z' = 9\sin 3y$.

8. Two persons are in a prison, incarcerated in the same room, so they communicate. The police propose them the following game: tomorrow they go to different rooms. To each of them it will be proposed a color: white or black. Each of them will must to guess the color of the other person. If at least one of them is right, they both will be freed. If both guess wrong, they will be immediately killed. Should they play this game and, if yes, what should be their strategy?

Variant 2

1. Give an example of a matrix 4×5 of rank 3. This matrix should not contain zeros as entries.

2. Find the volume of the parallelepiped spanned by the vectors $(1, 0, 1)$,$(1, 2, 3)$, and $(0, 1, -1)$.

3. How many four-digit numbers contain 8 but do not contain 5?

4. Let B be the adjacency matrix of a graph Γ. Prove that the entry (p, q) of B^k is the number of paths of length k between the vertices p and q.

5. Prove that if $Bw = -\lambda w, w \neq 0$, then $\det(B + \lambda E) = 0$ where E is the identity matrix.

6. Explain why g grows near p_0 if $g'(p_0) > 0$.

7. Draw the plot of the function $y(x) = \cos \frac{1}{x}$.

8. In a Banana Republic there was an election to the parliament where all people voted. Each person voting for the party "Orange" likes oranges. Among people voting for the others parties, 80% does not like oranges. What is the result of the party "Orange" on elections if exactly 36% of population likes oranges?

Variant 3

1. Give an example of a matrix 5×3 of rank 2. This matrix should not contain zeros as entries.

2. Find 2×2 matrices A, B such that

$$A \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot B = \begin{pmatrix} a-2c & b-2d+a-2c \\ c & d+c \end{pmatrix}$$

3. Draw the Euler-Venn diagrams for $(A \cap C) \setminus B$ and $(A \setminus B) \cap (A \setminus C)$.

4. Let us say that two four digit numbers are equivalent if they have at least three equal digits in the same positions (e.g. 1234 is equivalent to 1534.) Is it equivalence relation? Why?

5. Prove that for a $n \times n$ matrix B, if $\det B \neq 0$ then $By = a$ has only one solution $y \in \mathbb{R}^n$ for any given $a \in \mathbb{R}^n$.

6. Compute $\int_1^2 x dx$ using Darboux sums.

7. Draw the graph of $y(x) = \{1/x\}$ ($\{x\}$ is the fractional part of x).

8. A tram is called full if it has at least 70 persons. There are several trams on a road, some of them full. What is bigger, the percentage of full trams among trams or the percentage of people traveling in full trams among all people in all trams?

Variant 4

1. Give an example of two 3×3 matrices A, B such that $AB \neq BA$, and at least six entries of A are zeroes.

2. Find the 5×5 matrix B such that for each 5×5 matrix A, the matrix AB has five columns as A, but in a different order: the first column of A is the third column of AB, the second column of A is the second column of AB, the third column of A is the first column of AB, the fourth column of A is the fifth column of AB, the fifth column of A is the fourth column of AB.

3. Formulate the inclusion-exclusion formula for four sets and explain the idea of its proof.

4. Show that a connected graph with n vertices and without cycles has exactly $n-1$ edge.

5. Draw the unit ball for $\mathbb{R}^3$ for Chebyshev and Manhattan metric.

6. Solve (find a condition for p, q, r such that there exist solutions, and then find solutions in this case)

$$\begin{cases} \mu - \nu & = & p \\ \nu - \rho & = & q \\ -\mu + \rho & = & r \end{cases}$$

7. Compute $\int_1^2 x \cos(x) dx$.

8. Let 8 numbers be places on a circle, and each number is the average of its neighbors (so, for example, $x_3 = \frac{x_2+x_4}{2}$, $x_8 = \frac{x_7+x_1}{2}$, etc). Prove that all numbers are equal to each other.

References

Aleskerov, Fuad, Hasan Ersel, and Dmitri Piontkovski. *Linear algebra for economists*. Springer Science & Business Media, 2011.

Angrist, Joshua D and Jörn-Steffen Pischke. *Mostly harmless econometrics: An empiricist's companion*. Princeton university press, 2008.

Axelrod, Robert. *The complexity of cooperation*. 1997.

— "The dissemination of culture: A model with local convergence and global polarization". In: *Journal of conflict resolution* 41.2 (1997), pp. 203–226.

Backman, Olof and Christofer Edling. "Review Essay: Mathematics Matters: On the Absence of Mathematical Models in Quantitative Sociology". In: *Acta sociologica* 42.1 (1999), pp. 69–78.

Beltrami, Edward. *Mathematical models for society and biology*. Academic Press, 2013.

Bonacich, Phillip. "Power and centrality: A family of measures". In: *American journal of sociology* 92.5 (1987), pp. 1170–1182.

Bonacich, Phillip and Philip Lu. *Introduction to mathematical sociology*. Princeton University Press, 2012.

Bradley, Ian and Ronald L Meek. *Matrices and society: matrix algebra and its applications in the social sciences*. Vol. 501. Princeton University Press, 2014.

Breen, Richard and John H Goldthorpe. "Explaining educational differentials: Towards a formal rational action theory". In: *Rationality and society* 9.3 (1997), pp. 275–305.

Burt, Ronald S. *Structural holes: The social structure of competition*. Harvard university press, 2009.

Capecchi, Vittorio. "Mathematics and Sociology". In: *Applications of Mathematics in Models, Artificial Neural Networks and Arts*. Springer, 2010, pp. 1–78.

Coleman, James S. "Social capital in the creation of human capital". In: *American journal of sociology* 94 (1988), S95–S120.

Coleman, James Samuel et al. "Introduction to mathematical sociology." In: (1964).

Corfman, Kim P and Sunil Gupta. "Mathematical models of group choice and negotiations". In: *Handbooks in operations research and management science* 5 (1993), pp. 83–142.

Coxon, Anthony P.M. “Mathematical applications in sociology: measurement and relations”. In: *International Journal of Mathematical Education in Science and Technology* 1.2 (1970), pp. 159–174.

Edling, Christofer R. “Mathematics in sociology”. In: *Annual review of sociology* 28.1 (2002), pp. 197–220.

Edling, Christofer. *Interviews with mathematical sociologists*. 2007.

Elton, LRB. “Aims and Objectives in the Teaching of Mathematics to non-Mathematicians”. In: *International Journal of Mathamatical Education in Science and Technology* 2.1 (1971), pp. 75–81.

Everett, Martin G and Stephen P Borgatti. “Regular equivalence: General theory”. In: *Journal of mathematical sociology* 19.1 (1994), pp. 29–52.

Fararo, Thomas J. “Reflections on mathematical sociology”. In: *Sociological Forum*. Vol. 12. 1. Springer. 1997, pp. 73–102.

Fortunato, Santo. “Community detection in graphs”. In: *Physics reports* 486.3-5 (2010), pp. 75–174.

Fox, John. *A mathematical primer for social statistics*. 159. Sage, 2009.

Ghrist, Robert W. *Elementary applied topology*. Vol. 1. Createspace Seattle, 2014.

Hannan, Michael T and John Freeman. *Organizational ecology*. Harvard university press, 1993.

— “The population ecology of organizations”. In: *American journal of sociology* 82.5 (1977), pp. 929–964.

Hersh, Reuben and Ivar Ekeland. *What is mathematics, really?* Vol. 18. Oxford University Press Oxford, 1997.

Holme, Petter, Christofer R Edling, and Fredrik Liljeros. “Structure and time evolution of an Internet dating community”. In: *Social Networks* 26.2 (2004), pp. 155–174.

Kemeny, John G. *Mathematical models in the social sciences*. Tech. rep. 1972.

Kroneberg, Clemens and Andreas Wimmer. “Struggling over the boundaries of belonging: a formal model of nation building, ethnic closure, and populism”. In: *American Journal of Sociology* 118.1 (2012), pp. 176–230.

Lave, Charles A and James G March. *An introduction to models in the social sciences*. University Press of America, 1993.

Macy, Michael W and Robert Willer. “From factors to factors: computational sociology and agent-based modeling”. In: *Annual review of sociology* 28.1 (2002), pp. 143–166.

Polya, George. “Mathematical discovery: On understanding, learning, and teaching problem solving”. In: (1981).

Rapoport, Anatol. *Mathematical models in the social and behavioral sciences*. Wiley New York, 1983.

Rokhlin, V.A. "Teaching mathematics to non-mathematicians". In: *http://www.math.stonybrook.edu/~oleg/Rokhlin/LectLMO-eng.pdf* (1981).

Ruhnau, Britta. "Eigenvector-centrality—a node-centrality?" In: *Social networks* 22.4 (2000), pp. 357–365.

Schelling, Thomas C. "Dynamic models of segregation". In: *Journal of mathematical sociology* 1.2 (1971), pp. 143–186.

Strang, Gilbert. "Introduction to Linear Algebra. 4 edition." In: *Wellesley, MA: Wellesley Cambridge Press.* (2009).

Thurston, William P. "On proof and progress in mathematics". In: *For the learning of mathematics* 15.1 (1995), pp. 29–37.

Wasserman, Stanley and Katherine Faust. *Social network analysis: Methods and applications*. Vol. 8. Cambridge university press, 1994.

Weidlich, Wolfgang and Günter Haag. *Concepts and models of a quantitative sociology: the dynamics of interacting populations*. Vol. 14. Springer Science & Business Media, 2012.

Zarca, Bernard. *The Professional Ethos of Mathematicians*. 10.3917/rfs.525.0153. Paris, 2011. URL: https://www.cairn.info/revue-francaise-de-sociologie-1-2011-5-page-153.htm.