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and

their discoveries

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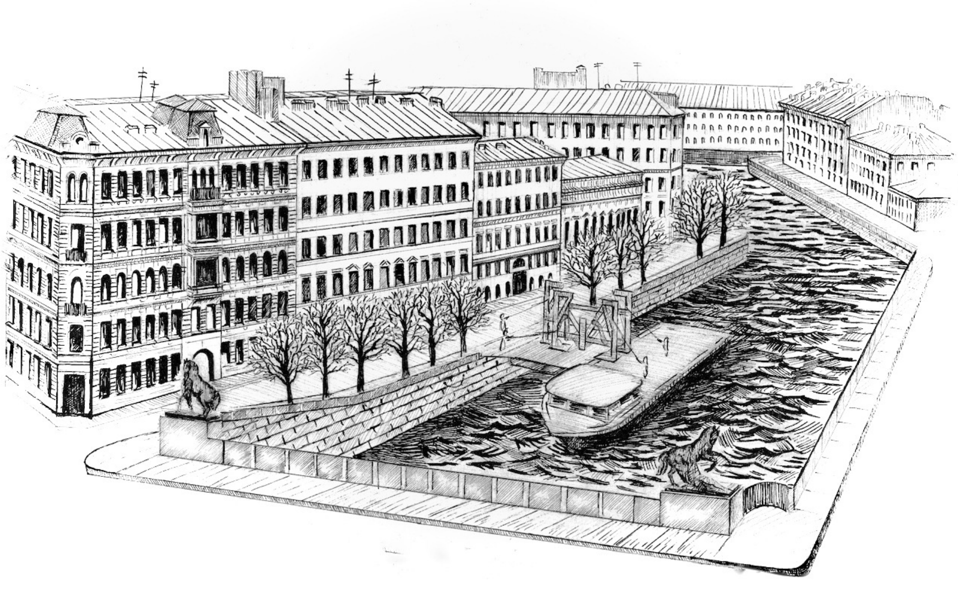

MCCME
MMXXIV

УДК 51(09)
ББК 22.1г
М34

М34 **Математики Санкт-Петербурга и их открытия.** Сборник статей (на англ. яз.). — М.: МЦНМО, 2024. — 496 с.: ил.

ISBN 978-5-4439-4703-7

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Translators: *Nodar Lakhuti, Lukeria Petrunina*
First page picture: *Elizaveta Plis*
Book cover, preparing illustrations for printing: *Alexander Ivanov*
Layout and typesetting: *Olga Shirokova*

The concept of the book: *Stanislav Smirnov*
Editor-in-chief: *Nikita Kalinin*

The most recent version of this book can be found on

https://sites.google.com/view/spbmath

Please, send corrections to mathspbthm@gmail.com

ISBN 978-5-4439-4703-7

“What is the most important thing a scientist should cultivate in himself? One should get rid of excessive ambition. One should not think that only a genius can be happy. One must learn to appreciate even a small achievement, to rejoice in it, and never overestimate oneself. One has to cultivate a love for work. One has to understand and cultivate the joy of learning, which is almost the same as the joy of life. Happiness is when your life’s work is needed.”

Sobolev, Sergei Lvovich

Contents

Foreword and acknowledgments

> Its use is not just that History may give everyone his due and that others may look forward to similar praise, but also that the art of discovery be promoted and its methods known through illustrious examples.[1]
>
> Gottfried Wilhelm Leibniz,
> *Historia et origo calculi differentialis.*

1. Mathematicians of the past, in general, experienced similar worries as we do. How to choose a research problem? Where to find a job? How to deal with administrative responsibilities? What to do in times of political instability, epidemics, famine, and war?

We take the reader through a book of answers to these questions, given by the mathematicians of the past — we learn what they did in a given situation, and how it turned out. Their biographies set a broader perspective; they allow us to see how the same aspirations play out differently in various historical periods.

This is the first way to read our book: the biographical notes, which make up half of the book, convey the life experiences of our colleagues from the past.

2. The reformist tsar Peter the Great wished to organize the education system in Russia in a European manner. Influenced (among others) by Leibniz, he created the Academy of Sciences and Arts in St. Petersburg. Peter the Great commissioned Blumentrost, his physician-in-ordinary (!), to find future collaborators in Europe.

Peter the Great passed away, but the Academy opened in 1725 under Empress Catherine I. To build science from scratch came several relatively young people, averaging about 30, including nineteen-year-old Euler, Russia's greatest mathematical fortune. Leonhard Euler found interesting mathematics in everything, be it cartography, blood circulation, or the study of the stability of ships. Among the early academicians were Christian Goldbach — who sparked Euler's interest in number theory and later worked as a cryptographer — and Daniel Bernoulli, who wrote in St. Petersburg his textbook on fluid

[1] This quote is translated from Latin by André Weil, in *History of Mathematics: Why and How*. Proceedings of ICM1978.

dynamics, one of the great achievements of eighteenth-century science, and popularized the Petersburg paradox.

Mathematical education in Russia received a strong impetus: schools and universities were established, textbooks were translated over the following decades. Euler's works were published for another 20 years after his death.

3. The next significant stage was marked by the mathematicians Bunyakovsky, Ostrogradsky, and Lamé. All of them were educated in Europe and made notable contributions to education in Russia. Like Euler, they did not shy away from any work: in addition to mathematics, Bunyakovsky worked in demography (for example, he used statistics to determine the size of the Russian army), Lamé calculated stresses for the domes and arches of St. Isaac's Cathedral, Ostrogradsky developed statistical methods for dealing with rejects in production and studied the theoretical issues of ballistics.

4. The next stroke of luck was Chebyshev, not much inferior to Euler in the breadth of his interests. Chebyshev frequently traveled to Western Europe to discuss mathematics, and then raised a whole galaxy of students so that there was someone to talk to in Russia.

> The older the school, the more valuable it is. For a school is a totality of centuries of accumulated creative techniques, traditions, and oral histories about scientists who have gone or are now living, their way of working, and their views on the subject of research. These oral traditions, which accumulate over the centuries and cannot be printed or communicated to those considered to be unsuitable — these oral traditions are a treasure whose effectiveness is difficult even to imagine and evaluate... Looking for any parallels or comparisons, the age of a school, its accumulation of traditions and legends, is nothing other than the energy of the school, in an implicit form.
>
> Academician N. N. Luzin

5. Then came the First World War, a time of upheaval. Near the end of the war, there came two revolutions. During a famine in St. Petersburg (then called Petrograd), the Sochocki family died of hunger and cold; Steklov and Krylov miraculously survived, while younger Vinogradov, Friedmann, and Besicovitch had moved to university in Perm, which kept changing hands during the civil war — but at least they were fed there!

Besicovitch and Tamarkin emigrated, fearing the new authorities, and the Steklov Institute, headed by Vinogradov, moved to Moscow. Many mathematicians died or left (some abroad, some to Moscow), but mathematical life in St. Petersburg (now Leningrad) did not stop: students matured — Linnik, Kantorovich, Alexandrov, Sobolev, Fock, Faddeev... and then the Second World War began.

After the war, Leningrad served as a place for people with "wrong" biographies (such as Rokhlin and Ladyzhenskaya), who had no place in Moscow. And once again, a large constellation of mathematicians emerged who have left their mark on world science.

6. In this book we write about mathematicians who lived or worked in St. Petersburg, including, for example, Georg Cantor, who was born in the city. An interesting coincidence: Cantor was baptized in the same Lutheran church where Euler was married. Chebyshev lived in the same house on Vasilevskiy Island where Cantor's childhood passed. Not far away, one finds Kovalevskaya's house. Behind the official biographies of scientists sometimes one does not see living personalities, while we, on the contrary, have tried to include in the texts as many personal stories as possible.

> The past not merely is not fugitive, it remains present. It is not within a few months only after the outbreak of a war that laws passed without haste can effectively influence its course, it is not within fifteen years only after a crime which has remained obscure that a magistrate can still find the vital evidence which will throw a light on it...
>
> Marcel Proust, *In Search of Lost Time*

7. Finally, the history of mathematics can also be looked at as a legacy composed of ideas. Mathematicians have long drawn inspiration from the classical works of their predecessors. Bunyakovsky engaged Chebyshev in the publication of Euler's works. In the process of editing, Chebyshev became interested in number theory and proved his famous estimates on the density of the distribution of prime numbers.

Besides biographical articles, the book includes mathematical notes — on Monge–Kantorovich's transportation problem, on the Vinogradov circle method, on Fedorov's crystallography, on Chebyshev's results in problems of cartography, and so on, 36 notes in all. Our authors have tried to show classical results in a new light whenever possible and appropriate. We hope that readers can gain inspiration and new insights from leafing through this book.

Nikita Kalinin

* * *

Is there something special and unique about St. Petersburg mathematicians and their mathematics? We certainly think so. Despite being a relatively young city, St. Petersburg, just over 300 years old, has established a prominent place for itself in the world of mathematics. It was therefore fitting that St. Petersburg was selected to host the ICM 2022, where we planned a grand celebration of mathematics, welcoming mathematicians from all around the globe. We wanted to give everyone a taste of local mathematical traditions, so we decided to prepare a short, coffee-table book to present some mathematical discoveries and personal anecdotes from around twenty St. Petersburg mathematicians. The goal was to make the content informative yet accessible, even for those who aren't particularly interested in the history or study of mathematics, showcasing the beauty of mathematical ideas and portraying their authors as relatable humans, rather than as cold-hearted calculators.

However, as often happens, life had other plans. The book was never published in its intended form, as the congress was moved online by the IMU EC. As we worked on what was initially meant to be a brief and light-hearted book, it began to evolve into something more comprehensive and profound, in a very different genre. While this new direction may appeal to a smaller audience, in our view it has become more interesting and informative. We still hope that this new format will attract many readers, not just mathematicians interested in their field's development, but even those well-versed in history who might find something new and worthwhile.

Should we be interested in the history of mathematical discoveries and the stories of the people behind them? We believe so, as these stories are not only engaging but can also teach us valuable lessons for our lives and studies today. Perhaps it's fitting that a new book in a new form emerged, especially as it coincides with the celebration of 300 years of science in St. Petersburg in 2024.

Stanislav Smirnov

Acknowledgements

First of all, we wish to thank all our authors for their patience and time.

For various help during this project we also thank Nikolai Andreev, Alexander Avdushenko, Fedor Bakharev, Dmitry Barinov, June Barrow-Green, Tatiana Blinkova, Sergey Demidov, Tatiana Dolzhenkova, Evgueni Doubtsov, Boris Frenkin, Dmitry Fuchs, Vitalii Gerbuz, Slava Goncharov, Christopher Hollings, Victor Kleptsyn, Alisa Knizel, Sergey Kryzhevich, Galina Kuteeva, Mikhail Lifshits, Marta Mazzocco, Anya Nordskova, Dmitry Pashchenko, Grigory Polotovsky, Mikhail Shifman, Evgeniy Rostovtsev, Reinhard Siegmund-Schultze, Kristina Staros, Alexander Tiskin, Anna Vasilyeva, Anatoly Vershik, Maxim Vsemirnov, Jean-Fred Warlin, Serge Yagunov, Ivan Yaschenko, Saint Petersburg University, and Saint Petersburg State Transport University, the archives of the Steklov Mathematical Institute in St. Petersburg and the library of the Steklov Mathematical Institute in Moscow.

Last but not least, we want to thank the MCCME publishing house for their volunteer help.

First years of mathematics in the Saint Petersburg Academy of Sciences

Christian Goldbach (1690–1764)

Christian Goldbach was a man of many talents: throughout his life he practiced law, music theory, he worked on deciphering letters of foreign diplomats while serving in the postal censorship offices and rose to the rank of Privy Councillor[1] under Empress Elizabeth. He studied the Riccati equation, series summation and convergence; he extensively corresponded with Euler, stimulating the latter's interest in number theory.

A descendant of an ancient Prussian family, he was born on 18 March 1690 in Königsberg (now Kaliningrad), the son of Pastor Bartholomeus Goldbach, a professor of history and rhetoric at the local university. While studying law at the University of Königsberg, Christian Goldbach spent much time learning mathematics, as the pages of his diary attest.

Upon completion of studies, the young man set off on a journey while looking for a way to fulfill his secret desire. He wrote to Teuerlain:[2]

> I had always dreamed of putting my energies into algebra and having an experienced teacher, who would introduce me to sources and stimulate me with new problems to move forward.

During long visits to European universities (1710–1714), he collected antiquities and books and visited libraries. Among those he met were Christian Wolff,[3] Gottfried Leibniz, Isaac Newton, the Bernoulli family, and Abraham de Moivre. Goldbach engaged in correspondence with some of them, discussing mathematical problems. Everywhere he was received with a warm welcome, probably due to his varied interests and his gift as a brilliant conversationalist.

He had a rare ability to instantly make acquaintances and then maintain connections via correspondence with people of all kinds. He defended his law thesis *On the Punishment of Kidnapping* in Groningen while staying there for two weeks.

When the Prussian king granted him the rank of *hofrat* (court counselor), Goldbach spent three years in Königsberg, corresponding with friends, before setting off again, this time traveling north, first to Sweden, performing unofficial diplomatic work along the way, then on to Denmark, Austria, and Venice.

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[1] A position roughly equivalent to vice-admiral or deputy minister.

[2] Teuerlein, David Andreas (1645–1728). A theologian and amateur mathematician.

[3] Christian von Wolff, (1679–1754). A German encyclopedist, actively involved in the search for candidates to form the Academy of Sciences in St. Petersburg.

From 1724, Goldbach's correspondence with the Nuremberg physicist Johann Gabriel Doppelmayr mentions the Academy being established in St. Petersburg and Johann Daniel Schumacher selecting candidates for it from among European scientists on behalf of the Russian tsar. Invitations to St. Petersburg had already been received by his friends (Jacob Hermann and Georg Bernhard Bilfinger, Daniel and Nicolaus Bernoulli), and Goldbach decided to travel there even without being invited. On his way from Riga, he wrote to Blumentrost,[4] with whom he was closely acquainted, and was not deterred by Blumenrost's reply that all positions at the Academy had been already taken. In August 1725, he was in St. Petersburg, ahead of Hermann and Bilfinger.

Goldbach's encyclopedic knowledge and conversational skills had been appreciated by the academic administration, and a position as Academy secretary and historiographer was quickly found for him. Goldbach also had time for mathematics since his secretarial duties, which included writing minutes of meetings, preparing academic work for publication, and keeping up with correspondence, hardly weighed heavily on him. True, his minutes were rather concise, but he was occasionally allowed to present his mathematical findings at the Academy Conference.[5] He also took part in the selection of candidates for positions at the Academy and established contacts with other international Academies.

At the same time, he was appointed Peter II's tutor in St. Petersburg at the request of Empress Catherine I. Goldbach even followed the Tsar's court to Moscow in January 1728 and lived there until 1732. While in Moscow, he took care of the academic affairs of the Academy in St. Petersburg, he wrote a preface to Volume I of *Commentaries of the Imperial Saint Petersburg Academy of Sciences*, an obituary to Nicolaus Bernoulli, and several mathematical papers.

Goldbach returned to St. Petersburg together with Anna Ioannovna's court[6] in January 1732, where he was appointed chairman of the Conference of the Academy of Sciences. He was in charge of all scientific affairs of the Academy for the next 18 months, before returning to his previous duties when a new president of the Academy was appointed.

In March 1742, early in Elizaveta Petrovna's[7] reign, Goldbach accepted an offer to join the Collegium of Foreign Affairs (located in Moscow). It is not entirely clear, however, what, as State Councillor (1742) and Privy Councillor (1760), Christian Goldbach was doing for 22 years as a diplomat.

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[4] Blumentrost, Laurentius, (1692–1755). Lieutenant-medic of Peter the Great, first president of the Academy of Sciences.

[5] The centerpiece of the Academy's scientific activity was its Conference, which met twice a week. The earliest minutes of the scientific meeting of the Conference date from 2 November 1725. That day the academician Jacob Hermann gave a report on the shape of the Earth.

[6] Anna Ioannovna (1693–1740) was the Empress of Russia from 1730 to 1740.

[7] Elizabeth of Russia, or Elizaveta Petrovna (1709–1762). Empress of Russia in 1741–1762.

At first, Goldbach often came to St. Petersburg on business, visited the Academy of Sciences during celebrations and public assemblies, delivered speeches and poems, and shone at high society receptions and at court. He did not return to St. Petersburg during the last ten years of his life (1754–1764). He corresponded regularly with Euler, often discussing mathematical problems. When Count Alexey Petrovich Bestuzhev-Ryumin joined the Collegium and established a service of censorship and perlustration of the encrypted correspondence of foreign diplomats, Christian Goldbach became the head of its deciphering service. Goldbach's successful decryption of ciphers and his development of new ones for secret correspondence were of enormous political importance to Russia.

Goldbach's finest hour came in 1744 when he decrypted a ciphered dispatch sent to Paris by the French ambassador, the Marquis de La Chétardie.[8] This became a textbook case in the history of cryptography [3]. Chétardie knew his letters were being opened but was convinced of the impossibility of decrypting his cipher. So he wrote light-heartedly about the Empress, saying she was wholly given over to pleasure, was frivolous, stupid, and vain. Bestuzhev-Ryumin,[9] who had become Chancellor by then, cleverly used this text in his fight against the "French" court party: by that time, he already had the deciphered texts of practically all the letters from the ambassador in his possession. He acted out the scene of deciphering the dispatch in front of Elizabeth "forcedly" uttering "reviling words."

On 6 June 1744, Chétardie was expelled from the country. On 26 July, Goldbach became an Active State Councillor.

An analysis of Christian Goldbach's scientific legacy, his correspondence with Nicolaus and Daniel Bernoulli, Leonhard Euler, Gottfried Leibniz, and Jacob Hermann, and an analysis of his works is available in [1]. By the time Goldbach arrived in St. Petersburg, several of his articles on arithmetic and algebra had been published in the German periodical *Acta Eruditorum.* During the Russian period of his life, six scientific articles were published in the first four volumes (1728–1733) of the newly established *Commentarii Academiae Scientiarum Imperiali.* These are papers on number theory, the integration of differential equations, and infinite series; in addition, a large number of Goldbach's results appear in his correspondence. In 1721, in correspondence with Nicolaus (II) Bernoulli, Goldbach presented some cases of the integrability of the Riccati equation

$$y' = cy^2 + bx^p y + ax^m.$$

[8] Jacques-Joachim Trotti, Marquis de La Chétardie, (1705–1759). French diplomat, French envoy to Russia in 1739–1742 and 1743–1744, helped Elizabeth, daughter of Peter I, arrange a coup d'état in 1741 and ascend the throne.

[9] Bestuzhev-Ryumin, Alexey Petrovich (1693–1766), a Russian statesman. At different times, he was an ambassador and chancellor, took part in many intrigues, and was twice sentenced to death. An advocate of Russia's rapprochement with England and Austria.

In 1729–30, in letters to Daniel Bernoulli, Goldbach gave the first example of a transcendental number,[10] and asked the question about the integrability in finite form of the differential binomial

$$\int x^m (a + bx^n)^p dx, \, m, n, p \in \mathbb{Q}.$$

After receiving Goldbach's letter, Bernoulli and Euler, living in the same apartment, even made a bet for one ducat (a 6–7 gramm gold coin) that Bernulli could not take this integral for $m = 0, p = -1/n$ in 15 days, but Bernoulli found it immediately. In 1730, Goldbach found all the cases m, n, p when this integral can be expressed in algebraic form.

Goldbach's correspondence with Leonhard Euler lasted for 35 years, and we owe the formulation of the famous Goldbach's problems (1742) to it: "Every odd integer greater than five can be represented as a sum of three prime numbers" (the ternary problem) and "Every even integer greater than two can be represented as the sum of two prime numbers" (the binary problem). For all sufficiently large odd numbers, the ternary problem was solved by Ivan Vinogradov (1937) and Yuri Linnik (1945), and completely by Harald Helfgott (2013), a mathematician of Peruvian origin. The Goldbach–Euler binary problem is still open.

Part of Goldbach's correspondence with Daniel and Nicolaus (II) Bernoulli (71 and 27 letters, respectively) and Euler (177 letters) was edited and published by Euler's great-grandson Paul Fuss in 1843 [4].

Christian Goldbach died on 20 November 1764. He was buried in the Lutheran section of the Sampsoniyevskoye cemetery in St. Petersburg.

Natalia Lokot

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[10] It was the number $\sum_{n=1}^{\infty} 10^{-(2^n-1)}$. Its transcendence was proved only in 1938, by Rodion Kuzmin [2], also a St. Petersburg mathematician. In 1930, Kuzmin proved the transcendence of the numbers a^b, where a is algebraic and b is a real quadratic irrational number. For the case of an arbitrary irrational algebraic b, this is Hilbert's seventh problem, which was solved in 1934 by Alexander Gelfond (Moscow) and Theodor Schneider (Frankfurt) independently.

The Chétardie cipher that Christian Goldbach cracked

From 1742, Goldbach worked in the "black office" (*Cabinet Noir*), where he deciphered letters of foreign ambassadors. This note discusses Goldbach's decipherment of the letter by the French ambassador, the Marquis de La Chétardie, of February 15, 1744. It was shown to Empress Elizabeth and led to the ambassador's expulsion. The structure of the cipher is described, and considerations are given as to how easy it is to crack.

Having arrived in Russia for the second time in 1743, the Marquis de La Chétardie counted on Empress Elizabeth's kind attitude towards him. He knew that his letters were intercepted by the Russians, but he was absolutely sure that his cipher could not be cracked, so he wrote openly everything he thought.

For example, in his letter of 15 February 1744, which we are going to discuss, he wrote:

> ...The Empress, by virtue of her laziness, laid everything on Bestuzhev...

Chétardie was using the so-called Great Cipher (Grand Chiffre), developed by Antoine Rossignol for the French court during the reign of Louis XIV, in the second half of the XVII century. This cipher uses three-digit numbers to encode words, letters, sounds, and syllables, with several numbers used interchangeably for the same frequently occurring syllables. Some numbers are meaningless and are added to make it harder to break the cipher.[1]

In 1742, Brevern tempted Christian Goldbach over from the Academy of Sciences to serve in the Collegium of Foreign Affairs, with a salary of 1500 rubles a year (approximately two or three professorship salaries nowadays). Bestuzhev–Ryumin later involved Goldbach in deciphering foreign correspondence. Goldbach did not crack the Chétardie cipher instantly, but by 1744, he had had two years' experience of cracking ciphers, and while the first few ciphers took him the whole first year to crack, this one he cracked within a fortnight.

[1] Note that it took the famous cryptanalyst Etienne Bazery, at the end of the XIXth century, several years to crack a similar cipher from the time of Louis XIV in an attempt to identify the man called "The Iron Mask."

The Russian Ministry of Foreign Affairs archive, where Goldbach's handwritten transcripts are kept, is closed to external visitors. However, it appears this was not always the case since Tatiana Soboleva, the author of the *History of Cryptography in Russia* (2002), quotes these transcripts abundantly.

Hence, I had to look into French archives.[2] Indeed, the letters from Chétardie are preserved in the *Archives du ministère des Affaires Étrangeres*, 2 rue Suzanne Masson, 93120 La Courneuve, France. On page 20 a reproduction of such a letter is presented.

The beginning of the letter, dated 15 February, looks like this:

335	632	679	498	283	249	202	97	996
752	786	983	95	155	900	591	179	23
478	987	742	597	36	659	933	894	126
527	97	99	813	865	780	898	958	432
507	302	514	694	611	510	661	56	414
506	406	359	95	358	712	562	715	900
219	51	498	111	823	880	466		

Since I had in possession several decrypted letters, I deduced the meaning of most of the numbers. In this excerpt, the first numbers 335 632 679 mean nothing and are added to complicate the task for an unwanted reader.

It is not easy to solve this cipher, even with a deciphered version in front of you. I thought at first that the words in one line corresponded to the numbers in the line below and tried to find matches. The assumption turned out to be wrong. The person who deciphered the letters wrote on top of the lines of the cipher, filling in the lines to the end, and the corresponding numbers might go a couple of lines below. Sometimes, the decipherment is done by meaning rather than *verbatim*; sometimes, you cannot make out what is meant (perhaps Chétardie confused a digit or two).

The matter progressed when, after several hours of studying the text, it struck me that the number 51 probably corresponds to the word *Tsarine*. After that, things went much faster — it became possible to match the numbers right after and before 51 with the corresponding text.

Note the old spelling — *logeoit* instead of the modern *logeait*, *seurement* instead of *sûrement*. Sometimes, Chétardie encoded a word exactly how it was spelled, sometimes as it was pronounced, and sometimes, he used abbreviations when the word was clear from the context.

For example, *deux* = 136 (d') 755 (eu), *faits* = 216 (fait) 816 (s), *ulcéré*= 339 (u) 574 (l') 438 (cé) 283 (ré).

[2] I am grateful to Jean-Fred Warlin for photographing the letters for me.

Let us read the letter of 15 February further. We will write one word per line (with an approximate translation):

498 (En) 283 (re) 249 (marq) 202 (ant)	let us note
97 (que)	that
996 (tout)	everything
752 (est)	is
786 (junk)	
983 (dans)	in
95 (la) 155 (même)	the same
900 (si) 591 (tu) 179 (ation)	situation/position

. . . .

359 (par) 95 (la) 358 (m) 712 (ort) 562 (de) 715 (Brevern)
Par la mort de Brevern — after the death of Brevern.[3]

The text on the illustration on page 20:

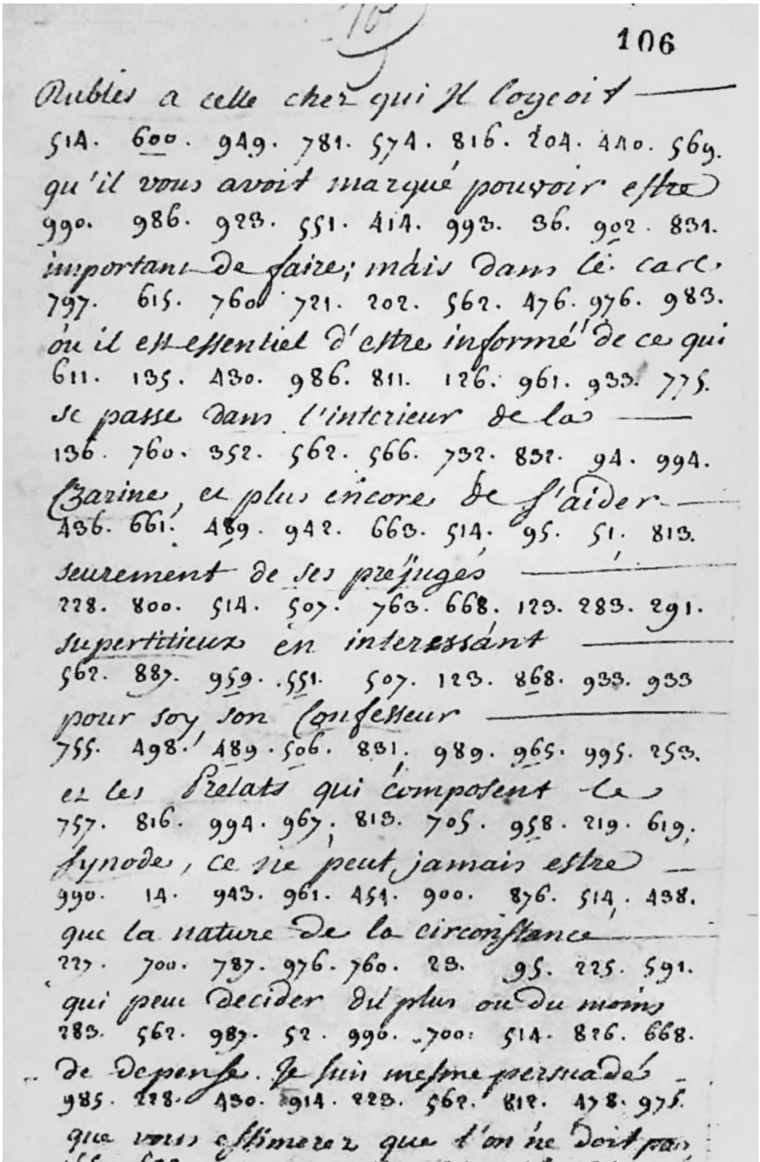
> [600] roubles à celle chez qui il logeait, qu'il vous avait marqué pouvoir être important de faire, mais, dans les cas où il est essentiel d'être informé de ce qui se passe dans l'intérieur de la Tsarine, et plus encore de s'aider sûrement de ses préjugés superstitieux en intéressant pour soi son confesseur et les prélats qui composent son synode, ce ne peut jamais être que la nature de la circonstance qui peut décider du plus ou du moins de dépense. Je suis même persuadé que vous estimerez qu'on ne doit pas...

It concerns Chétardie bribing someone who lives with Mikhail Golitsyn. To know more about what is going on in the Tsarina's entourage, it is necessary to bribe her clergyman and the [Most Holy] Synod to be able to take advantage of her superstitious nature. How much money will be needed cannot be predicted in advance [in modern terms, Chétardie asks for funding].

So, how could Goldbach crack such a cipher? Let him speak for himself (from a letter from Goldbach to Bestuzhev–Ryumin):

> My dear Sir!
>
> Having brought to Your Excellency the first fruits of the third key, I hope that, instead of reproaching me for any slowness in this, there will be more reason to wonder at my haste, should one ever please to compare the key itself with the letters unscrambled and when it would be clear that it was required to examine every number or every figure very diligently, so it would be possible to learn the content of just one letter. But since all this work has already been done, I am in a position to give one piece a day, provided that I am not prevented from doing so by other things. As for the fourth and fifth keys, of which I also have several items [letters] in my hands, I find the aforesaid keys incomparably more difficult than the first ones...

[3] Carl Hermann von Brevern, (1704–1744). A minister of Empress Elizabeth, the president of the Academy of Sciences in 1740–1741.



106

Rubles a celle chez qui Il logeoit
514. 600. 949. 781. 574. 816. 204. 440. 569.
qu'il vous avoit marqué pouvoir estre
990. 986. 923. 551. 414. 993. 36. 902. 831.
important de faire; mais dans le cas
797. 615. 760. 721. 202. 562. 476. 976. 983.
où il est essentiel d'estre informé de ce qui
611. 135. 430. 986. 811. 126. 961. 933. 775.
se passe dans l'interieur de la
136. 760. 352. 562. 566. 732. 832. 94. 994.
Czarine, et plus encore de s'aider
436. 661. 489. 942. 663. 514. 95. 51. 813.
seurement de ses prejuges
228. 800. 514. 507. 763. 668. 123. 283. 291.
superstitieux en interessant
562. 887. 959. 551. 507. 123. 868. 933. 933
pour soy son Confesseur
755. 498. 489. 506. 881. 989. 965. 995. 253.
et les Prelats qui composent le
757. 816. 994. 967. 813. 705. 958. 219. 619.
synode, ce ne peut jamais estre
990. 14. 943. 961. 451. 900. 876. 514. 438.
que la nature de la circonstance
227. 700. 787. 976. 760. 23. 95. 225. 591.
qui peut decider du plus ou du moins
283. 562. 987. 52. 996. 700. 514. 826. 668.
de depense. Je suis mesme persuadé
985. 228. 430. 914. 223. 562. 812. 478. 975.
que vous estimerez que l'on ne doit pas
155. 683. 97. 99. 124. 938. 746. 934. 97.

Once at least a few numbers are correctly solved (or the indicative content of the letter is known), further deciphering becomes so much easier. We can assume that Goldbach knew French, had an idea of the current political situation, and understood that numbers encode not separate characters but syllables or even words. We can also assume that Goldbach received all of

Chétardie's letters (the techniques for intercepting letters, opening and copying them were well established by then).

Even so, it was not at all clear how such a cipher could be broken: on his return to France, on September 27, 1744, Chétardie had his servant-secretary Dupré, who had access to the cipher, taken to the Bastille, suspecting treason on his part. The secretary was interrogated for five months before being found to be completely innocent and a naive simpleton, and released (Louis XV even awarded him compensation for his time in prison).

However, it is possible to make a few guesses even without access to the cipher. For example, a frequency analysis of the numbers shows that some numbers occur much more frequently than others. Here are some that occur more than 20 times in this letter:

95 la (35 times)
97 que (31 times)
204 a (33 times)
283 raî or r, ré (27 times)
451 le (24 times)
507 s' (21 times)
562 de or dé (39 times)
813 et (22 times)
989 pour (22 times).

So, while frequency analysis is not directly applicable, Goldbach could match these numbers to the most frequent French syllables, significantly reducing the enumeration of variants.

Furthermore, there are numbers in the letter, and they are not encrypted, e.g., *dépêches des 18 et 25 janvier* looks like 444 984 18 813 25 246. The letters did not always reach their addressees, so Chétardie resent some letters and mentioned the date of the letters previously sent. That is how we learn that 813 is *et* (and) and 246 is *janvier* (January).

Syllables in French words are rarely repeated, but in *superstitieux* 123(su) 868(pers) 933(ti) 933(ti) 55(eux), the syllable ti is repeated. Undoubtedly, Goldbach had a list of words with repeated syllables, as it was one of the well-known(!) methods of breaking ciphers. Sometimes, Chétardie would start or end a letter with a paragraph of unencrypted text, for example, saying that he has already sent this letter and that he duplicates it just in case. If he was not careful enough, such letters may not have been identical — which would have given information as to which numbers stood for the same thing. Another possibility is that Goldbach could have gotten hold of a draft letter (taken from a waste bin, perhaps), which could be compared to all the encrypted letters, and that would have given enough information to decrypt the cipher.

Thus, although breaking such a cipher seems to be a completely hopeless task at first, Goldbach did have ways in which he could make a start, for Chétardie, confident that the cipher could not be broken, was unlikely to be careful about the frequency of numbers, references to dates, or repeated syllables.

Regardless of the method of deciphering, at three o'clock in the morning on June 17, 1744, soldiers led by General Ushakov came to Chétardie. They ordered him to leave Moscow within a day and Russia within eight days and forcibly took from him Elizabeth's portrait, which had been given to him by the Empress herself.

In January 1744, a new contract with Goldbach was entered into, precisely due to his success in deciphering. The minutes of the reports to Elizabeth on 3 January 1744 state:

> ... 18. Her Imperial Majesty [Elizabeth] deigns to hear and most graciously approves the draft contract concluded by State Councilor Goldbach regarding his entry into Russian service. Following his humble report, it is proposed that he, Goldbach, be awarded up to 1000 roubles as a reward for his diligent work and special skill in deciphering secret letters, which Her Imperial Majesty has most graciously approved.

Nikita Kalinin

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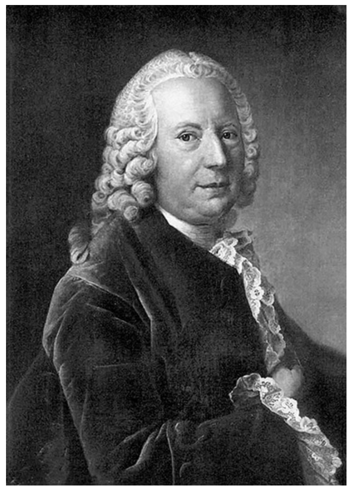
Daniel Bernoulli (1700–1782)

Daniel Bernoulli, one of the great scientists of the XVIIIth century, was born on the 29th of January, 1700, in Groningen, Holland, to Johann Bernoulli, a professor of mathematics at the local university, who later became one of the greatest mathematicians in Europe. Johann spent a lot of time teaching mathematics to his sons Daniel and Nicolaus (1695–1726) and instilled in them a lifelong love for the subject. In 1705, the family moved to Basel, where Johann was offered a post as a professor of mathematics at the university, which became vacant after the death of his elder brother, the famous mathematician Jacob Bernoulli. Daniel's childhood was peaceful, typical of the family of a successful scientist like his father.



Daniel graduated from the Basel Gymnasium at the age of thirteen, and three years later he was awarded a master's degree in philosophy. At his father's insistence, Daniel began to study medicine, as the practice of medicine has always ensured material well-being and a dignified position in society. It is worth noting that Daniel's father himself, on the advice of his elder brother Jacob, studied medicine, became a Doctor of Medicine, and alternated lecturing in mathematics at the University of Basel with medical practice.

In 1718, Daniel went to Heidelberg to continue his studies of medicine under the eminent physician Nebel and moved to Strasbourg the following year to improve his knowledge of anatomy and surgery. He returned to Basel in 1720, defended his dissertation *On Breathing*, and was awarded the degree of Licentiate of Medicine.

Daniel's serious studies of mathematics started in his native Basel in 1720–1723, but he did not abandon medicine. In 1723, Daniel went abroad again, this time to Venice, to complete his medical studies under the guidance of the famous Italian physician Michelotti. He successfully combined his medical studies with mathematics. In 1724, Bernoulli published his first mathematical work *Mathematical Exercises*. He got help from his friend, a noble Venetian, who printed several copies at his own expense. The work was devoted to defending the ideas of Daniel Bernoulli's father and uncle against the unfounded attacks of Italian scientists. This research made him famous in Italian scientific circles, and the Academy of Sciences in Bologna included him on its membership list.

From 1724, the Académie des Sciences de Paris began to run competitions for finding the best solutions to various scientific problems, awarding generous prizes to the winners. In this way, the Academy incentivized scientists to solve the most urgent problems while contributing to the development of science. The first competition was devoted to the subject "On the best way to construct an hourglass or water clock used on ships." Daniel Bernoulli sent his solution to Paris and was awarded a prize. Overall, between 1724 and 1757, he was awarded prizes ten times and was second only to Euler in the number of prizes he received.

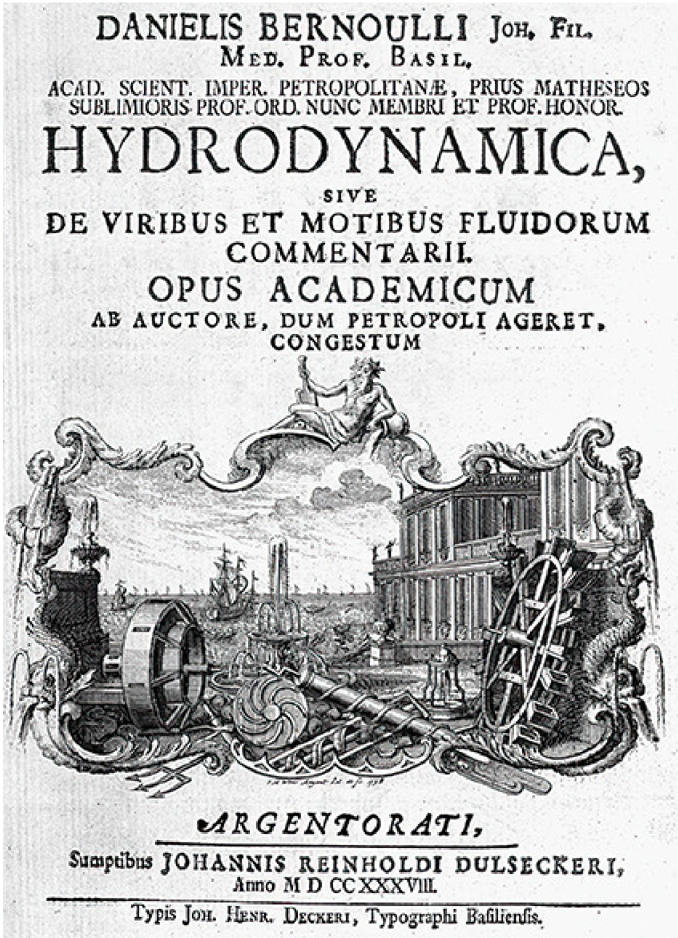
In 1725, the St. Petersburg Academy of Sciences was established, and Nicolaus and Daniel Bernoulli were invited to join. They arrived in St. Petersburg on the 27th of October 1725. Daniel got the position of chair of physiology with a salary of 800 rubles a year, and Nicolaus got the chair of mathematics with a salary of 1000 rubles a year. Unfortunately, Nicolaus, a brilliant mathematician, was not in good health and died seven months later, just after his wedding. Empress Catherine I personally expressed her condolences to Daniel on the death of his brother.

Daniel Bernoulli's activities in St. Petersburg were extraordinarily productive. During his first year at the Academy, he made more than ten presentations on solving problems in mathematics and mechanics, related to physiology. In 1729, he began to work on *Hydrodynamica* and carried out numerous experiments to test his hypotheses.

Unfortunately, the harsh St. Petersburg climate was affecting his health, and in 1733 he tendered his resignation to the authorities of the Academy. On the 24th of July of that year, he left St. Petersburg after eight years of service. He returned to Basel, where he took up the chair of anatomy and botany. In 1750, he was offered the chair of physics at the University of Basel and held it until the last years of his life. In 1734, Bernoulli published *Hydrodynamica* in Strasbourg, the work that brought him worldwide fame.

It should be noted that Daniel's association with the St. Petersburg Academy did not cease after his departure. In 1737, Daniel Bernoulli was elected an honorary member of the St. Petersburg Academy of Sciences with an annual pension of 200 rubles. Most of his papers were published in St. Petersburg: overall, between 1731 and 1775, 33 out of 41 papers saw the light of day there.

Bernoulli's main works relate to hydrodynamics, the kinetic theory of gases, and the theory of periodic motion. Bernoulli, along with D'Alembert and Euler, laid the foundations for the theory of partial differential equations.



DANIELIS BERNOULLI JOH. FIL.
MED. PROF. BASIL.
ACAD. SCIENT. IMPER. PETROPOLITANÆ, PRIUS MATHESEOS
SUBLIMIORIS PROF. ORD. NUNC MEMBRI ET PROF. HONOR.
HYDRODYNAMICA,
SIVE
DE VIRIBUS ET MOTIBUS FLUIDORUM
COMMENTARII.
OPUS ACADEMICUM
AB AUCTORE, DUM PETROPOLI AGERET,
CONGESTUM

ARGENTORATI,
Sumptibus JOHANNIS REINHOLDI DULSECKERI,
Anno M D CCXXXVIII.
Typis JOH. HENR. DECKERI, Typographi Basiliensis.

The cover page of the *Hydrodynamica.*

Probability theory also occupied an important place in his research. His most famous paper, *An Attempt at a New Theory for Calculating the Probability of Random Variables*, was published in St. Petersburg in 1738. In this paper, Bernoulli introduced the concept of "moral expectation." He applied this notion to a problem that was called the "St. Petersburg game."

The problem is put as follows: two players, Paul and Peter,[1] are playing. Peter tosses a coin; if it is *tails*, he pays Paul a ducat and the game ends; if it is *heads*, the game continues. If it's *tails* the second time, Peter pays Paul two ducats, and the game ends; if it's *heads* again, the game continues. The third time, it's either four ducats or the game continues, then either eight ducats or

[1] This might have been an allusion to the preeminent apostles Peter and Paul, who have an infinite time to play.

the game continues, and so on. For the game to start, Paul has to give Peter a certain amount of money in advance for the sake of fairness, which must equal the mathematical expectation of the winnings. In this case, the possible values of the winnings are 1, 2, 4, 8, etc. Thus, the mathematical expectation of Paul's winnings is $\frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 + \frac{1}{8} \cdot 4 + \ldots$, thus being infinite. However, the moral expectation of his winnings, according to Bernoulli's theory, will be finite. (Bernoulli's main argument is that the utility of money is multiplicative, not additive, and the value of one ducat for someone who has only ten of them is much greater than for someone who has a thousand). This sophistry has been called the "St. Petersburg paradox." An explanation of the paradox of the "Petersburg Game" was given by Alexander Khinchin in 1925, in his work *On the Petersburg Game.*

Bernoulli's scientific achievements were highly esteemed by his contemporaries. In his home country, he was elected Rector of Basel University twice. He was also elected a member of many foreign academies and scientific societies, including the Berlin Academy of Sciences in 1747, the Paris Academy of Sciences in 1748, and the Royal Society[2] in 1750. Daniel Bernoulli was among seven foreign scientists honored by Empress Catherine II with the gift of a personal copy of the gold medal minted in commemoration of the victory over the Ottoman Empire.

Daniel Bernoulli lived a long life; he never married. He coined the terms "hydrodynamics" and "steady state," which became universally accepted and are still in use today. In his later years, Bernoulli turned to charity. At his own expense, he built a small hotel for traveling students and scientists, where they could find food and shelter. He was a modest and well-balanced man, well-respected not only by his colleagues but by everyone who knew him.

On the 17th of March, 1782, Daniel Bernoulli died in Basel at the age of 82.

Larisa Konovalova

[2] The Royal Society of London for Improving Natural Knowledge.

Bernoulli's principle in hydrodynamics

Daniel Bernoulli is often regarded as the father of hydrodynamics. "It seems that the first person to use this term was M. Daniel Bernoulli, who gave this title to his *Treatise of fluid motions*, printed in Strasbourg in 1738 [6]. If the title was new, it must be admitted that the work was also new. M. Daniel Bernoulli seems to be the first who reduced the laws of fluid motion to safe and non-arbitrary principles, which none of the authors of hydraulics had done before him. The same author had already given in 1727, in the *Memoirs of the Academy of Petersburg* [5], an essay on his new theory." (D'Alembert, [12]).

As a mathematician and physicist, Bernoulli used analysis, particularly differential calculus, to formalize some fundamental laws of fluid mechanics and put them into equations.

1 A problem of vessels and drain

His memoir *Hydrodynamica* [6], dedicated to the Duke of Courland and Semigallia Biron, brings together various theoretical contributions and comments on the subject of fluid forces and movements, corroborated by numerous experiments.

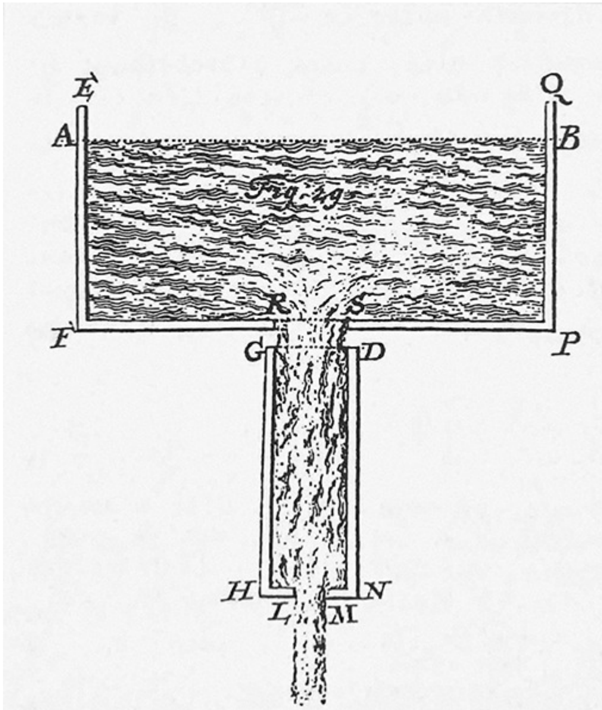
> These include fluids flowing through vessels with various shapes of openings or pipes, driving effect by rotation or translation of the vessels, and even an (unrealistic) application to navigation.

These include fluids flowing through vessels with various shapes of openings or pipes, driving effect by rotation or translation of the vessels, and even an (unrealistic) application to navigation.

1.1 A counter-intuitive result. The starting point of all his computations, now referred to as Bernoulli's principle, relates the velocity of a fluid to its pressure at a given point. It can be stated as follows

Bernoulli's principle. *In a horizontally flowing fluid, the pressure of the fluid at points where its velocity is high is lower than the pressure of the fluid at points where its velocity is low.*

Thus, in a horizontal pipe with sections of different diameters, the pressure of the water in the sections where the water is flowing fast is lower than in the sections where the water is flowing slowly. This may seem counter-intuitive,



as one would tend to associate high velocity with high pressure. However, this principle simply translates to the fact that water will accelerate if there is more pressure behind it than in front of it.

1.2 A counterpart of Huygens principle in mechanics. Bernoulli derived this principle by analogy with the living force principle of Huygens in mechanics [20]. In his own words, the proof is the following :

"The potential elevation of the system, each part of which is moved by any velocity, shows the vertical height which the center of gravity of this system reaches, if each particle, by the upward movement raised by its velocity, is understood to rise as far as it can.

The vertical height marks the actual descent by which the center of gravity descends after each particle has been at rest.

The potential elevation is necessarily equal to the actual descent when all motion remains in the spread matter." (Bernoulli, [6])

The potential elevation is therefore connected to the velocity, which is itself computed in terms of the flux: "The movement of fluids is very close to such that everywhere the velocity is reciprocally proportional to the corresponding size of the vessel" (Bernoulli, [6]). The actual descent is expressed in terms of the pressure using the laws of hydrostatics.

Bernoulli was then able to use these conservation laws on infinitesimal volumes and introduced geometric considerations to compute, for instance, the drain rate of a vessel.

2 Modern formulation

Bernoulli's original text is quite difficult to access today: the terminology no longer corresponds to current standards, the hypotheses are not always clearly formulated (except in the form of rules for applications), and the notations are very confusing.

The principle is nevertheless explained nowadays in a simple form in all fluid mechanics courses.

2.1 The pressure, a macroscopic observation of internal interactions. The pressure involved in Bernoulli's principle is the internal pressure of the fluid, which is exerted in all directions as the fluid flows. Note that this pressure is not the same as the pressure the fluid would exert on an object that would get in its way and stop its movement.

It is interesting to note that Bernoulli's memoir actually contains the first elements of the kinetic theory of gases: fluid particles have a state of thermal agitation that is all the more vivid the higher the pressure. The pressure is, in fact, the result of the many shocks between these fluid particles.

2.2 A global energy balance. It is the pressure of the neighboring portions of the fluid that is the source of the force providing the fluid with the work necessary for its acceleration.

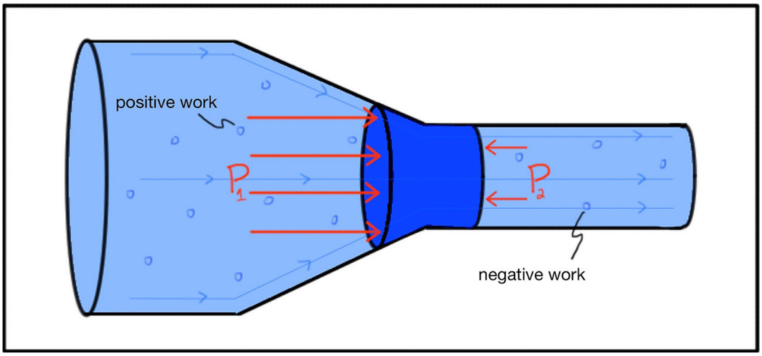
Consider a pipe in which water is flowing in a laminar way from left to right. When the volume of water, shown in dark blue in the figure below, reaches the narrow part of the pipe, its speed increases. The pressure force generated by the pressure on the left of the volume of water in question pushes it to the right and provides it with positive work since it pushes it in the direction of its movement. The pressing force generated by the pressure to the right of the volume of water under consideration pushes it to the left and provides it with negative work since it pushes it in the opposite direction to its displacement.

It is known that the water must accelerate (for conservation of the volume flow), and therefore the total work supplied by the pressure forces to the portion of fluid considered must be positive.

The work W_F of a force F (parallel to the motion) is written Fd, d being the distance traveled when the force is applied. The pressure force is given by PS, P being the pressure and S being the area to which the pressure is applied. Thus, we write :

$$W_{\text{pressure}} = PSd\,.$$

The area S_ℓ on the left is larger than the area S_r on the right. But when the volume of the water under consideration enters the narrow part of the pipe, it



One of the experements from Bernoulli's Hydrodynamica book.

deforms and lengthens: the pressure force due to P_ℓ on the right-hand side is therefore applied over a greater distance d_ℓ than the pressure force due to P_r on the left-hand side. The total volume of the section of water studied remains constant $S_\ell d_\ell = S_r d_r$. Therefore, the pressure P_ℓ of the water on the left must be greater than the pressure P_r of the water on the right for the sum of the work done on the volume of the water studied to be positive overall.

3 Euler equations versus Bernoulli equation

Bernoulli's principle can be made more quantitative by writing a detailed energy balance. This leads, in particular, to clarifying the assumptions on the flow.

3.1 An inviscid incompressible flow. Bernoulli's work mainly concerns water, which is considered incompressible. Section 8 of *Hydrodynamica* proposes actually an extension to the case of "elastic fluids," taking into account the effects of compressibility (which was a very original contribution), but we will not treat this case here.

Bernoulli also specifies in his manuscript that he neglects all the effects of viscosity, and in particular the adhesion to the boundary (which he calls "friction" or "tenacity" of the fluid).

For an incompressible inviscid fluid of density ρ, Euler's (later) work [15] shows that the dynamic equation governing the flow can be written as

$$\begin{cases} \underbrace{\nabla_x \cdot u = 0}_{\text{incompressibility constraint}} \\ \rho \cdot \underbrace{(d_t u + (u \cdot \nabla_x) u)}_{\text{acceleration along the flow}} = \underbrace{-\rho g e_z - \nabla_x P}_{\text{forces}} \end{cases} \tag{1}$$

where $x = (x_1, x_2, z)$ is the 3D vector of spatial coordinates, $u = u(t, x)$ is a 3D vector field representing the bulk velocity of the fluid at time t at a given position x, g is the gravity and $P = P(t, x)$ is the (scalar) pressure. From the mathematical point of view, P is just the Lagrange multiplier associated with the incompressibility constraint. Indeed, taking the divergence of the second equation, we get $\nabla_x \cdot ((u \cdot \nabla_x)u) = -\Delta_x P$, meaning that P depends in a non-local way on u and its derivatives and on the boundary conditions.

Note that the Eulerian description of the fluid with u and P is, in a way, the point of view of an observer who stands at a given position (as opposed to the Lagrangian description following the fluid particles).

3.2 The Bernoulli equation. In his work, Bernoulli also restricts his attention to a steady state flow, which implies that the Eulerian velocity field u does not depend on time $u = u(x)$. Recall that this does not mean that a given fluid particle has a constant velocity, but that at any given point in the domain occupied by the fluid, an observer always sees the same flow. The fluid particles, on the other hand, are advected by the flow: they move along the streamlines with a velocity that will be precisely given by Bernoulli's relation.

The last assumption, which is not formulated in a very precise and rigorous way, concerns the geometry of the flow. Bernoulli assumes for simplicity that the fluid is "divided into layers perpendicular to the direction of motion", and that "the particles of fluid [are] moved with the same velocity, so that everywhere the velocity of the fluid is reciprocally proportional to the corresponding magnitude of the vessel." This assumption can be relaxed by considering a small tube of fluid around a field line, but a necessary condition is that these lines remain approximately parallel, i.e. that the flow is laminar.

The definition of laminar flow is actually not completely consensual. We will limit ourselves here to the case of irrotational flows $\operatorname{rot} u = 0$. Since u is also divergence-free, the convection term can be rewritten

$$(u \cdot \nabla_x)u = \nabla_x \frac{|u|^2}{2} - u \wedge (\operatorname{rot} u) - u(\nabla_x \cdot u) = \nabla_x \frac{|u|^2}{2}.$$

which is also a potential term. We therefore end up with Bernoulli's equation

$$\nabla_x \left(\rho \frac{|u|^2}{2} + \rho g z + P \right) = 0\,.$$

For general laminar incompressible flows, the conservation holds only along the flow lines :

$$u \cdot (u \cdot \nabla_x)u = (u \cdot \nabla_x)\frac{|u|^2}{2},$$

so that, given two points l and r on the same flow line

$$\rho \frac{|u_\ell|^2}{2} + \rho g z_\ell + P_\ell = \rho \frac{|u_r|^2}{2} + \rho g z_r + P_r\,.$$

This implies, in particular, Bernoulli's principle (with $z_l = z_r$).

However, the assumption of a permanent laminar flow in the geometry considered by Bernoulli is questionable. It is briefly discussed in his dissertation, but the physical phenomena involved are extremely complicated and are actually still poorly understood.

4 Eddies and turbulence

This final section provides an overview of the complexity mentioned above.

4.1 Creation of vorticity. As shown previously, an irrotational flow satisfies the Bernoulli relation. On the other hand, it is easy to verify that an incompressible inviscid flow (with impermeability condition $u \cdot n = 0$ on the walls) does not create vorticity. Indeed, in two dimensions of space, the vorticity $\omega = \nabla_x^{\perp} \cdot u$ is simply transported

$$d_t\omega + (u \cdot \nabla_x)\omega = 0,$$

and therefore conserved. In three dimensions of space, the vorticity $\Omega = \operatorname{rot} u$ satisfies

$$d_t\Omega + (u \cdot \nabla_x)\Omega = \Omega \cdot \nabla_x u.$$

There is an additional "stretching" term, but if the vorticity is initially zero, it remains so (provided that the solution is smooth enough to make sense of the stability estimate).

Bernoulli indicated in his memoir that his model is locally flawed, especially near the opening and the boundaries of the vessel, due to viscosity (which is never exactly zero) and wall adhesion. The Euler equations (1) were generalized for viscous fluids by Navier [24] and Stokes [27]

$$\begin{cases} \nabla_x \cdot u = 0 \\ \underbrace{d_t u + (u \cdot \nabla_x)u}_{\text{acceleration}} = \underbrace{-g e_z - \nabla_x \dfrac{P}{\rho}}_{\text{forces}} + \underbrace{\nu \Delta_x u}_{\text{viscosity}} \end{cases} \tag{2}$$

where the viscosity term (with $\nu > 0$) accounts for the non-conservative internal forces (friction between neighboring fluid particles having different speeds). These equations do not seem at first sight to be very different from (1), especially when the Reynolds number (proportional to $1/\nu$) is very large. In particular, we have the transport-diffusion equation for vorticity Ω

$$d_t\Omega + (u \cdot \nabla_x)\Omega - \nu\Delta_x\Omega = \Omega \cdot \nabla_x u,$$

which should preserve the irrotationality of the flow.

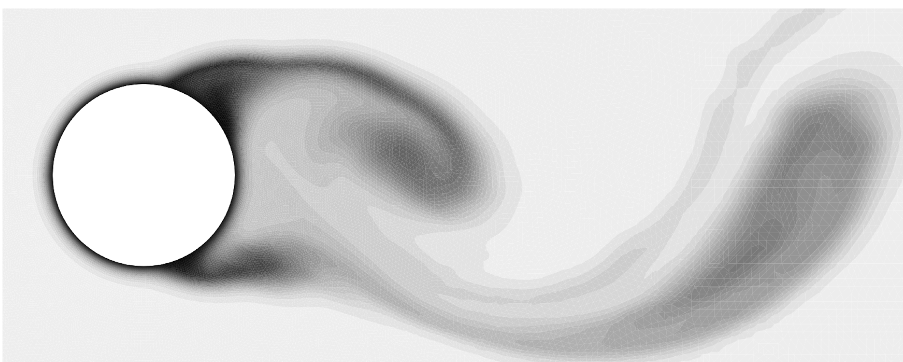
However, as the Navier–Stokes equations are of order 2 (the Laplacian involves second derivatives), one must impose more conditions on the boundaries, which allow in particular to model the effect of adhesion to the wall ($u = 0$ on the walls). This condition is, in general, different from the prediction given by Euler's equations for an inviscid fluid. It is therefore expected that even weakly viscous fluids ($\nu << 1$) behave very differently from ideal fluids ($\nu = 0$), especially in the vicinity of the walls. Observations show that small vortices are generated near the walls and can potentially destabilize the whole fluid.

4.2 Transition to turbulence. Prandtl was the first to study these boundary effects in more detail, introducing the notion of boundary layer and characterizing it by means of multi-scale expansions [26]. The main idea is that far from the boundary the viscous flow should resemble the idealized flow u_E governed by (1), which is then connected to the boundary condition by a fairly steep profile defined on a layer of size $\sqrt{\nu}$. To understand this connection, the idea is to zoom in on the direction orthogonal to the boundary at scale $\sqrt{\nu}$. For a stationary flow, in the simple geometry where the domain is 2D and the boundary is $x_2 = 0$, the equation for this profile $u = (v_1(x_1, x_2/\sqrt{\nu}), \sqrt{\nu} v_1(x_1, x_2/\sqrt{\nu}))$ reads

$$\begin{cases} \nabla_x \cdot v = 0 \\ (v \cdot \nabla_x) v_1 - d^2_{x_2 x_2} v_1 = -d_{x_1} p_E(x_1, 0) \\ v(x_1, 0) = 0, \quad \lim\limits_{x_2 \to \infty} v(x_1, x_2) = u_E(x_1, 0) \end{cases} \tag{3}$$

where (x_1, x_2) are the tangential and normal coordinates and u_E, p_E are the velocity and pressure obtained by the Euler equation (1).

Unfortunately, this equation is not always mathematically well posed. The stability of the boundary layer (and thus the validity of the two-scale expansion) is conditional on the fact that the gradient of pressure imposed by the internal flow $d_{x_1} p_E(x_1, 0)$ does not become positive and too large. Physically, in the case of flow around an obstacle, it is observed that the boundary layers that form upstream of the obstacle progressively lose their momentum and



A numerical simulation of a flow around an obstacle (credit : Bertrand Maury), so we see the mesh. The flow is laminar ahead of the obstacle (on the left), and becomes turbulent (with Von Karman alleys) past the obstacle. The grey level corresponds to the vorticity intensity.

cannot climb the adverse (positive) pressure gradient downstream, so they separate from the wall. In other words, the interaction with the boundary is expected to create vorticity, and this vorticity is then propagated within the fluid in the form of small vortices at all scales. This has two extremely important consequences. On the one hand, even if the viscosity is small, the presence of these small structures makes the flow quite strongly dissipative. On the other hand, the velocity field loses its regularity, and the Navier–Stokes equations (2) themselves can become unstable.

Understanding this transition mechanism to turbulence is a major challenge for physicists and mathematicians. Without making an exhaustive list of mathematical contributions on this subject, we can summarize the state of the art as follows

• the multi-scale expansion proposed by Prandtl is justified in some situations when the boundary layer is stable, for instance, with a negative gradient of pressure [25, 22, 17] (see also [8, 9, 18] and the references therein). The separation of the boundary layer has been described formally by Goldstein [19], and recently studied mathematically in [13].

• the instability of the flow in presence of small-scale structures is so strong that one even loses the uniqueness of solutions for the Euler equations [14] as well as for the Navier–Stokes equations [7, 1]. This means that, for flows of this complexity, it is very difficult (if not impossible) to follow precisely all the details of the velocity field.

• the enhanced dissipation and inviscid damping due to the combination of the small viscosity and the presence of small-scale structures are established for particular flows (typically shear flows) in the absence of boundaries (see the review [2] for a simple presentation of these mechanisms).

• in regimes when turbulence is fully developed, Kolmogorov proposed in 1941 a statistical approach [23], predicting, in particular, the distribution of eddies at different scales (see [16] for instance). Although completely heuristic, these predictions are consistent with the critical regularity for the Navier–Stokes equations (the threshold for the uniqueness [10, 21]).

However the prediction of the emission of vortices at the boundary layer (destabilization of the laminar flow), the probabilistic study of the Navier–Stokes equations to rigorously derive the Kolmogorov model (fully turbulent regime), or the mathematical description of the transition between these two regimes remain very challenging open problems. They probably require a very different approach from the usual point of view adopted in PDEs, where one could take advantage of the instability and mixing properties as is the case for example in the KAM theory of dynamical systems (see [3, 4] for results in this direction).

Acknowledgements

The author thanks L. Chevillard, A.-L. Dalibard, I. Gallagher, and B. Maury for their very useful comments on a preliminary version of this manuscript.

Laure Saint-Raymond

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Leonhard Euler (1707–1783)

The breadth of Euler's interests and the variety of his activities were not inferior to those of Leonardo da Vinci: besides being a mathematiciand and a physicist, he was an architect (he made calculations for the innovative Kulibin[1] bridge project), a geographer (Euler was in charge of the compilation of the first Atlas of the Russian Empire), he was involved in military science, navigation, astronomy, etc.

Euler obtained fundamental results and laid the foundations for many branches of mathematics: number theory (the Riemann zeta function and continuous fractions; he also introduced the concept of a primitive root of unity), analysis (elliptic functions, series summation), calculus of variations (the Euler–Lagrange equation), special functions, surface geometry (the definition of curvature by normal cuts, equations of geodesics, the concept of conformal mapping). His results in mechanics are no less significant; it would suffice to mention Euler angles and the hydrodynamic Euler equation.

Leonhard Euler, one of the greatest mathematicians ever, was born in Basel, Switzerland, on April 15th, 1707, into the family of a poor Protestant priest. Basel was a center of European education and culture at the time. The University of Basel, founded in the XVIth century, was a hotbed of enlightenment ideals. During the mid-XVIth century, the Bernoulli family moved there from the Netherlands, a fact that would play an important role in Euler's life.

Leonhard Euler was initially educated by his father, who, as a young man, had successfully studied mathematics under Jacob Bernoulli. The pastor was preparing his son for a career in the clergy but taught him mathematics as well. While in his last year of grammar school, Euler attended lectures in mathematics at the university, where one of the greatest mathematicians of

[1] Ivan Kulibin (1735–1818), famous Russian self-taught mechanic.

the time taught: Johann Bernoulli (1667–1748), the younger brother of Jacob Bernoulli. Johann Bernoulli noticed the talented youth and advised him to study the primary sources independently. He also allowed him to come to his house on Saturday afternoons to discuss any difficult parts encountered. During these sessions, Euler met his sons Nicolaus (1695–1726) and Daniel (1700–1782).

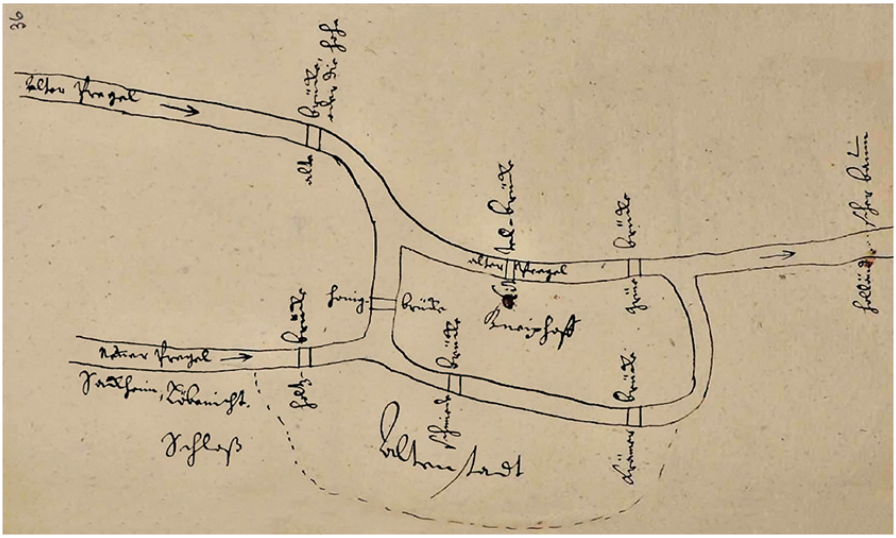
In 1723 Euler was awarded a Master of Arts degree. For the test, he gave a speech in Latin comparing the philosophies of Descartes and Newton. At his father's request, Euler studied theology. Both father and son realized that in Switzerland an academic career was unpromising: the number of applicants for professorships far exceeded the number of vacancies. In 1727, Euler applied for the position of chair of physics at the University of Basel, but without success.

Russia, St. Petersburg (1727–1741). Peter the Great wanted to establish an Academy of Sciences in Russia as early as the last years of the XVIIth century. He discussed his plans with the great Leibniz three times. Peter did not live to see the Academy established, because only in 1724 did the Senate decide to establish the Academy of Sciences, and it was inaugurated by Catherine I in 1725.

The St. Petersburg Academy of Sciences stood out from other European academies in several ways. Firstly, it had a fixed budget of 24 000 rubles per year and generous remuneration for its professors; by comparison, the country's total budget was about 8 million rubles. Secondly, the academy had a more universal character, consisting of three classes: mathematics, physics, and the humanities. Thirdly, the academy had a school and a university.

There were not enough Russian scientists, so the government invited foreigners. Invitations were sent to the sons of the famous Johann Bernoulli — Nicolaus and Daniel. Nicolaus became a professor of mathematics, and Daniel became a professor of physiology. Euler, seeing his friends departing to Russia, had, in his own words, "an indescribable desire to go with them to St. Petersburg." Daniel promised to put in a good word for him and fulfilled his promise: Euler was offered a position as an *adjunct* in the department of physiology, with a salary of 200 rubles a year. Leonhard was not embarrassed by the fact that he was to practice medicine. In those days medicine was not perceived as a science far removed from mathematics; for example, Johann Bernoulli alternated between mathematics and medical practice.

In 1727, twenty-year-old Euler, having received 130 rubles to cover travel expenses from the St. Petersburg Academy of Sciences, left for Russia. He became immediately involved in the intensive work of the Academy, and his papers published in the *Commentaries of the St. Petersburg Imperial Academy of Sciences* quickly earned him fame and a place of honor among mathematicians in Europe. In 1730, Euler was already a professor of physics, with a salary of 400 rubles. Two years later, Daniel Bernoulli left for Basel,



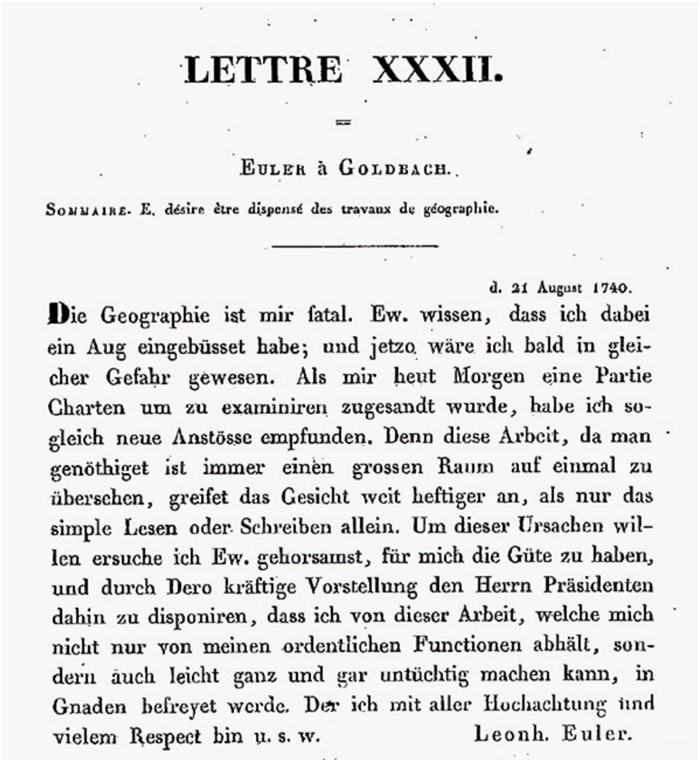
From the letter from Euler to Ehler, March 9th, 1736: find a walk through the city that would cross each bridge once and only once. The starting and ending points of the walk need not be the same. This is knows as “The problem of the 7 bridges of Königsberg”.

and Euler took the chair of mathematics with a salary of 600 rubles per year. Euler became one of the most important figures in the Academy of Sciences; he constantly presented papers, published articles, and did not refuse any assignments. He was involved in drawing up a general map of Russia and engaged in cartography, but mathematics was the main thing in his life. It was during these years that he emerged as a great mathematician. However, the intense work had a detrimental effect on his health, and at the age of 27, he went blind in his right eye.

On the 7th of January, 1733, Euler married Katharina Gsell, the daughter of an academic painter from Switzerland. They lived happily for over 40 years, his wife bearing him 13 children, but only three sons and two daughters survived.

Euler was a non-confrontational and deeply religious person. He was kind and could get on well with anyone. He worked under any circumstances and in any environment: “A baby on his lap, a cat on his back — that’s how he wrote his immortal works,” as Dieudonné Thiébault said.[2]

[2] “[Euler] a fait faire aux sciences mathématiques, des pas de géant; et ses immenses travaux ne lui coûtoient rien : c’est au milieu de sa famille, et du bruit que des enfans peuvent fàire; c’est en jouant lui-même avec celui qu’il prenoit sur ses genoux, et avec un angola monté sur son épaule, qu’il a composé quelques-uns de ces Mémoires que l’Europe a admirés et admirera toujours.” Dieudonné Thiébault, *Mes souvenirs de vingt ans de séjour à Berlin*, vol. 5, p.13 (Paris, 1804). *Angola* is a Turkish Angora cat.



LETTRE XXXII.

EULER à GOLDBACH.

SOMMAIRE. E. désire être dispensé des travaux de géographie.

d. 21 August 1740.

Die Geographie ist mir fatal. Ew. wissen, dass ich dabei ein Aug eingebüsset habe; und jetzo wäre ich bald in gleicher Gefahr gewesen. Als mir heut Morgen eine Partie Charten um zu examiniren zugesandt wurde, habe ich sogleich neue Anstösse empfunden. Denn diese Arbeit, da man genöthiget ist immer einen grossen Raum auf einmal zu übersehen, greifet das Gesicht weit heftiger an, als nur das simple Lesen oder Schreiben allein. Um dieser Ursachen willen ersuche ich Ew. gehorsamst, für mich die Güte zu haben, und durch Dero kräftige Vorstellung den Herrn Präsidenten dahin zu disponiren, dass ich von dieser Arbeit, welche mich nicht nur von meinen ordentlichen Functionen abhält, sondern auch leicht ganz und gar untüchtig machen kann, in Gnaden befreyet werde. Der ich mit aller Hochachtung und vielem Respect bin u. s. w. Leonh. Euler.

A letter to Goldbach, where Euler complains that his work on cartography cost him an eye.

By 1740, Euler's health had deteriorated due to constant overwork and the harsh climate in St. Petersburg. At that time, the Prussian King Frederick II intended to convert the Royal Society into an Academy of Sciences, and to this end he invited Euler to Berlin. Euler accepted the offer and on 29 May 1741 he resigned from the St. Petersburg Academy of Sciences. However, he retained his honorary membership in the St. Petersburg Academy with a pension of 200 rubles a year. In June of 1741, Leonhard Euler left St. Petersburg for Berlin.

Prussia, Berlin (1741–1766). Euler's Berlin period lasted for a quarter of a century, from 1741 to 1766. Euler's relationship with the Prussian King went wrong. In Berlin, it was thought that the duty of a scientist was, among other things, to decorate parlors and to entertain guests with elegant conversations. Euler did not do that; he engaged in mathematics. In 1744, the Berlin Academy of Sciences was inaugurated, but Euler was not offered the position of president of the Academy after Leibniz left but was appointed director of the mathematical department of the Academy.

Over the years in Berlin, he published 109 papers in the *Proceedings of the Petersburg Academy of Sciences* and 127 in the *Proceedings of the Berlin*

Academy of Sciences. He rendered numerous services to the Russian Academy of Sciences, took care of the replenishment of their libraries, and corresponded extensively with Russian academicians. It should be noted that throughout his life the great mathematician felt the deepest gratitude to the Russian Academy of Sciences, knowing that his move to St. Petersburg was of decisive importance for his life.

In a letter to Schumacher[3] on November 18th 1749 Euler wrote:

> I and all the others who were lucky to be at the Russian Imperial Academy for some time must admit that everything we have become is due to the favorable circumstances in which we found ourselves. As for myself, in the absence of such excellent circumstances, I should have been obliged mainly to turn to other occupations in which, by all accounts, the only thing I could indulge in was penny-pinching. When his Royal Majesty asked me recently where I had learned what I know, I answered, telling the truth, that I owed everything to my stay at the Academy of St. Petersburg.

Euler's financial situation was more than modest. He was constantly concerned about these financial difficulties, and in the early 1750s he set up a boarding school at his house for his Russian students.

Euler made every effort to ensure that his family had a comfortable life. In 1753 he managed to buy a beautiful small estate in Charlottenburg, with a house and a garden. In 1756, the Seven Years' War between Prussia and Russia began. Life was becoming more expensive, money was losing value, yet wages were not increasing. The advancing Russian army destroyed the estate in Charlottenburg, and the Russian officers whom Euler knew in St. Petersburg, including his godson, persuaded Euler that he, as an honorary member of the Academy, could demand compensation. Euler wrote letters to St. Petersburg, including some to Lomonosov,[4] but the matter dragged on. Only Catherine II, who ascended the throne after the coup of 1762, ordered that Euler be compensated for all the damages (1200 rubles, his annual salary in St. Petersburg) and asked him to return to St. Petersburg.

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[3] Schumacher, Johann Daniel, (1690–1761). The director of the Library of the Russian Academy of Sciences and later secretary of the Russian Academy of Sciences, responsible for dealing with all financial and economic matters. He was born in Alsace and came to Russia with Pierre Lefort, nephew of Peter the Great's admiral. In 1721, Peter the Great sent him to France, England, and the Netherlands, to persuade scholars recommended by Christian Wolff to come to Russia. He was married to the daughter of Peter the Great's cook.

[4] Lomonosov, Mikhail Vasilyevich, (1711–1765). The first Russian scientist, physicist, chemist, metallurgist, and creator of the kinetic molecular theory of heat. Lomonosov was of peasant stock, so he had to forge documents to enter the Slavic Greek Latin Academy in Moscow, the only higher education institution in Russia at the time. In 1735, he enrolled as a student at the University of the Academy of Sciences (he subsequently became its member in 1745). In 1736 he went to Marburg and was a pupil of Ch. Wolff. In 1740, on his way back to Russia, he was commandeered into the Prussian army but escaped. In 1755 he drew up a project for Moscow University, which was renamed Lomonosov University in 1940 in his honor.

Meanwhile, Euler's position in Berlin weighed him down. In 1759, Maupertuis, president of the Berlin Academy of Sciences, died. Note that his salary was double that of Euler. Euler hoped that Friedrich would offer him the post, but in vain. Euler's relations with Friedrich took a turn for the worse in the early 60s; the humiliating position he was put in was also pushing him towards a breakup: the great mathematician received a salary of 1,200 thalers *per annum*, or approximately 400 rubles.

In December 1765, Euler wrote to Chancellor Count Vorontsov requesting to be made a member of the St. Petersburg Academy of Sciences. Euler put forward the following demands: the post of vice-president with an annual salary of 3000 rubles for himself, provision of a flat free of billeting,[5] as well as a chair of physics for his eldest son with a salary of 1000 rubles per year, and provision of decent jobs in medicine and artillery for his middle and younger sons.

Vorontsov relayed Euler's conditions to Catherine II, to which the empress replied "Of course, I find him quite worthy of the position he desires ... At the present state of the Academy, there is no money for the salary of 3000 rubles, but for a man of such merit as Mr. Euler I shall add to the academic salaries from the state revenues, which together would amount to the requested 3000 rubles. He will have official quarters and not the slightest shadow of a soldier. Although the Academy does not have a free chair of physics with a salary of 1000 rubles for the eldest son, I will nevertheless assign it [the salary] to him, as well as an official private medical practice to the second son, and a guaranteed position should he wish to enter public service. The third son will have a position without any difficulty. I am sure that my Academy will rise from the ashes thanks to such an important acquisition, and I congratulate myself in advance on having returned the great man to Russia." In 1766, Euler was back in Russia.

Russia, St. Petersburg (1766–1783). Leonhard Euler returned to St. Petersburg on the 17th of July 1766. The next day, the empress received Euler and his two elder sons. Catherine II granted him 8000 rubles in silver to buy a house. Euler bought one on the Nikolayevskaya embankment (Lieutenant Shmidt Embankment, nowadays), where he lived till the end of his days. School No. 27 is housed in that building these days, and a commemorative plaque hangs on its walls.

Euler immediately became involved in the work of the St. Petersburg Academy of Sciences. He brought many manuscripts back with him that he had not had time to publish in Berlin. Euler, still full of new ideas to pursue, worked with the same zeal as before.

Euler was interested in the problem of blood movement through the arteries. The topic was important for military surgery. In January of 1737, he took part

[5] A billet is a place where soldiers are housed temporarily. Rural and urban residents were supposed to provide free accommodation for soldiers because the Russian army didn't have barracks during the whole of the 18th century, and being released from this duty as a resident was rare. Thus, an apartment free of billeting was quite a luxury.

in a discussion of Josias Weitbrecht's work, *On the Movement of Blood*, at the Academy. In 1742, Euler formulated and solved the problem of the flow of fluid in an elastic tube for the first time.

In a paper in 1775, Euler wrote about his investigations of the flow of blood through the arteries. Due to considerable mathematical difficulties, he was unable to solve the resulting system of equations and uttered the famous phrase: "If God had wanted us to understand the flow of blood through the arteries, he would not have invented such complex equations."

In 1774, Euler's wife died and was buried in the Smolensky Lutheran Cemetery. In order not to change his way of life, Euler married her half-sister Abigail Gsell. Euler's eyesight deteriorated dramatically. Catherine II summoned the famous oculist, Baron Wentzel, for Euler personally, and Wentzel successfully performed an operation to remove the cataract. Euler was ordered to rest his eyes for a few days after the procedure, but he could not restrain himself from continuing his work. As a result, Euler became permanently blind.

Blindness began to be coupled with deafness, but nothing stopped Euler from working: he started dictating his writings. During the 12 remaining years of his life, he dictated 10 large books and over 400 articles. He managed to write very clearly on a black table with white chalk. In the later years of his life, academic publications could not keep up with the flow of works authored by the blind scholar. Euler jokingly promised the director of the Academy, Count Orlov, that his works would fill the *Commentaries* of the Academy for 20 years following his death. Indeed, the Academy continued publishing his works for 47 years after his death, bringing the number to 771.

The 18th of September, 1783, started as an ordinary day in Euler's life: he was teaching mathematics to his grandson, doing calculations. At about 5 p.m., he felt the onset of an acute headache. Before he lost consciousness, he said: "I am dying." At 11 p.m. Euler was gone. According to Condorcet, "he ceased to live and calculate." The great mathematician was buried at the Smolensk cemetery next to the grave of his first wife. In 1956, his ashes were transferred to the Necropolis at the Alexander Nevsky Lavra.

Leonhard Euler is the most prolific mathematician in history. Five hundred thirty books and articles were published during his lifetime. In 1910, Eneström compiled a bibliography containing 886 titles. For over a hundred years, the Swiss Naturalists Society was publishing a collection of Euler's works: seventy-two volumes were published by the early 1980s, with eight additional volumes of Euler's scientific correspondences planned for publication. The very last of these volumes saw the light of day in 2023.[6]

Larisa Konovalova

[6] https://www.springer.com/series/4854.

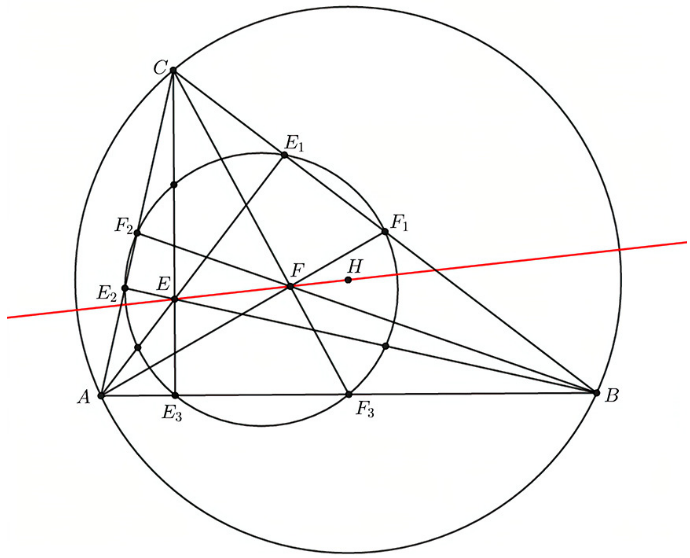
Leonhard Euler's triangles and elliptic curves

Leonhard Euler left us with an immense legacy of works in mathematics, physics, and many other subjects, and much of modern mathematics starts with an idea of Euler. Here, we will focus on his celebrated paper on elementary geometry *Easy solutions to some difficult geometric problems* [4], published in St. Petersburg in 1767, on Euler's return to Russia after a twenty-five-year stay in Berlin. This paper is the starting point for many theorems about triangles that illustrate the beauty of mathematics in a way that is accessible to high school students. But it also inspired, as we shall see, interesting developments in algebraic geometry, where other ideas tracing back to Euler also play a role.

Let us recall what the paper is about. Euler considers four centres of a triangle ABC: the orthocentre E (in Euler's notation) is the intersection of the altitudes; the centroid or centre of gravity F is the intersection of medians; the incentre G and the circumcentre H are the centers of the inscribed and of the circumscribed circle, respectively. He shows that E, F, H lie on a line, which is now called the Euler line, and that $EF : FH = 2$. He then shows how to reconstruct the triangle from any of the triangles EFG, EGH, FGH: out of the edge lengths of these triangles he constructs a cubic equation whose roots are the edge lengths of the original triangle. The property that the cubic equation must have three positive roots gives constraints on the distances between the centers, which he discusses in the case of the isosceles triangle. One of the consequences of Euler's calculations is the so-called *Euler relation* $R^2 - d^2 = 2rR$ involving the distance $d = GH$ between incentre and circumcentre and the radii r, R of the inscribed and of the circumscribed circle. From this relation the *Euler inequality* $R \geq 2r$ readily follows.

At this point we need to spoil the exposition with a necessary remark. Euler discusses various relations between distances in the triangle but does not explicitly consider the radius of the circumscribed circle. So, neither Euler's relation nor Euler's inequality appears in this paper. On the other hand, the inequality $R \geq 2r$ does appear in a 1746 article of the British mathematician William Chapple [2], published in a supplement to the Gentleman's Magazine, albeit with a questionable proof.

Certainly, the Euler line is in the paper, and it later turned out that other many further remarkable centers also lie on this line [14], in particular the center of the Feuerbach nine-point-circle. But here we would like to follow



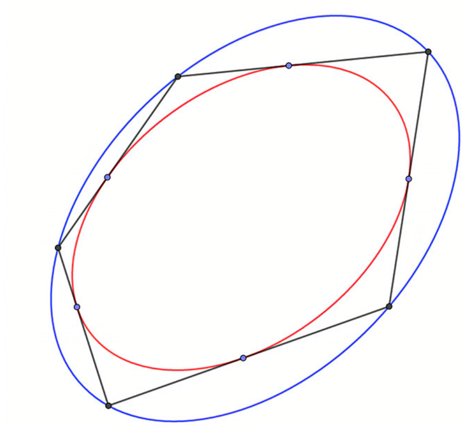
Euler's line.

another thread and for this we have to introduce another Swiss[1] émigré in St. Petersburg. Euler's sight had deteriorated since 1738 to reach complete blindness by 1772, so he invited the seventeen-year-old Nicolaus Fuss, who had studied with Daniel Bernoulli in Basel, to assist him in his work. Fuss participated in the publication of more than 200 papers that Euler wrote between 1772 and 1783, the year of his (Euler's) death. Fuss later married Euler's granddaughter Albertine Euler, became an esteemed academician and contributed in various capacities to developing science and education in Russia. We owe him a moving biographical obituary of Euler. In 1794 Fuss proves a theorem [7] generalizing Euler's relation to bicentric quadrilaterals: the distance d between the centers of the circumscribed and inscribed circles of a bicentric quadrilateral is related to their radii R, r by the formula

$$(R^2 - d^2)^2 = 2r^2(R^2 + d^2).$$

Incidentally, in the same volume of the Proceedings of the Academy there are still six articles by Euler, the last of which with the interesting title "*Is 1000009 a prime number or not?*" (it is not since, as Euler shows, it can be written as the sum of two squares in two different ways!). Fuss then tries to find a similar relation for polygons with more sides. In the 1798 the volume of the Proceedings (four articles of Euler here) he publishes the solution to this

[1] Since the author of this text suspects that one of the reasons for asking him to write about Euler is that he is Swiss, he will unashamedly bias the exposition injecting a couple of further Swiss characters into the story.



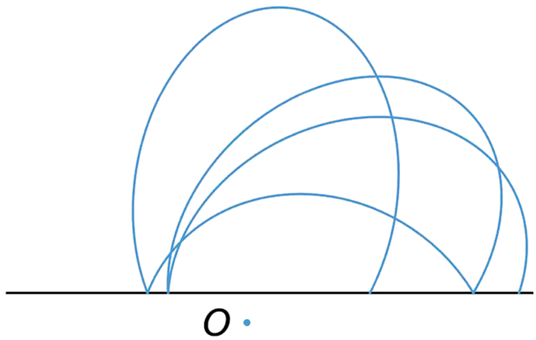
Poncelet's theorem. One conic is inscribed in a pentagon while the second conic is circumscribed about it.

problem for up to eight sides, but only in the special case where one vertex of the polygon is assumed to lie on the line connecting the centers of the two circles [8]. Jakob Steiner (another Swiss mathematician) poses the general problem in the second volume of Crelle's Journal in 1826 [16] and gives then a solution without proof for up to eight sides (but omitting the heptagon). An interesting coincidence is that the paper following [16] is Niels Abel's "Recherches sur les fonctions elliptiques" marking the beginning of the theory of elliptic curves, which turned out to be the key to solve the general case.

Carl Gustav Jacobi tells this story in 1828 [13] and comments that Fuss incurred in the "unfortunately not frequent" error of formulating a claim in a special case, while it actually holds in the general case. Indeed, in the meantime, in 1822, Poncelet's book [15] had appeared. In it, it is shown that if a pair of conic sections have the property that one is inscribed in a polygon and the other circumscribed about it, then there are infinitely many polygons with this property, and any point of the circumscribed conic can serve as a vertex. No Swiss connection here but a Russian one: Jean-Victor Poncelet participated in Napoleon's Russian campaign (which some of the readers will know as the Patriotic War of 1812) and was captured at the Battle of Krasny. He wrote his book as a prisoner of war in Saratov between 1812 and 1814.

Jacobi shows that Fuss's equations are equivalent to Steiner's and offers a general analytic solution for an arbitrary number of sides in terms of elliptic integrals. Later Arthur Cayley[2] [1] gave an algebraic solution, giving for each n an equation for the locus of pairs of conic sections admitting an n-sided polygon inscribed in one and circumscribed about the other. He uses the

[2] ... who lived the first 8 years of his live in Saint Petersburg.



The trajectory of a particle bouncing at a straight line.

theory of elliptic functions of Abel and Jacobi, but reminds us that his result is really a consequence of Euler's addition formulae for elliptic integrals [3, 5].

An explanation of these results in modern terminology was given more recently by Phillip Griffiths and Joseph Harris [12, 10, 11]. Given two generic conic sections, they observe that the variety of pairs consisting of a point of the first conic and a line through it which is tangent to the second conic is a complex elliptic curve (which is topologically a torus $S^1 \times S^1$) with two involutions: one replaces the point by the other intersection point of the line with the first conic and the other replaces the tangent with the other tangent through the same point. The composition of these involutions is the translation by an element of the elliptic curve and we have a polygon if and only if this element has finite order. Thus, Cayley's result can be understood as the description of the locus of points of finite order in a family of elliptic curves.

The author's interest in this story came from a recent result of Giovanni Gallavotti and Ian Jauslin [9] who reconsider a dynamical system introduced by Ludwig Boltzmann in 1868: a particle in a plane subject to an attractive central force with an inverse-square law bounces elastically at a straight line not going through the center. As shown by Gallavotti and Jauslin, the system has, in addition to the energy, a second independent conserved quantity. It turns out [6] that if for given values of the two conserved quantities one orbit is periodic, then all orbits with the same values of the conserved quantities are periodic. Again, an elliptic curve with two involutions provides the explanation of this phenomenon.

Giovanni Felder

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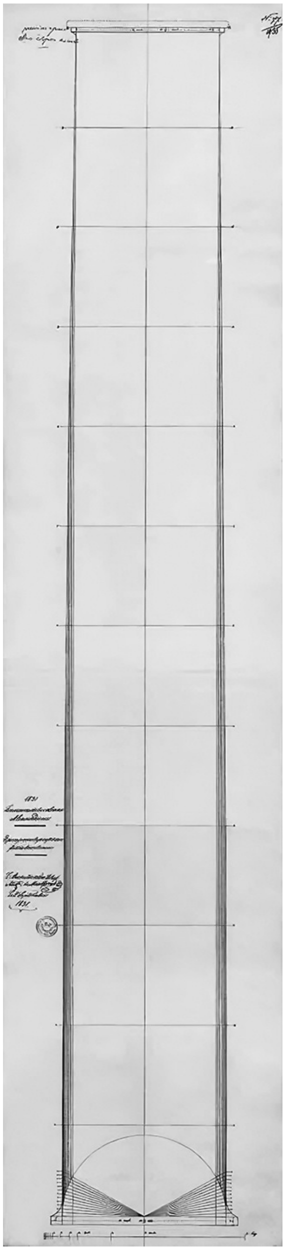
Gabriel Lamé (1795–1870)

The French scientist Gabriel Lamé was a mathematician, mechanic, engineer, corresponding member of the St. Petersburg Academy of Sciences, and member of the Paris Academy of Sciences. He lived and worked in St. Petersburg from 1820 to 1831. His scientific interests took shape there: they covered a wide range of areas in mathematics and physics, from the number theory to the theory of elasticity and mathematical physics, from purely theoretical studies (such as the proof of Fermat's theorem for $n = 7$) to very specific applications such as building suspension bridges. He developed the general theory of curvilinear coordinates. In cooperation with the French Embassy, St. Petersburg State University set up the Lamé Chair in St. Petersburg, where French scientists can visit for three months. Gabriel Lamé is one of the 72 scientists whose names are immortalized on the Eiffel Tower.

Gabriel Lamé was born in Tours in western France. He received his primary education at the Lycée Louis-le-Grand in Paris. At 16, he had to leave the Lycée and work as a petty clerk in a law firm. One day, he got his hands on a book by Legendre, *Elements of Geometry*, and was so fascinated that he returned to the Lyceum, successfully graduated, and entered the École Polytechnique. After that, he studied for another three years at the École Nationale Supérieure des Mines de Paris (also known as Mines ParisTech), where he met and befriended the soon-to-be-famous physicist Émile Clapeyron (Benoit Paul Émile Clapeyron), with whom he was inseparable for many years.

After the War of 1812, great efforts were made in Russia to rebuild engineering structures, buildings, and roads. There was a shortage of engineers, so teachers from the Institute of the Transport Engineers Corps, founded in 1809, were sent off to various locations to supervise these projects. The director of the institute, Agustin Betancourt (he was from Spain), had to bring in European specialists as teachers. In 1819, Pierre-Dominique Bazaine, a professor at the institute, went to France to invite two young promising engineers, and he chose Lamé and Clapeyron. "Tempted by the promise of many advantages and a high degree of freedom," ([1], p.241) they accepted the offer. Lamé and Clapeyron gave lectures in higher mathematics, physics, astronomy, mechanics and machine design, and applied chemistry. They also taught "A course in new discoveries and improvements in the arts." Transport engineer Andrei Delvig recalled ([3], p. 130) that Lamé "was a decent man, deeply learned, of pleasant appearance and graceful form, who lectured eloquently and was well versed in what he taught."



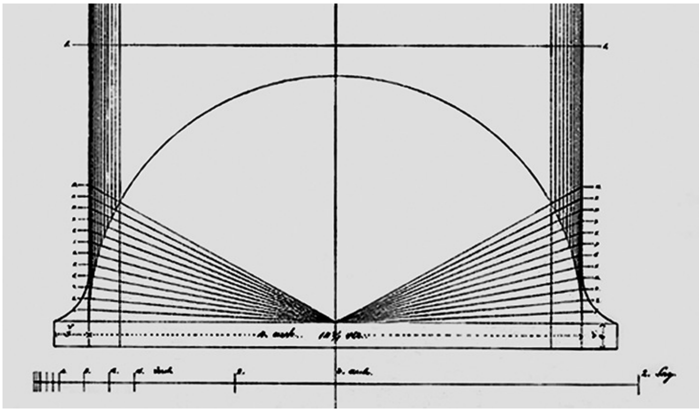
Lamé's scientific interests were related to the work carried out by the engineers of the institute: the construction of St. Isaac's Cathedral, the St. Petersburg-Moscow road, chain bridges, etc.

Lamé took part in the construction of the Alexander Column. In 1829, the design for a 47.5-meter-high triumphal column in honor of Russia's victory over Napoleon, submitted by Auguste de Montferrand, was approved. For it to be pleasant to look at, the shape of the column was to be made slightly barrel-shaped. The line of curvature of the column, i.e., the outline of the outer contour, was calculated by Lamé. He wrote:

> Taste, in accordance with the rules of art, requires that the column should gradually diminish in diameter as it rises in height and that this diminution should occur along a smooth curve.

The column is made of a solid granite monolith and is supported by its own weight. Transporting and lifting it was also an engineering challenge. Even today, admiring the beauty and grandeur of the Alexander Column, one may recall Gabriel Lamé and his "blessed choice of curvature," using Montferrand's words ([4], p. 14).

Lamé and Clapeyron developed the designs for suspension bridges over the Yauza River, the Moscow River, and the Luga River in Yamburg. They wrote several papers on suspension bridges and the construction of rope polygons. Lamé and Clapeyron are believed



The height of the column was divided into 12 equal parts, and the column was made thinner at the bottom according to the pattern presented in the picture. Sketch of Lamé, archives of the Institute of the Transport Engineers Corps. (Montferrand, Auguste Ricard. Folder 14, list 77. Epure pour le profil au fût de la colonne.

to have pioneered the ideas of graphostatics, introducing the concept of the rope polygon as a tool of investigation. Related to problems in the theory of elasticity, Lamé also worked on the theory of series, in particular the Fourier series. Under the influence of Ostrogradsky, Lamé took an interest in the problem of heat propagation. This is evidenced by his article on the laws of cooling and solidification of a liquid sphere, published in 1831 (Ann. Chim. et Phys., V. 47. P. 250–256). Modern researchers believe it to have been the first work in the theory of free boundary problems.

Lamé was also interested in railway construction. In 1830, he was sent to the inauguration of the Liverpool and Manchester railway to learn from the experience of its construction. He was awarded the 3rd class of the Order of Saint Stanislaus for his report on the trip.

After eleven years of a successful career in Russia, Lamé nevertheless decided to leave. The exact reason for his return is still unknown, but it may have been the July Revolution in France in 1830, which made the Russian government suspicious of the French. Lamé himself (a Saint-Simon supporter) was uncomfortable with the political situation that developed during the reign of Nicholas I in Russia. Lamé's letters also help us understand the reasons for his abrupt departure. As early as 1828, he wrote:

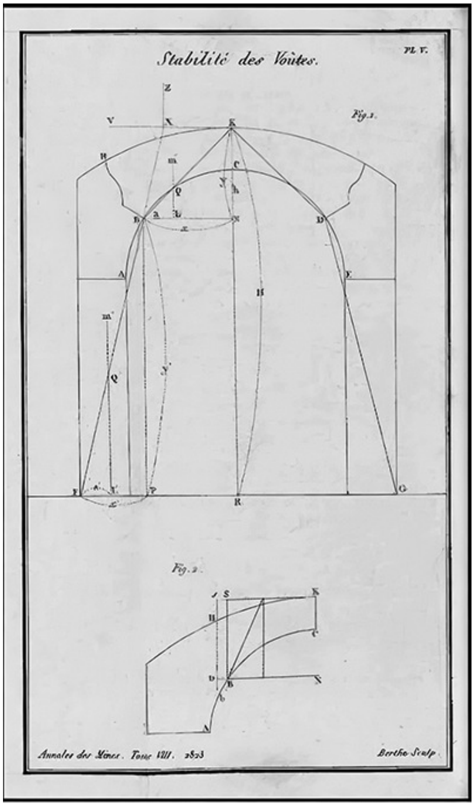
> French families are leaving Russia. Our social circle is shrinking drastically, and unless we fit in with Russian society, which does not suit us and which we are not suited to, our salon will turn into a desert. This prospect does not frighten me. I like solitude enough. But my wife, who does not have math to work hard on, and who cannot even spend time with her husband when he is working, is perfectly entitled to start complaining, and that worries me.

In 1831, Lamé, a professor, a major in the Russian service, and a corresponding member of the St. Petersburg Academy of Sciences, resigned (officially, due to illness) and returned to Paris. While working in Russia, Lamé published 18 papers on mathematics, structural mechanics, theoretical mechanics, and applied mechanics. His writings show that the main scientific fields to which Lamé devoted his later life, such as the theory of elasticity, the analytic theory of heat, and the theory of curvilinear coordinates, were outlined in St. Petersburg. It was in Russia that he obtained his first results in mathematical physics.

After his departure from Russia, Lamé remained engaged in railway construction for several years. Together with Clapeyron, he drew up the first prospective plan for the railway network in France, and after some time, they started building the Paris-Saint-Germain railroad.

Lamé soon abandoned his engineering career, however. In 1832, he began teaching physics at the Ecole Polytechnique, then lectured at the Sorbonne. In the following decades, he produced major works in mathematical physics that made him famous.

Lamé announced a proof of Fermat's Last Theorem at the same time as Augustin Louis Cauchy. Both had been publishing their supposed proofs in parts until Ernst Kummer pointed out an error they both had in common:



A figure from Lamé's work *On stability of vaults*, [5].

Lamé and Cauchy assumed the unique factorization property (factoriality) for some rings for which it did not actually hold.

In 1863, Lamé was forced to abandon his work due to hearing loss. His last years were difficult. Andrei Delvig recalled that he saw Lamé shortly before his death, quite old, deaf, and sickly, but Lamé "spoke with pleasure of his time in Russia and remembered it with gratitude." ([3], p. 130)

Margarita Voronina

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Mikhail Vasilyevich Ostrogradsky (1801–1862)

Mikhail Ostrogradsky[1] had a significant influence on the teaching of mathematics in the military and engineering schools of the Russian Empire. He solved the wave propagation problem in a cylindrical basin, obtained a formula for transforming the integral over a three-dimensional body into an integral over its surface, introduced the notion of a conjugate operator, and studied variational principles, including those in non-conservative mechanical systems.

He was born on September 24 (September 12 Old Style) 1801 on his father's estate in Poltava province and lived in the countryside with his parents until he was eight years old. His younger brother Andrei recollected that from childhood Mikhail liked to measure everything, from frogs to the depth of a well, and used to carry around a piece of string with him all the time for that purpose.

In 1809, Mikhail entered Poltava Gymnasium. At the same time, according to the custom of Russian nobility, Ostrogradsky was enlisted into public service in the Poltava governor's office. Later, Ostrogradsky was prepared to enter Kharkov University, though he dreamed of a military career, and in 1817 he became a university student. Under the influence of a keen mathematics teacher, Andrei Pavlovsky, he began to work diligently on the textbooks by himself and in 1818 he passed the examinations for the three-year course of study. Then, on his own petition, he left the university with a certificate to enter military service.

A year later, however, Ostrogradsky returned to the university "for improvement in the sciences pertaining to applied mathematics." He suffered in the conflict between the rector Timofei Osipovsky, mathematician and materialist, and the new trustee of the university Zachary Korneev, who came "to dispel the gloom of delusion, established by the conceit of reason, and to plant Christian virtues in the hearts." Ostrogradsky did not attend lectures on Divine Law.

[1] Spelling variants: Ostrogradskiy, Ostrogradskii, Ostrogradskii.

The teacher wrote a complaint to the trustee who forwarded it to the minister of education. As a result, Ostrogradsky was punished “for freethinking”: deprived not only of a candidate degree awarded to him in April 1821 but also of the diploma of 1818.

Soon, to get better at mathematics, Ostrogradsky asked his father to let him go to France. In a quiet province, such trips were considered very dangerous at the time, as Ostrogradsky’s biographers mention; relatives and neighbors thought the young man was being sent to his death and that his father, having given permission, had lost his mind. In autumn 1822, Ostrogradsky (on the second attempt, because the first time he was robbed by a fellow traveler) reached Paris where he attended several courses in mathematics with Pierre-Simon Laplace, Joseph Fourier, and Augustin-Louis Cauchy. There is little information about his stay in Paris. He did not keep diaries; his correspondence was sparse and uninteresting, mainly containing requests for money. In his first letter, he wrote that he had already toured the entire city, the city is lovely, but its beauties are exaggerated, the carriages are very bad and usually harnessed with two horses.

In November 1826, Ostrogradsky presented his first work, *Memoir on Wave Propagation in a Cylindrical Vessel*, to the Paris Academy of Sciences. It was printed in 1832. There are two versions of why he wrote this work. The first one is that Ostrogradsky wandered alone along the banks of the Seine and, being bored, watched the movement of waves which inspired him to study wave motion. According to the second one, in 1826, he did not get money from his father in time for some reason, and since he owed money for the hotel and food, on the complaint of his host, he was put in the debtor’s prison in Clichy, where he wrote the paper. Ostrogradsky sent it to Cauchy, who presented it to the Paris Academy of Sciences with a most flattering review, and that was why it was printed in France. Moreover, Cauchy himself bought Ostrogradsky out of prison.[2] In Paris, Mikhail Ostrogradsky became close with Viktor Bunyakovsky.

With time, his father stopped sending him money altogether, so Ostrogradsky had to return to Russia at the beginning of 1828, and on foot at that — he was robbed again on the way. The fact that he had been in France, where he could have breathed in the revolutionary spirit, most likely explains the secret police surveillance that he was under for some time on his return, which lasted until it became clear that he did not have any political agenda.

He submitted several papers to the Academy of Sciences in St. Petersburg. These works, along with his popularity in France, led to Mikhail Ostrogradsky being elected an adjunct of the Academy of Science on December 17, 1828.

[2] Cauchy, a strong supporter of Jesuits and member of the Society of St Vincent, did as he had been taught, i.e., he ransomed a pauper from prison because Ostrogradsky had committed no crimes, but he was poor. Cauchy helped Ostrogradsky obtain a place in the college of Henry IV, which allowed the young scholar to improve his financial situation.

By that time, three of his works on mathematical physics and mathematical analysis had been published in the Academy publications. In August 1830, Ostrogradsky was elected Extraordinary Academician and a year later Ordinary Academician in Applied Mathematics, and in 1855, after the death of Academician Pavel Fuss, he took up the chair of Ordinary Academician in Pure Mathematics. Mikhail Ostrogradsky took an active part in the life of the Academy: he wrote scientific papers, made reports, and reviewed the submitted works. However, most of his time was devoted to pedagogical work, to which he gave more than 30 years of his life.

Although Ostrogradsky remained in St. Petersburg, his influence was quite tangible throughout Russia. In 1830, he submitted a report to the Academy of Sciences in which he defined his goal: to promote applying theoretical knowledge to practical needs. This position was most clearly reflected in the higher engineering schools of St. Petersburg. Ostrogradsky taught at the Naval Cadet Corps (from 1828), the Institute of the Transport Engineers Corps (from 1830), the Main Pedagogical Institute (from 1832), the Main Engineering School (from 1840), and the Main Artillery School (from 1841).

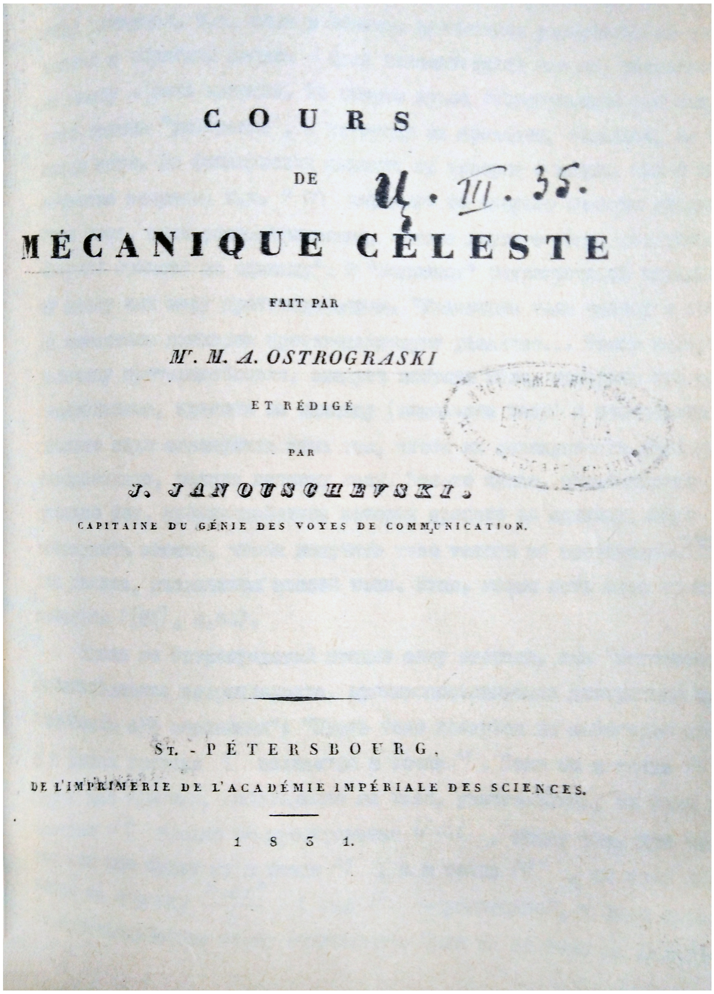
In 1829, Ostrogradsky gave his first course in mechanics in St. Petersburg. It was a series of public lectures on celestial mechanics in which he generalized the methods underlying Lagrange's analytical mechanics. The course lasted from November 1829 to March 1830. Despite the high cost of the course, 30 people attended the lectures. To put the number into context: the subscription cost one hundred roubles, while in the 1830s wheat flour cost up to 2 roubles per *pood* (16 kg), a cartload of hay — up to 8 roubles, a hundred eggs — 3.5 roubles.

The Institute of Transport Engineers bought five subscriptions to these lectures and made them available to the institute's assistants. One of them, Ignaty Yanushevsky, an institute graduate in 1828, recorded these lectures. In the report to the director of the institute on the necessity to publish the lectures, Yanushevsky wrote [1]:

> Ostrogradsky's course by the novelty and generality of ideas it contains would be respected not only in Russia, but also abroad, and most of all it would be useful for our officers, who, after leaving the institute, the best institution in terms of mathematics in Russia, often have neither time nor means to follow new discoveries.

These lectures by Ostrogradsky were lithographed in French, see the illustration on page 57. It could be said that they have reached out to us thanks to Yanushevsky.

Ostrogradsky gave various courses in mathematical sciences. At the Naval Cadet Corps (later the Naval Academy), he led mathematics and descriptive geometry classes. In addition to compulsory courses, Ostrogradsky read *Algebraic and Transcendental Analysis*. The lectures were public and attracted quite a lot of attendees — more than 60 people. The notes of these lectures



COURS

DE

MÉCANIQUE CÉLESTE

FAIT PAR

Mr. M. A. OSTROGRASKI

ET RÉDIGÉ

PAR

J. JANOUSCHEVSKI.

CAPITAINE DU GÉNIE DES VOYES DE COMMUNICATION.

ST.-PÉTERSBOURG.

DE L'IMPRIMERIE DE L'ACADÉMIE IMPÉRIALE DES SCIENCES.

1831.

The first page of the Ostrogradsky's lectures on Celestial mechanics, archives of The Institute of Transport Engineers.

were published in 1837. When he joined the Main Pedagogical Institute, Ostrogradsky emphasized that mathematics should be taught at least thrice weekly. The scholar compiled his own course and needed an assistant to repeat the material there. These requirements were met. Among Ostrogradsky's most prominent pupils from the Pedagogical Institute were Ivan Vyshnegradsky, Alexander Tikhomandritsky, Nikolai Budaev, Petr Roshchin, Egor Sabinin, Daniil Delarue, and others.

One of Ostrogradsky's favorite educational institutions was the Institute of Transport Engineers Corps. He was invited there in 1830 to teach mechanics, along with the famous mathematician Viktor Bunyakovsky. The letter from the chief manager of transport routes to the rector of the institute stated: "I order Your Excellency to invite two adjuncts of the St. Petersburg Academy of Sciences, Ostrogradsky and Bunyakovsky. The former will teach analytical mechanics and astronomy, and the latter will teach differential and integral calculus and synthetic statics" [2]. At the institute, mathematical education was always paramount. Thus, at the session of the conference (council) of the institute in 1834, the following resolution was adopted: "Higher mathematics should be ranked to the first category of sciences, i.e., to the sciences every engineer needs." The provision of 1843 states rather tellingly: "Those engineers, who do not master higher mathematics, should not be graduated as engineers, only as architects."

In 1859, at the meeting of the Institute's Council, it was noted that "the urge for analysis is learned by them (students) through the study of higher mathematical sciences" [3]. The point here is that engineers take a responsible approach to the assigned task, carefully analyze it, and only then make decisions. It was the students of this institute who recorded Ostrogradsky's lectures on celestial mechanics in 1831 and on analytical mechanics in 1836 and 1857. The latter, the third lithographed course by Academician Mikhail Ostrogradsky of 1857, is less well known. A copy of the course is kept only in the scientific and technical library of The Emperor Alexander I St. Petersburg State Transport University (formerly the Institute of the Transport Engineers Corps). The comparison of Ostrogradsky's lectures, published twenty years apart, allows us to trace the dynamics of the development of analytical mechanics.

Mikhail Ostrogradsky quickly gained exceptional authority as a scientist, lecturer, and educator. Valerian Panaev, a graduate of the institute in 1844 and a famous engineer, left interesting memoirs about the institute, its professors and scientists. He wrote: "Every pupil was eagerly looking forward to the happiness and achieving the great honor of listening to lectures of Ostrogradsky... The clarity and brevity of his expositions were amazing. He did not torture the listener with calculations but constantly kept him in an alert state regarding the essence of the question" [4]. Ostrogradsky stimulated listeners' work in mathematics and mechanics by his mere presence, the effect of which could be seen in a large number of papers and translations produced by students of Ostrogradsky in the 1830s–1850s "at the behest of the soul" (i.e., of their own volition) kept in the archives [5].

Transport engineer Alexander Durnovo wrote [6]:

> Hardly any other professor has been the subject of as many stories, anecdotes, and legends as Ostrogradsky. And he was really original and peculiar... M.V.'s presentation of his subject was excellent — precise, clear, and even artistic. He never repeated himself, and if he had to speak sometimes about something he had already explained before, he always did it in a new way, with new arguments and new

> methods of making conclusions. Often, he got carried away, and always in one direction — into the field of military history. Fascinatingly, with great enthusiasm, he explained to his listeners the exploits of the great generals, making chalk drawings ... and proving the strong correlation between a commander's talent and his mathematical mindset.

By decree of Nicholas I, an Educational Committee was established in November 1848 to determine priorities in technical education and to oversee "its progress in the educational institutions under the Ministry of Transport" [7]. At the time, these educational institutions included the Institute of Transport Engineers Corps, the Civil Engineering College, and various schools. In December 1848, Mikhail Ostrogradsky was elected a member of the committee and was appointed the primary supervisor for mathematical science teaching in the military schools of St. Petersburg and the educational institutions under the Ministry of Transport. His responsibilities included considering new programs and teaching methods, new guidelines for all academic subjects, reviewing books and textbooks purchased for the schools, and selecting the teaching staff. Given that Ostrogradsky taught in five St. Petersburg schools and was associated with Moscow University and the Moscow Technical College (now the Bauman Moscow State Technical University), his influence on teaching mathematics can hardly be overestimated.

Indeed, Ostrogradsky's ideas spread throughout the country. For example, his students from the Main Pedagogical Institute, where "professor Ostrogradsky, using his own notes, taught higher algebra, differential, integral calculus, and calculus of variations, analytic geometry, and the theory of mechanics with applications" [8], received appointments at the institutions of the Ministry of Public Education in St. Petersburg, Warsaw, Caucasus, Derpt (now Tartu in Estonia), Belarus, Kiev, and Kazan educational districts as well as in various gymnasiums and colleges in Siberia and the Urals. His pupils from other educational institutions also moved all over the country.

To give a more complete picture of Mikhail Ostrogradsky's multifaceted activities, one might point out his work on various expert committees. For example, in 1835, he was a member of committees on the consideration of projects for the water supply of St. Petersburg "using passage pipes," on the research of firing "regulated grenades," and on review of the experiments of academician Boris Jacobi (Moritz Hermann von Jacobi) for the "application of electromagnetic force to movement of ships" (and other means of transport).

Ostrogradsky's scientific works pertain to analytical mechanics, fluid mechanics, the theory of elasticity, celestial mechanics, mathematical analysis, and differential equations. In analytical mechanics, his results are related to developing the principle of virtual displacements, variational principles of mechanics, and other problems. The theory of heat propagation in liquids was in fact first constructed by Ostrogradsky. A number of formulas and methods are named after him. Mikhail Ostrogradsky was also concerned with the theory of magnetism as well as with problems of the calculus of variations, integration of algebraic functions, number theory, algebra, geometry, and probability theory.

He was friends with the great Ukrainian poet Taras Shevchenko, and there is evidence that the latter lived in Ostrogradsky's flat after his return from exile. Mikhail loved the Ukrainian culture and often used Ukrainian words in everyday speech and lectures. Not wishing to puzzle out inarticulately written works of Lobachevsky (Gauss compared them to a thick forest, through which one cannot find the way without examining every tree first), Ostrogradsky gave a negative review of them. After his mother's death, Mikhail Ostrogradsky resumed participating in church services.

In 1834, Mikhail Ostrogradsky was elected a foreign member of the American Academy of Arts and Sciences, of the Turin Academy in 1841, of the Accademia Nazionale dei Lincei in Rome in 1853, and of the Paris Academy of Sciences as a corresponding foreign member in 1856.

Mikhail Ostrogradsky died on 1st January 1862 (20th December 1861 Old Style) of a tumor in his back; he was buried in his ancestral manor. Aleksei Krylov wrote:

> The sturdiness of Mikhail Vasilievich's body could be envied by Taras Bulba himself, for even the then septic surgery and semi-shamanic medicine needed four months to bring him to the grave.

Margarita Voronina

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Gauss–Green–Ostrogradsky divergence theorem

The divergence theorem may be classically formulated as follows: *Let A be a bounded connected open set with (piecewise) smooth boundary ∂A in $\mathbb{R}^n$, $n \geq 1$, and $f = (f_1, f_2, \ldots, f_n)$ a smooth vector field in an open set containing the closure of A. We have*

$$\int_A \operatorname{div} f \, d\mathcal{H}^n = -\int_{\partial A} f \cdot \nu \, d\mathcal{H}^{n-1}, \tag{1}$$

where ν is the interior unit normal to ∂A, $\mathcal{H}^k$ is the $k-$dimensional Hausdorff measure and $\operatorname{div} f := \sum_{i=1}^n D_i f_i$, $D_i := \dfrac{\partial}{\partial x_i}$. Recall that, for any positive real number k, $\mathcal{H}^k$ allows us to measure any set in $\mathbb{R}^n$ and, for integral k's, it agrees with Lebesgue's $k-$dimensional measure on smooth $k-$dimensional surfaces in $\mathbb{R}^n$.

In 1-dimension, equation (1) simply reads as $\int_a^b f \, dx = f(b) - f(a)$; in any dimension it is equivalent, $i = 1, \ldots, n$, to

$$\int_A D_i f \, d\mathcal{H}^n = -\int_{\partial A} f \nu_i \, d\mathcal{H}^{n-1}. \tag{2}$$

In particular, (1) trivially holds if A is a product of n bounded intervals. Finally, from (1) yields $\int_A D_i f \, dx = 0$ for any f with compact support in A, hence the following *duality formula*

$$\int_A D_i f \, \phi \, d\mathcal{H}^n = -\int_A f D_i \phi \, d\mathcal{H}^n \tag{3}$$

whenever f, ϕ are smooth functions in A and ϕ has compact support in A.

The divergence theorem goes also under the names of Gauss, Ostrogradsky, Green and combinations such as Gauss–Green's or Gauss–Ostrogradsky's theorem. This is not surprising since special cases may be traced back to the beginning or early periods of multidimensional calculus, and credits are often misplaced. A reference for the history of the divergence theorem is [5].

In the 18th century both Lagrange and Laplace used the fundamental theorem of calculus and iteration to reduce domain integrals into boundary integrals. Special cases of (1) occur in the papers of Gauss in 1813, 1833, and 1839. In 1826 Mikhail Ostrogradsky presented a paper to the Paris Academy of Sciences where he formulated and proved the divergence theorem in dimension

3; similarly he did in a paper presented to the Paris Academy in 1827 and to the St. Petersburg Academy in 1828. The last presentation was the only one published in 1831; the other two survived only in manuscript form.

The divergence theorem also appears in a paper of Poisson presented in 1828 and published in 1829 without reference to Ostrogradsky, though there seems to be evidence that Poisson had seen Ostrogradsky's paper. In the same year 1828, Green published privately a paper, that remained unnoticed for quite some time, where he used formulas of the type

$$\int_A (u\,\Delta v + Du\cdot Dv)\,dV = \int_{\partial A} u\,\frac{\partial v}{\partial \nu}\,dS$$

in 3 dimensions, formulas nowadays named *Green's identities*; instead, there is no explicit mention of what we call *Green's theorem*:

$$\int_C (L\,dx + M\,dy) = \pm\int_D \Big(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\Big)\,dxdy$$

where C is a closed curve in $\mathbb{R}^2$ and D the enclosed domain. Such a formula appears, instead, without proof in a paper of Cauchy, 1846, and later with proof in the inaugural dissertation of Riemann, 1851, in relation to *Cauchy's theorem* for functions of complex variables.

Finally, it is worth mentioning the connection between Ostrogradsky–Gauss–Green formulas and *Stokes' formula*

$$\int_S (\mathrm{curl}\,\sigma)\,dA = \pm\int_\Gamma (\sigma\cdot\tau)\,ds$$

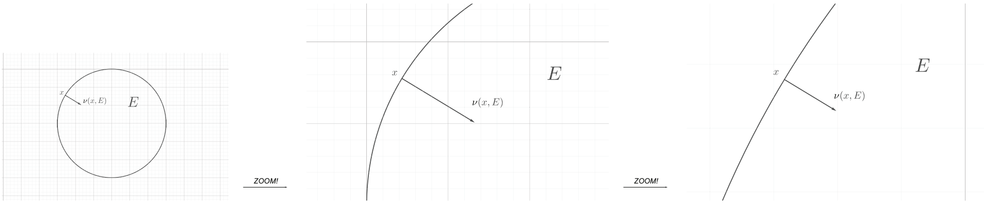
where ds is the element of length of the boundary curve Γ of a surface S in $\mathbb{R}^3$ and τ is the unit tangent vector to Γ, the sign depending on orientation. A long process — involving many mathematicians, in particular Ostrogradsky, Hankel, Volterra, Poincaré, and Elie Cartan — leads to the *Stokes theorem* for differential forms: *For smooth oriented $(k+1)-$surfaces S and $k-$differential forms ω we have:*

$$\int_S d\omega = \int_{\partial S}\omega. \tag{4}$$

Special instances of this are: Stokes's theorem when ω is a 1-form in 3-space, Green's theorem when ω is a 1-form in 2-space and the divergence theorem when ω is a 2-form in 3-space.

The previous theorems are both technically and conceptually relevant in mathematical analysis, geometry, and physics. Of the enormous pertinent literature, I will illustrate here only some aspects of the theory of *sets of finite perimeter* that De Giorgi developed in a series of papers starting in early 1950, see [2]. As general references for proofs and more, I refer the reader to [1], [3], and [4].

Let E be a Lebesgue-measurable set in $\mathbb{R}^n$, $n\geq 1$. According to De Giorgi, E is a *set of finite perimeter* or a *Caccioppoli set* if the distributional derivatives



Points of the reduced boundary of a Caccioppoli set and the associated normal are identified by a measure-theoretic blow-up procedure.

of its *characteristic function* (or *indicatrix*), χ_E, are signed Radon measures μ_i. This means, according to the duality formula (3), that $\int \chi_E D_i\psi dx = -\int \psi d\mu_i$, for every smooth ψ with compact support, the $\mu_i's$ being signed Radon measures; equivalently, according to Riesz theorem, that the linear map $\lambda(\phi) := \int \chi_E \mathrm{div}\,\phi\, d\mathcal{H}^n$ is equibounded for $|\phi| \leq 1$. Writing $D_i\chi_E$ for μ_i and $|D\chi_E|$ for the*total variation* of the vector-valued measure $D\chi_E$, that is the measure that at every Borel set B takes the value

$$\sup\left\{\int_B \mathrm{div}\,\phi \,\middle|\, \phi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n),\, |\phi(x)| \leq 1\, a.e.\right\},$$

the *perimeter* of E is then defined as the total variation $|D\chi_E|$ evalued at $\mathbb{R}^n$

$$P(E) := |D\chi_E|(\mathbb{R}^n).$$

From now on, I shall denote by $\mathcal{P}$ the class of Caccioppoli sets. From (1) one easily infers that bounded sets A with smooth boundaries of finite $\mathcal{H}^{n-1}$ measure are in $\mathcal{P}$, and $P(A) = \mathcal{H}^{n-1}(\partial A)$ holds. For a generic $E \in \mathcal{P}$ we know from the theory of differentiation of measures that at $|D\chi_E|$-a.e. x there exists the Radon–Nikodym derivative of $D\chi_E$ with respect to $|D\chi_E|$

$$\frac{dD\chi_E}{d|D\chi_E|} = \lim_{r\to 0^+} \frac{D\chi_E(B(x,r))}{|D\chi_E(B(x,r))|} =: \nu(x,E). \tag{5}$$

Therefore, for any $\phi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$ we can write

$$\int_E \mathrm{div}\,\phi\, d\mathcal{H}^n = -\int \phi\cdot\nu(x,E)\,|D\chi_E| \tag{6}$$

and even take the second integral over ∂E, since the *support* of $|D\chi_E|$ lies in ∂E. Formula (6) sounds like the extension of the divergence theorem to Caccioppoli sets. But it is not precisely so, because there are sets for which $P(E) < \infty$ but $\mathcal{H}^{n-1}(\partial E) = \infty$ (see Fig. 2), and we are missing a relation between boundary and $\nu(x,E)$. All that is specified by the celebrated *rectifiability theorem* of De Giorgi that follows. Introduce the *reduced boundary of* E, $\partial^- E$, defined as the set of points at which $\nu(x,E)$ exists and $|\nu(x,E)| = 1$; we have: *$\partial^- E$ is $(n-1)-$rectifiable, that is, apart from a $\mathcal{H}^{n-1}$-zero set, $\partial^- E$ is* *the union of Borel sets of* $(n-1)-$*dimensional manifolds with tangent planes normal to* $\nu(x, E)$ *for* $\mathcal{H}^{n-1}-a.e.\, x \in \partial^- E,$ *moreover*

$$D\chi_E = \nu(x, E)\mathcal{H}^{n-1}|_{\partial^- E},$$

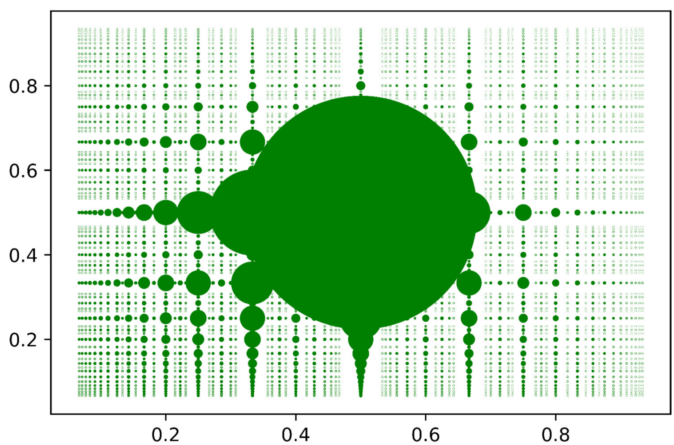
and the divergence theorem holds in the form:

$$\int_E \operatorname{div} \phi \, d\mathcal{H}^n = -\int_{\partial^- E} \phi \cdot \nu(x, E) \, d\mathcal{H}^{n-1}.$$

Furthermore, we have $\partial^- E \subset \partial_\mu E$ where $\partial_\mu E$ is the *measure-theoretic boundary* of E, consisting of the points which are neither of density one (rarefaction points) for E nor of density zero for E (rarefaction points for the complement of E), and, moreover, $\mathcal{H}^{n-1}(\partial_\mu E \setminus \partial^- E) = 0$. Finally, a theorem of Federer states that E is a Caccioppoli set if and only if $\mathcal{H}^{n-1}(\partial_\mu E) < \infty$.

The class of Caccioppoli sets and the perimeter enjoy a number of functional properties. By definition, (i) the perimeter $P(E)$ *is lower semicontinuous* with respect to the convergence *in mean*, that is, the convergence of the indicatrices in L^1. Moreover, one proves (ii) *approximation*: for every $E \in \mathcal{P}$ there is a sequence of sets $E_j \in \mathcal{P}$ that are polygonal (a set is *polygonal* if its boundary is contained in finitely many hyperplanes) or bounded with smooth boundaries, such that the $E_j's$ converge in mean to E and $P(E_j) \to P(E)$; (iii) *compactness*: from a sequence of sets in $\mathcal{P}$ with equibounded perimeters and supports we may choose a subsequence that converges in mean to a set in $\mathcal{P}$.

Therefore, we may think of Caccioppoli sets as of the limits in mean of sets with smooth boundaries and of the perimeter as of *Lebesgue's extension* of the



An approximation to a Caccioppoli set that has everywhere dense boundary; an example of such a set is given by the union of circles with centers at all rational points in the unit square.

classical measure of the boundary:

$$P(E) := \inf \Big\{ \liminf_{j\to\infty} P(E_j) \Big\},$$

where the infimum is taken among sequences of sets E_j with smooth boundaries converging to E in mean. This distinguishes the class of Caccioppoli sets as the *natural* class in which to study variational problems involving the area of codimension one surfaces. One such problem is that of finding a set of minimum perimeter among sets of prescribed full measure, which has a simple positive answer in the class $\mathcal{P}$, and then characterize the optimal sets as hyperspheres, which is the *isoperimetric property* of the hypersphere in its dual form: *Denote by C a hypersphere in $\mathbb{R}^n$; then $P(E) \geq P(C)$ for all $E \in \mathcal{P}$ with $\mathcal{H}^n(E) = \mathcal{H}^n(C)$, moreover equality holds if and only if $E = C$.*

A rough presentation of De Giorgi's proof of the isoperimetric property of the hypersphere follows.

1. Steiner's symmetrization procedure, which classically preserves the full measure and decreases the boundary measure of smooth sets, extends to Caccioppoli sets.

Given a bounded set $E \in \mathcal{P}$ and a direction η in $\mathbb{R}$, let H be $(n-1)-$plane orthogonal to η. I recall that *Steiner symmetrization* of E in the direction η is the set $E^{s,\eta}$ enjoying the property that its intersection with (almost) any straight line L orthogonal to H is a segment, symmetric about H, the length of which equals the $1-$dimensional measure of $L \cap H$. The following holds: $\mathcal{H}^n(E^{s,\eta}) = \mathcal{H}^n(E)$ and $P(E^{s,\eta}) \leq P(E)$; moreover, if equality holds, then E is *normal* with respect to H or in the direction η, that is, the intersections of straight lines orthogonal to H with E are segments.

2. On the account of the approximation theorem, to prove that $P(B) \geq P(C)$ for all $B \in \mathcal{P}$ with $\mathcal{H}^n(B) = \mathcal{H}^n(C)$ it suffices to prove it in case B is a polygonal set Π. If so, Π is enclosed in a large ball $B(0, R)$ and, by compactness and semicontinuity, the perimeter has a minimizer in the class $\mathcal{E}$ of sets $B \in \mathcal{P}$ with $\mathcal{H}^n(B) = \mathcal{H}^n(\Pi)$ and supports in $B(0, R)$. Let E be one of such minimizers. Steiner symmetrized $E^{s,\eta}$ is also a minimizer in the same class $\mathcal{E}$, hence E is normal in the direction η. Since this holds for all directions and all minimizers, we conclude that every minimizer is normal with respect to any direction or every minimizer is convex.

3. Let E be a bounded and convex set and let H be an $(n-1)-$plane passing throught the barycenter of E. After a rotation we may assume that H is the hyperplane $x_n = 0$ and we may represent E, by setting (y, x_n) instead of $(x_1, \ldots, x_n)$, as the set $f_1(y) \leq x_n \leq f_2(y)$, $y \in D$, where D is a convex domain in $\mathbb{R}^{n-1}$ and f_1, f_2 are Lipschitz functions. One also sees that

$$P(E)-P(E^{s,x_n}) = \int_D \sqrt{1+|Df_1|^2}\,d\mathcal{H}^{n-1} + \int_D \sqrt{1+|Df_2|^2}\,d\mathcal{H}^{n-1} - 2\int_D \sqrt{1+\left|D\frac{f_2-f_1}{2}\right|^2}\,d\mathcal{H}^{n-1}.$$

According to Minkowski's inequality we then infer $P(E)-P(E^{s,x_n}) \geq 0$ and equality holds if and only if $D(f_1-f_2)=0$ in D. This yields that E is symmetric in any direction, hence E is a hypersphere.

In particular, we have proved that the sphere is the unique solution to the problem of finding a set of smallest perimeter among sets of given full measure with smooth boundaries.

In the classical context, De Giorgi's proof may be read as: If the set E is optimal, then E is a sphere. This essentially amounts to what Steiner did and believed to be a proof of the isoperimetric property of the sphere. In the measure theoretic context the existence of a solution is proved, in Steiner's classical context it is assumed.

As we know nowadays, this makes Steiner's proof not complete: before accusing somebody of murder, having excluded everybody else, it is better to be sure that we are in a case of murder. By assuming the existence of certain objects, one may in fact prove all kind of nonsense.

According to Perron, assuming that there is a largest positive integer, n, we may prove that n must be 1; in fact if n were larger than 1, then $n^2 > n$, contradiction. Steiner never accepted that his proof of the isoperimetric property was incomplete, although Dirichlet pointed out the gap to him. The irony of history is that in the nineteenth century distinguished mathematicians such as Gauss, Dirichlet, Riemann, Neumann, made a similar error, see [6].

Mariano Giaquinta

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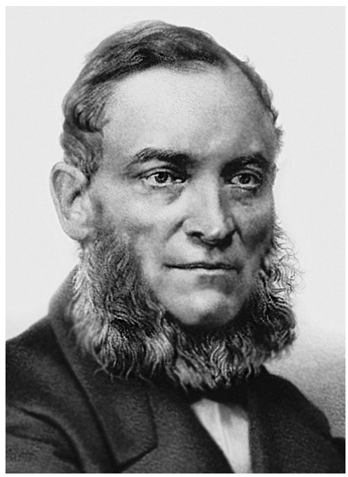
Viktor Yakovlevich Bunyakovsky (1804–1889)

Viktor Bunyakovsky[1] was a Russian mathematician, demographer, member of the Academy of Sciences (1828), and its vice-president 1864–1889. He made significant contributions to the development of probability theory and statistics, and actively participated in the establishment and subsequent reforms of mathematical education in the Russian Empire.

He was born in the town of Bar, Podolsk Governorate, into the family of a lieutenant colonel of cavalry, who died in Finland during the Russian-Swedish war of 1808–1809. Having lost his father at an early age, Bunyakovsky was brought up in the family of General Count Alexander Tormasov, a participant in the Patriotic War of 1812.



In 1820, together with Tormasov's son Alexander, Bunyakovsky went abroad to study. He studied in Coburg in Bavaria, where he took private lessons, then at the Academy in Lausanne, before moving to Paris to study at the Collège de France and the Faculty of Sciences of the Sorbonne (1824–1825). He attended lectures by Augustin-Louis Cauchy, Pierre-Simon Laplace, Joseph Fourier, Siméon Denis Poisson, Adrien-Marie Legendre, and André-Marie Ampère. In Paris, he was awarded bachelor's and licentiate degrees (1824), and in 1825, he received a doctorate in mathematics for his work on analytical dynamics and the theory of heat. He became friends there with Mikhail Ostrogradsky.

After returning to Russia, Bunyakovsky taught at the First Cadet Corps (1826–1831) and St. Petersburg University (1846–1859). He was a professor at the Mining Institute and the Institute of Transport Engineers Corps. In 1830, he became an Extraordinary Academician, and in 1864, he was elected vice-president of the Academy of Sciences and held this position until his death.

Bunyakovsky investigated particular issues of integration and elementary number theory and wrote a historical review of various "proofs" of the parallel postulate. He did not accept Lobachevsky's theory, although he spoke of it

[1] Spelling variants: Bunyakovskii, Bunyakovskiy.

with respect. Bunyakovsky's main achievements were in probability theory, statistics, and demography.

Having reworked the lectures of Laplace that he had listened to in Paris, Bunyakovsky wrote the best textbook of its time, *Foundations of Mathematical Probability Theory* (1846).

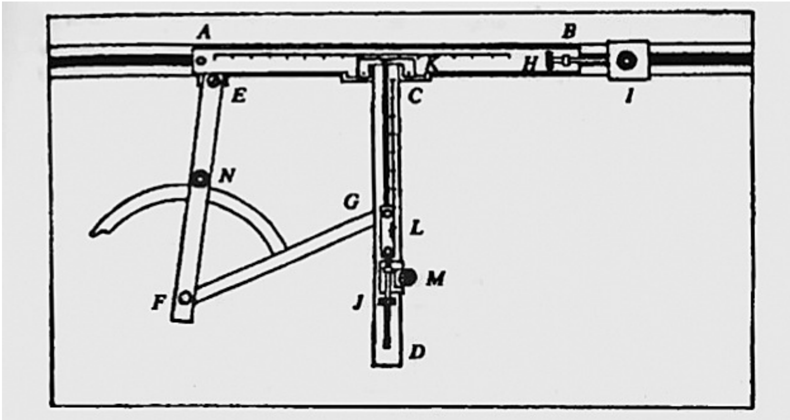
Bunyakovsky mainly was concerned with applied issues of demography and insurance; however, he did influence Pafnuty Chebyshev in the field of probability theory. The mathematical foundations of insurance in Russia had already been laid by Leonhard Euler and Nicolas Fuss, whom Bunyakovsky refers to in his papers. From 1858, Bunyakovsky was the government's chief expert on statistics and insurance and helped organize a retired insurance saving association in the navy, an organization to which members contributed money (usually 6% of their salary) and from which an annual pension was paid to a member or (in case of his death) his family members after a certain period of participation or time in service.

The rational planning of the old-age insurance saving association required an understanding of how much and what payments were to be made, on average: Bunyakovsky investigated relevant mathematical models. In 1869, he published a series of four notes on the subject. In *Notes on a Question About Life Pensions*, read at the meeting of the Physics and Mathematics Division of the Academy of Sciences on December 10, 1868, he gave a mathematical solution to the problem of how to "determine the age X, at the reaching of which a person becomes entitled to a pension, conditioning that age so that the annually increasing total of pensioners would not exceed some given limit with time." Mathematician Andrei Markov participated in 1890 in the calculations for the association, for which he was issued a commendation from the Ministry of Finance. Incidentally, Markov, a gymnasium student, wrote a letter to Bunyakovsky. The latter found some mistakes in Markov's research but did praise the future academician.

The mortality records of the Orthodox population were practically the only source of demographic data available at the time (there were no censuses); it was from these that Bunyakovsky had to calculate the age distribution of Russia's population. Having studied the issue, Bunyakovsky concluded:

> Our extremely unfavorable situation regarding mortality laws in comparison with other European nations, which is still accepted as an undoubted fact, in my opinion, is simply a scientific misunderstanding, which arose and was retained solely because those who addressed this issue did not go deep enough into its essence.

The old mortality table method exaggerated mortality for a country with an increasing population. Considering it erroneous, Bunyakovsky proposed a new method of compiling mortality tables. The difference between his method and the previous one was that the numbers of deaths by age were compared with the number of births of the generation to which the deaths of that age belonged.



The *équerre* schema.

Bunyakovsky calculated, among other things, the probable size of the Russian army. In those days, such a value could only be estimated using statistics.

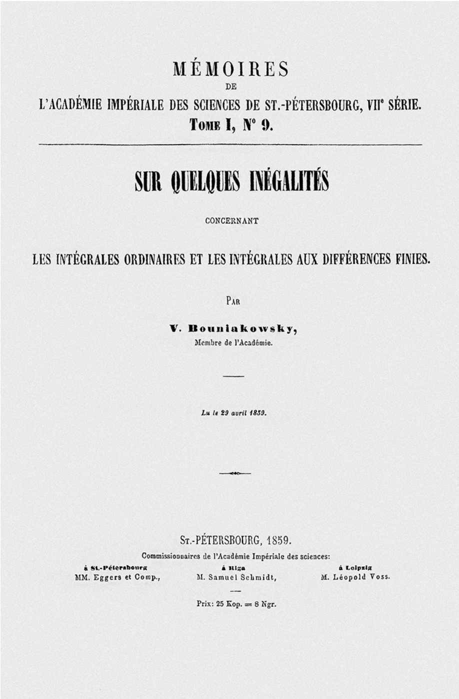
He also gained fame as an inventor. In 1855, he invented new models of pantographs (devices that make scaled-down copies of drawings and maps), planimeters (devices that measure areas), and the so-called *équerre*, which proved to be more accurate and cheaper than their existing equivalents. The *summing équerre* was designed to use the method of least squares: by moving rulers and clamping screws, one could obtain $\sqrt{x^2 + y^2}$ from the numbers x, y, which made it possible to calculate the sum of the squares of the given numbers quickly.

In 1867, building on the working principle of the traditional Russian abacus, Bunyakovsky invented "self-count," a device for repeatedly adding and subtracting large numbers. Self-counts were to calculate the average monthly or annual values of meteorological elements: for example, instead of dividing the sum by 30, each summand was counted with the factor $1/30$. The original self-counts have survived only in The Polytechnic Museum and The National Museum of the Republic of Karelia.

In 1889, Viktor Bunyakovsky, together with Pafnuty Chebyshev and Vasily Imshenetsky, got Sofia Kovalevskaya elected to the Academy of Sciences. Thus, she became the first woman among the corresponding members of the Academy of Sciences.

Cauchy–Bunyakovsky–Schwarz inequality. The inequality

$$\left(\sum_{i=1}^{n} u_i v_i\right)^2 \leq \left(\sum_{i=1}^{n} u_i^2\right)\left(\sum_{i=1}^{n} v_i^2\right)$$



MÉMOIRES
DE
L'ACADÉMIE IMPÉRIALE DES SCIENCES DE ST.-PÉTERSBOURG, VII^e SÉRIE.
TOME I, N° 9.

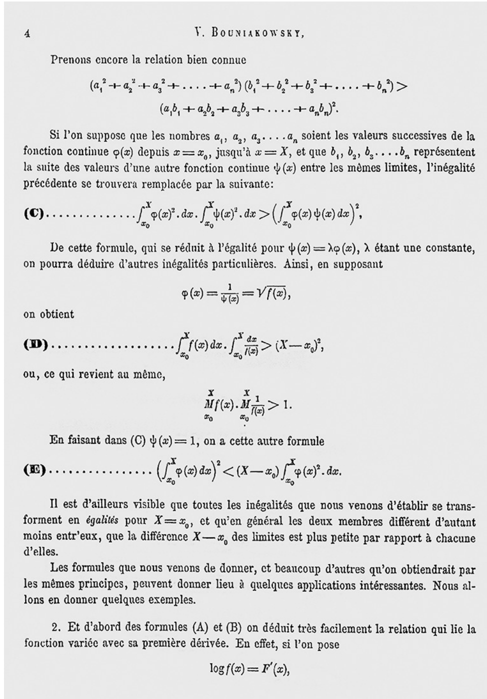
SUR QUELQUES INÉGALITÉS
CONCERNANT
LES INTÉGRALES ORDINAIRES ET LES INTÉGRALES AUX DIFFÉRENCES FINIES.

PAR
V. Bouniakowsky,
Membre de l'Académie.

Lu le 29 avril 1859.

ST.-PÉTERSBOURG, 1859.
Commissionnaires de l'Académie Impériale des sciences:
à St.-Pétersbourg MM. Eggers et Comp., à Riga M. Samuel Schmidt, à Leipzig M. Léopold Voss.

Prix: 25 Kop. = 8 Ngr.



4 V. BOUNIAKOWSKY,

Prenons encore la relation bien connue

$$(a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + b_3^2 + \ldots + b_n^2) > (a_1 b_1 + a_2 b_2 + a_3 b_3 + \ldots + a_n b_n)^2.$$

Si l'on suppose que les nombres $a_1, a_2, a_3 \ldots a_n$ soient les valeurs successives de la fonction continue $\varphi(x)$ depuis $x = x_0$, jusqu'à $x = X$, et que $b_1, b_2, b_3 \ldots b_n$ représentent la suite des valeurs d'une autre fonction continue $\psi(x)$ entre les mêmes limites, l'inégalité précédente se trouvera remplacée par la suivante:

$$\textbf{(C)} \ldots\ldots \int_{x_0}^{X} \varphi(x)^2 . dx . \int_{x_0}^{X} \psi(x)^2 . dx > \left(\int_{x_0}^{X} \varphi(x)\psi(x)\,dx\right)^2,$$

De cette formule, qui se réduit à l'égalité pour $\psi(x) = \lambda\varphi(x)$, λ étant une constante, on pourra déduire d'autres inégalités particulières. Ainsi, en supposant

$$\varphi(x) = \frac{1}{\psi(x)} = \sqrt{f(x)},$$

on obtient

$$\textbf{(D)} \ldots\ldots \int_{x_0}^{X} f(x)\,dx . \int_{x_0}^{X} \frac{dx}{f(x)} > (X - x_0)^2,$$

ou, ce qui revient au même,

$$\overset{X}{\underset{x_0}{M}} f(x) . \overset{X}{\underset{x_0}{M}} \frac{1}{f(x)} > 1.$$

En faisant dans (C) $\psi(x) = 1$, on a cette autre formule

$$\textbf{(E)} \ldots\ldots \left(\int_{x_0}^{X} \varphi(x)\,dx\right)^2 < (X - x_0) \int_{x_0}^{X} \varphi(x)^2 . dx.$$

Il est d'ailleurs visible que toutes les inégalités que nous venons d'établir se transforment en *égalités* pour $X = x_0$, et qu'en général les deux membres diffèrent d'autant moins entr'eux, que la différence $X - x_0$ des limites est plus petite par rapport à chacune d'elles.

Les formules que nous venons de donner, et beaucoup d'autres qu'on obtiendrait par les mêmes principes, peuvent donner lieu à quelques applications intéressantes. Nous allons en donner quelques exemples.

2. Et d'abord des formules (A) et (B) on déduit très facilement la relation qui lie la fonction variée avec sa première dérivée. En effet, si l'on pose

$$\log f(x) = F'(x),$$

The cover page of Bunyakovsky's paper and the page where the inequality for integrals was used.

for sums appeared in the work of Cauchy in 1821, then Bunyakovsky proved (as an intermediate step, see the illustration above) the inequality

$$\left| \int_{\mathbb{R}} f(x)g(x)dx \right|^2 \leq \int_{\mathbb{R}} |f(x)|^2 dx \cdot \int_{\mathbb{R}} |g(x)|^2 dx$$

for integrals in his 1859 work. Bunyakovsky's work was published in French in the *Transactions of the Saint Petersburg Academy of Sciences*. Hermann Schwarz obtained the same inequality 30 years later, in 1888. In the USSR/Russia, this inequality is known as the Cauchy–Bunyakovsky inequality.

Nikita Kalinin

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Chebyshev’s era

Pafnuty Lvovich Chebyshev (1821–1894)

Pafnuty Lvovich Chebyshev[1] is the founder of the 19th-century Russian mathematical school in Saint Petersburg. He contributed to number theory, probability, approximation theory, analysis, geometry, algebra, differential equations, abelian integrals, cartography, and astronomy. His results on the distribution of prime numbers stand as the basis of analytic number theory. His work on probability includes significant extensions of the law of large numbers and of the central limit theorem, and makes him one of the principal founders of modern probability theory. He was the first mathematician to recognize the importance of a general theory of orthogonal polynomials. We owe him fundamental results on the approximation of a real analytic function by polynomials. He had a very strong interest in mechanical engineering and conceived several machines and devices. His work highlights the importance of mathematics in the applied sciences, and conversely, it shows how practical problems may motivate theoretical research.

Pafnuty Chebyshev was born on May 16, 1821, in the village of Okatovo, about 80 km south of Moscow, district of Borovsk,[2] Kaluga province. The village was a part of the property of his father, a former army officer of noble descent. Chebyshev received his education at home until age 11, first from family members and then private tutors. In 1837, he enrolled in the faculty of physics and mathematics of Moscow University. One year later, he wrote a paper titled *Calculation of the roots of equations* in which he gave a method of approximation for the roots of an algebraic equation of degree n. This was the first of a series of works he published in approximation theory, a topic in which he became engaged for the rest of his life.

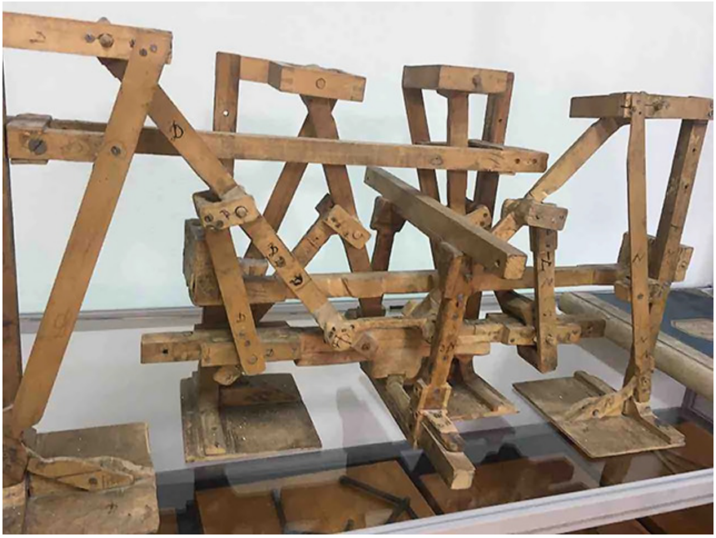
At Moscow University, Chebyshev's talents attracted the attention of N. D. Brashman, his teacher of Mechanics who became his mentor and for whom Chebyshev always retained a profound respect, both as a mathematician

[1] Pronounced as Chebyshov, with the stress on "o".
[2] Now Zhukovsky district.

and a person. He graduated in 1841 with a candidate dissertation titled *On the numerical solution of algebraic equations of higher degree.* In 1843, he published the first of a series of 17 papers that appeared in Liouville's journal (*Journal de mathématiques pures et appliquées*). This paper, entitled *Note on a class of multiple definite integrals*, attracted the attention of Eugène Charles Catalan even before its publication, since the latter wrote a sequel to Chebyshev's results which was published in the same issue of the journal. In 1844, Chebyshev published a paper in Crelle's journal (*Journal für die reine und angewandte Mathematik*), titled *Note on the convergence of Taylor's series*, in which he pointed out a gap in a statement by A.-L. Cauchy involving the integration of a series of functions. In 1845, he defended a *Magister* thesis, titled *An attempt for an elementary analysis of the theory of probability*, opening up a series of works on a topic in which he became a world leader.

In 1847, Chebyshev settled in Saint Petersburg where he started to teach at the Imperial University. In 1847, he defended a doctoral dissertation on number theory, titled *The theory of congruences*, concerning a topic in which he became interested after reading Leonhard Euler's works. According to B. N. Delone [3], this dissertation contained the first non-trivial results on the distribution of prime numbers since the works of Euclid, and their importance is only comparable to those obtained by Bernhard Riemann on this topic. Part of this thesis was published in 1852 in Liouville's journal under the title *On the function which determines the totality of prime numbers smaller than a certain limit.* It contains new results on the growth and the limiting behavior of the function $\pi(x)$ of the number of primes less than x for large x, incidentally invalidating several statements made by Legendre and giving alternative results and proofs. By then, Chebyshev had become a highly respected mathematician in Europe, and his prestige and influence increased.

Motivated by the steam engine constructed by James Watt in 1763, Chebyshev published a paper titled *On the theory of mechanisms, known under the name of parallelograms* in 1854. This was the first of around ten papers he wrote on the theory of mechanical linkages. Roughly speaking, Watt produced a mechanism that, from a combination of circular motions, produces a rectilinear one. Chebyshev was interested in this question because it involves the conception of mechanical devices and approximation theory. He also wanted to obtain an exact solution to the problem of transforming a rectilinear motion into a circular one, not only approximations. The problem was solved in 1871 by Yom Tov Lipman Lipkin, one of his young collaborators, and two years later it was given another solution by the French engineer Charles-Nicolas Peaucellier [6]. In his work on this topic, Chebyshev inaugurated the general study of transforming one motion into another one through mechanical linkages. By the end of the 19th century, interest in the theory of linkages had declined. Still, it became very active again in the 1980s, under the impulse of William Thurston: see the survey by Alexei Sossinsky [8]. Among the

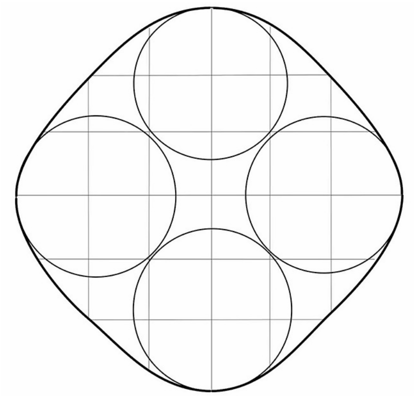


Chebyshev's plantigrade-machine.

devices that Chebyshev invented are mechanical linkages for a wheelchair and for a rowing boat. Incidentally, it is in his 1854 paper on linkages that the Chebyshev polynomials appeared for the first time.

Chebyshev was fluent in French, which was not unusual among 19th-century educated Russians. In the early 1840s, he started building contacts with renowned mathematicians in Western Europe and he established close friendships with Charles Hermite, Joseph Bertrand, Catalan, Leopold Kronecker, Edouard Lucas, and many others. Alexander Vasiliev [9] and Vladimir Possé [7] mention that he spent almost all his summers abroad, mainly in Paris. His *Collected Works* contain a report on a 3-month stay he made in France in 1852, in which he records that he conversed with Irénée-Jules Bienaymé, Cauchy, Liouville, Hermite, Victor Amédée Lebesgue, Alphonse de Polignac, Joseph Alfred Serret, and other mathematicians. The meetings took place in the evening, since during the day he was busy visiting industrial plants. In his report, he notes his observations on the windmills in Lille, on the metallurgical plant in Hayange, on the paper mills in Coronne, on the foundry and the cannon factory in Ruelle, on a turbine in a windmill in Saint-Maur, on a water mill in Meaux, on an arms factory in Châtelleraut, etc. From Paris he made a small trip to London, where he conversed with Arthur Cayley and J. J. Sylvester, and visited the Royal Polytechnic Institute, where he examined models of various machines. He also went to Brussels, where he visited the museum of engines, and on his way back to Russia he made a stop in Berlin and had several discussions with Peter Gustav Lejeune Dirichlet.

In 1853, Chebyshev was elected adjunct at the Saint Petersburg Academy of Sciences. His first task was to assist V. Ya. Bunyakovsky, who noticed his strong capacity for work, in publishing an edition of Euler's works on number theory. He became an ordinary academician in 1859.



A unit of tissue dressing the half–sphere. From Chebyshev's unfinished article on cutting garments, see [4] for details.

From childhood, Chebyshev was handicapped by a withered leg; he walked with the help of a stick and he was excluded from most of the children's games. As we have noted, he was fascinated by mechanical devices, and kept this passion until the end of his life. One of his first constructions was a computing machine, which he built with his own hands. His invented devices are displayed in the Conservatoire National des Arts et métiers in Paris, at Saint Petersburg University, the Saint Petersburg Academy of Sciences, and elsewhere.

Chebyshev was a member of most European Academies of Sciences. In 1874, he was elected a foreign member (the first Russian after Peter the Great) of the Paris Académie des Sciences. He became a member of the *Association française pour l'avancement des sciences*, a learned society created in 1872, aiming to promote relations between the various sciences. Chebyshev participated in four of its annual meetings: Lyon (1873), Clermont-Ferrand (1876), Paris (1878), and La Rochelle (1882), each time presenting several works on various topics (geographical maps, the cutting of garments, his calculating machine, etc.). Abstracts of his talks are reproduced in his *Collected Works*.

In 1882, Chebyshev resigned from his professorship and started dedicating all his time to research. One day a week, his house was open to young scientists who wished to discuss their results with him or seek advice. In the summer of 1893, he made his last extended visit to Paris. He died on November 26, 1894. His tomb can be visited in the basement of the Church of the Transfiguration in the village of Spas-Prognanye, 10 km from the science city of Obninsk. In the nearby village of Mashkovo, there is a school named after Chebyshev, which hosts a museum containing a collection of original photographs and objects that belonged to him, including an arithmetic machine that he constructed. There are several biographies of Chebyshev, e.g., [7, 9, 10]. For a guide to his life and work, see [2]; on his contacts with Western European scientists, see [1].

Athanase Papadopoulos

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On Chebyshev's work on geography

Pafnuty Lvovich Chebyshev, like his most famous predecessor, Leonhard Euler, was involved in a great variety of research topics, including number theory, probability, the theory of elliptic integrals, differential geometry, approximation theory, mechanics, and others. He was always driven by the desire to find effective solutions with practical algorithms and good approximations, in case a precise solution could not be found. His biographers report that while he spent a significant amount of his time studying the works of mathematicians from the past like Euler, Lagrange, Gauss and Abel, he avoided reading the works of his contemporaries, considering that this would be an obstacle to having original ideas.

The mathematical theory of cartography, i.e., the art of map drawing, is an applied field that raises interesting theoretical questions in geometry and analysis, and it is not surprising that this topic became one of Chebyshev's domains of interest. It was also an ideal ground for applying his ideas in approximation and interpolation theories.

In 1856, Chebyshev wrote two papers on cartography carrying the title *Sur la construction des cartes géographiques*, the same title as two memoirs that Lagrange published 57 years before [8]. It is probable that Chebyshev became interested in this field by reading Lagrange's memoirs, but also those of Euler. Indeed, the latter was intensively involved in geography; at the Saint Petersburg Academy of Sciences he had the official charge of geographer, and he published several works in cartography. The reader might refer to [1] for a commented edition of the works of Euler and Lagrange on geography. Chebyshev was part of an editorial committee, headed by Bunyakovsky, for an edition of Euler's works on number theory, and there is no doubt that he skimmed Euler's works on various topics, including cartography.

The problem of drawing geographical maps is that of mapping on a Euclidean plane, and with minimal distorsion, a subset of a curved surface, which is usually a sphere representing the Earth. It was known by mathematicians since Greek antiquity that it is not possible to require that such a mapping preserves distances up to a scale. In other words, such a mapping has necessarily a "distortion." The main question was that of finding geographical maps whose distortion is minimal in a sense that had to be made precise. It is not surprising that Chebyshev, who was working on practical and theoretical problems involving approximation and optimization techniques, became interested in this topic.

Chebyshev based his investigations of geographical maps on Lagrange's work, presented in the two memoirs [8], itself is an extension of the works of Euler and Lambert on cartography [8, p. 641]. The setting is that of a general conformal, that is, angle-preserving, map from a subset of the sphere into the plane. In fact, Lagrange worked in the setting of a spheroid, that is, a figure obtained by rotating an ellipse along an axis; it was known at this epoch that the Earth was rather spheroidal than spherical. Chebyshev considered the simpler case where the Earth is spherical. He used a formula established by Lagrange for a quantity he called the *magnification ratio*. This is the ratio between a length element at a point on the sphere and its image by the conformal map.

The coordinates of a point on the sphere are denoted by (s, t) where s is the co-latitude of a point, that is, its distance to the North pole, and t its longitude, that is, the angle made by the plane containing the meridian passing through this point with the plane containing a fixed meridian chosen as an origin for longitudes. The fact that the map is conformal implies that the magnification ratio, as the complex derivative of the map, does not depend on the direction. In a plane containing a meridian, Lagrange took rectangular coordinates (p, q) with $p = \cos s$ and $q = \sin s$, which makes the length element on the sphere equal to $\sqrt{ds^2 + q^2 dt^2}$. The Euclidean plane is equipped with coordinates (x, y) and the length element equals $\sqrt{dx^2 + dy^2}$. The magnification ratio, in this notation, is

$$m = \frac{\sqrt{dx^2 + dy^2}}{\sqrt{ds^2 + q^2 dt^2}}.$$

The problem is then to determine (x, y) in terms of s and t in such a way that the deviation of the magnification ratio m from its integral over the region considered is minimal. The problem becomes a problem in the calculus of variations, a field where Lagrange was a pioneer. Using a new variable satisfying $du = ds/q$ and passing to complex notation, Lagrange reached the formula

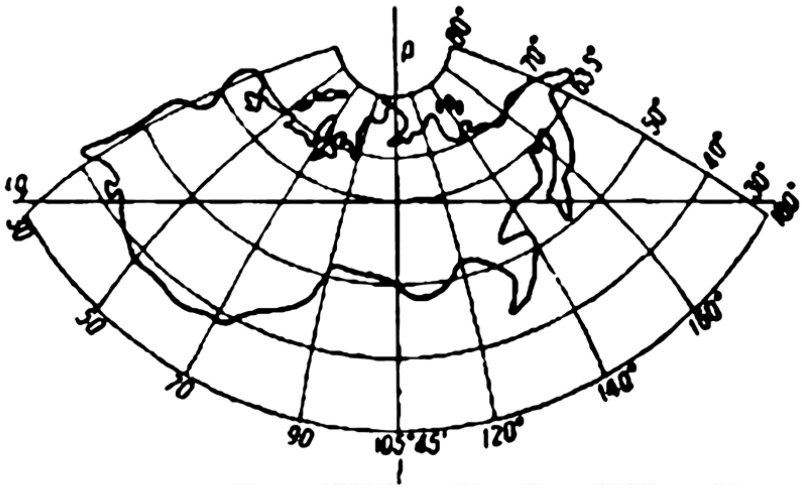
$$m = \frac{\sqrt{f'(u+it)F'(u-it)}}{\frac{2}{e^u + e^{-u}}}$$

where f' and F' are arbitrary functions. This is the formula that Chebyshev used, and his main observation was that this gives

$$\log m = \frac{1}{2}\log[f'(u+it)] + \frac{1}{2}\log[F'(u-it)] - \log\frac{2}{e^u + e^{-u}},$$

and that the sum U of the first two terms is a real harmonic function, that is, it satisfies Laplace's equation $\dfrac{\partial^2 U}{\partial u^2} + \dfrac{\partial^2 U}{\partial t^2} = 0$.

From this, he reduced the search for the best map to finding a solution to the Laplace equation, defined on a region with given boundary values. His



The map of the USSR made using Chebyshev's projection method.

main theorem can be stated as follows: *For a given country, or a region of the sphere, the projection for which the magnification ratio has the smallest variation is one for which the magnification ratio on the boundary of the region to be represented is constant.*

Chebyshev also indicated a method for the computation of the magnification ratio, whose effectiveness depends on the complication of the boundary of the region to be mapped.

Regarding Chebyshev's interest in geography, one may also mention his "rule for the approximate evaluation of distances on the surface of the earth", contained in Vol. II, p. 736 of his Collected Works [2].

Chebyshev did not write the details of the proof of his theorem. The theorem was considered as being difficult, and, several years later, various mathematicians wrote proofs of it. In 1911, Gaston Darboux published a paper carrying the same title as Chebyshev's, giving details of the proof, based on Chebyshev's idea of using the Laplace equation, and applying Green's formula [5].

In 1894, D.A. Gravé found a proof of Chebyshev's theorem and presented it at the annual meeting of the *Association française pour l'avancement des sciences*, a French learned society of which Chebyshev had been a member. In 1911, he published a paper [7], titled *Sur un théorème de Tchébychef généralisé*, in which he gave a proof of a slightly more general result. In fact, Gravé was a student of Chebyshev, and under his influence, he had already published in 1896 a paper on cartography carrying again the title *Sur la construction des cartes géographiques*, in which he studied a related problem, that of area-preserving mappings from the sphere to the plane [6].

A modern proof of Chebyshev's theorem, also based on his ideas, was given about a century after Chebyshev found it, by John Milnor [9], who also pointed out further developments and highlighted the importance of the case where the region on the sphere which is mapped is geodesically convex. Milnor writes in his paper: "This result has been available for more than a hundred years, but to my knowledge, it has never been used by actual map makers."

Chebyshev's second paper on geography, which he read at a ceremonial meeting of the Imperial University of Saint Petersburg on February 8, 1856, was also the occasion for him to express his view on the concord between theory and practice in mathematics. He wrote there:

> Since the oldest antiquity, Mathematical Sciences have been the subject of particular attention. They still attract more interest because they influence the arts and industry today. The reconciliation between theory and practice brings the most beneficial results. Practice is not the only side benefiting from these relations: conversely, sciences develop under the influence of practice. The latter discovers new objects of study for the former, and brings new points of view for subjects that are known since long times.
>
> Despite the high degree of development that Mathematical Sciences attained, thanks to the works of the greatest geometers of the last three centuries, practice clearly shows that they are incomplete in many ways. Indeed, it addresses to Science questions that are essentially new, thus leading the search for methods which were unknown till then. If a theory gains a significant profit from new applications from an old method or new developments, it will gain even more from the discovery of new methods. Thus, science finds in practice a safe guide.

Let us conclude this article with another quote from Chebyshev ([11, p. 45–46]):

> Mathematics already traversed two epochs, the one where the problems were set by the gods (for instance, the problem of the duplication of the cube) and the other one where the semi-gods, like Fermat, Pascal, and others established them; today we entered the third period, where the questions to be solved are raised by the needs of humanity.

Athanase Papadopoulos

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Alexander Nikolaevich Korkin (1837–1908)

Alexander Korkin graduated from gymnasium (i.e., grammar school) and university, even though his father was a serf on a taxable estate.[1] Korkin, together with Egor Zolotarev, classified the maximum values of minima of quadratic forms in four and five variables. They were also the first to explicitly construct the quadratic form corresponding to the E_8 lattice.

The eldest of Pafnuty Chebyshev's pupils, Korkin was born in the village of Zhidovinovo, Totemsky District, Vologda Governorate, into a family of state serf peasant. Serfs didn't have the right to study at schools (including specialized high schools preparing students for a university track), much less at universities. His father, a literate and intelligent man, was determined to give his son an education at any cost. He took his son to Vologda, where he persuaded a gymnasium teacher to take Alexander into the family to prepare him for studies at the gymnasium.

In the family of his first teacher, Ivanitsky (incidentally, a pupil of Viktor Bunyakovsky), Korkin learned German and French, and was keen on reading, natural science, and mathematics. Then, his father succeeded in getting the Vologda Treasury Chamber to exclude his son from the taxable estate so he could study at a gymnasium. For his son's sake, he donated 200 rubles to the gymnasium church, which was a lot of money at that time. Unfortunately, Korkin's father did not have long to rejoice in his son's successes: Korkin Sr died in 1849, leaving his family with almost no money.

In 1853, Alexander Korkin graduated from the gymnasium with honors and entered St. Petersburg University the following year. There, he attended lectures of Osip Somov, Viktor Bunyakovsky, and Pafnuty Chebyshev. The

[1] During the 19th century in Imperial Russia, there was a distinction between taxable and non-taxable estates. The Code of the Law of the Russian Empire of 1832 defined four main types of estates: dvoryans (nobility), clergy, urban dwellers and peasants. The estates of urban dwellers and peasants were taxable by the State, and serfs working on these had more freedom but still relatively few rights. The serfs who worked on non-taxable estates belonging to nobility and clergy had less freedom and fewer rights.

faculty awarded him a gold medal for his first scientific paper on conditional extrema, *The Theory of Maxima and Minima of Functions* (1856). Upon leaving university in 1858, Alexander Korkin had to apply again to the Treasury Chamber with a request to be excluded from the taxed estate, was approved as having a candidate's degree,[2] and started teaching mathematics in the First Cadet Corps (from 1858 till 1861).

In 1860, he defended a master's thesis *On the Definition of Arbitrary Functions in Integrals of Partial Differential Equations* (for lack of money, he could not print it, so he wrote a small number of copies by hand in lithographic ink) and moved to St. Petersburg University, where he worked for nearly 50 years. In 1861, following a request by Chebyshev, he was elected adjunct at the university in the department of pure mathematics. In 1862–1864, Korkin was sent to Berlin and Paris to prepare for a professorship, where he attended lectures by Michel Chasles, Gabriel Lamé, Joseph Liouville, Joseph Bertrand, Karl Weierstrass, Leopold Kronecker, and Ernst Kummer. Alexander Korkin defended his doctoral thesis, entitled *On systems of first order partial differential equations and some questions of mechanics*, in 1868 (the reviewers were Somov and Chebyshev). He was then elected, first, as an extraordinary professor in the Department of Pure Mathematics, then as a full professor in 1873, and then as an honored professor in 1886.

After the deaths of Somov, Bunyakovsky, and Chebyshev, Korkin took over the instruction of the most important courses. Korkin was credited with setting up a course at the university called *The Integration of Differential Equations and Calculus of Variations*, which he developed after Chebyshev left in 1882. In this course he introduced a considerable section on ordinary differential equations and their systems, including his own way of solving them, and on partial derivative equations. At his home, Korkin gave a course on partial differential equations for selected students. At the university, such a specialized course was introduced to the curriculum only later by Vladimir Steklov.

For more than 30 years, Korkin also worked at the Naval Academy. Aleksey Krylov, who attended a course of his lectures on differential and integral calculus at the Academy in 1888 and also specialized courses on particular branches of mathematical physics in 1891, wrote that "there were many courses on differential and integral calculus both in Russian and in foreign languages, but Korkin did not adhere to any of them and did not read so much as dictated to us his entirely original course, which is distinguished by the particular precision of definitions, brevity, naturalness, and elegance of derivations of all formulas <...> which is necessary for technicians who study mathematics as a tool for practical applications" [5, p. 109].

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[2] The Candidate's degree was the first academic degree in the Russian Empire between 1835 and 1884. It was awarded to students who graduated from university with honors. The following degree was a Master's degree.

In the whole of the second and partially the third chapters of his monograph *On Some Differential Equations of Mathematical Physics with Applications in Technical Matters* (1913), Krylov based the material directly on Korkin's lectures. Leaving his position at the Naval Academy, Korkin recommended Krylov as the most capable student to take his place.

The main topics of Korkin's research were the integration of differential equations and their systems (the subject of his master's and doctoral theses), partial differential equations and their applications to mechanics, and number theory. In number theory, Korkin worked on the theory of quadratic forms and congruence theory. Together with Zolotarev, he managed to solve the difficult problem of the exact evaluation of the minima of positive quadratic forms in four and five variables.

In *Sur les formes quadratiques*, Korkin and Zolotarev were the first to consider the E_8 lattice while constructing an integral quadratic form associated with it.

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \end{bmatrix}$$

Generating vectors for the E_8 lattice.

Many prominent Russian mathematicians thought of themselves as Korkin's students; a number of them, including Egor Zolotarev, Aleksey Krylov, Alexander Lyapunov, Andrei Markov, Sergey von Glazenap, Dmitry Grave, Ivan Ivanov, and Nikolai Günther, considered themselves Korkin's immediate pupils. Korkin had an excellent command of French, which he used to write many of his works and letters to foreign mathematicians; he also had a good knowledge of Latin. He was fond of astronomical calculations; in particular, he made some corrections to a textbook on spherical astronomy by Aleksey Savich.

Every evening, St. Petersburg students and professors would gather in Alexander Korkin's flat. In his autobiographical notes, Dmitry Grave recalled these evenings with great warmth:

> Sitting on the sofa, Korkin held interesting conversations, as he was an intelligent and educated man. It was particularly interesting when he was talking about mathematics. I must admit that both my theses

> derived from these conversations, although Chebyshev and Markov also played a major role in my doctoral thesis.

Alexander Korkin is buried in the Smolensk Orthodox Cemetery.

Galina Sinkevich

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Julian Karol Sochocki (1842–1927)

Julian Sochocki[1] is known to all mathematics students for his theorem on the behavior of an analytic function near its essential singularity. He was a professor at St. Petersburg University and a founding member and later chairman of the St. Petersburg Mathematical Society.

Julian Karol (Julian Vasilievich) Sochocki was born in Warsaw (the capital of the Kingdom of Poland, a part of the Russian Empire at the time) on January 24, 1842, in the family of a low-ranking official. Having finished school with honors, 18-year-old Sochocki went to St. Petersburg and on September 15, 1860, was enrolled in the Physics and Mathematics Department of St. Petersburg University. St. Petersburg was being shaken by student unrest. The university was closed from 10 April to 10 October 1861, and rallies and protests did not cease. On 12 October Sochocki was imprisoned in the Petropavloskaya fortress for taking part in student riots and five days later transferred to Kronstadt, where he was detained until 6 December. After that, he left for his homeland, having thus completed only one and a half semesters.

In Warsaw, Sochocki took part in the January Uprising of 1863–1864, helping to transport weapons in a hay cart. After the uprising was suppressed in 1864 Sochocki returned to St. Petersburg but was not allowed to be reinstated to the university. He studied on his own. In 1865, he took his master's exams in mathematics and mechanics and presented his thesis (an equivalent of a modern graduation paper) on elliptic functions, for which he received his degree of candidate in mathematics in 1866.

In 1868 Sochocki presented his master's thesis (an equivalent of a modern Ph.D. thesis) *Theory of integral residues with some applications* containing his famous theorem on the behavior of a function near an essential singularity. The

[1] Spelling variant: Sokhotski.

theorem is now called the Casorati–Sochocki–Weierstrass theorem because it was also independently published by the Italian mathematician Felice Casorati (1868) and the German mathematician Karl Weierstrass (1876). As Sochocki noted in the preface, his research style "does not shy away from the simplicity with which all general cases can be obtained," which distinguished him from his contemporary researchers.

In 1869, Sochocki started working in the Institute of Civil Engineers, first as a privatdozent, then as an associate professor. In 1873 he defended his doctoral thesis *On definite integrals and functions used in series expansion*, which contains the so-called Sochocki formulae (later called the Sochocki–Plemelj formulae), still used in quantum physics today. While being a professor at the Institute of Civil Engineers, Sochocki also lectured at St. Petersburg University, where he became a professor in 1882 and professor emeritus in 1893.

Sochocki's *Higher Algebra* (1882–1888) and *The Theory of Definite Integrals* (1901) were recommended as teaching aids at St. Petersburg, Kazan, and Kharkov universities. Having published the work in Russian, Sochocki himself translated it into Polish and published it in his homeland. He was elected a corresponding member of the Krakow Academy in 1894. Many famous mathematicians were among his students: Egor Zolotarev, Georgy Voronoy, Andrei Kiselev, Ivan (Jaan) Depman, Ivan Ptaszycki, Aleksei Adamov, Eugene Borisov, Ivan Ivanov, Wiktor Staniewicz, Andrei Zhuravski.

The last years of Sochocki's life were difficult. His wife and three children died in starving Petrograd. Here are two emphatic documents. The first is a notice sent to Sochocki by the Rector of the University, Alexander Ivanov [10]:

> Professor Yu.V. Sochocki of the First Petrograd University was enrolled on April 10, 1919, on an enhanced food ration [...]
>
> Food will be distributed at the Distribution Desk of the Commissariat of Public Education (Anichkov Palace, Room 15, 1st front door from Fontanka, through the gates).
>
> Bread will be given out once a week. Please bring your own crockery for the sunflower oil.

And here is a letter dated July 10, 1922, from Sochocki himself to the Commission on the Improvement of the Welfare of Scientists [11]:

> Having spent three winters in a row in an unheated flat, my health has become quite deteriorated. At the present time, in view of my weakness and the approaching winter, I am forced to apply to the Commission on the Improvement of the Welfare of Scientists with the request to arrange for me, if possible, suitable accommodation, where I could spend the winter months in decent conditions, without any danger to my life.
>
> Yu.V. Sochocki. July 10, 1922, Fontanka 126, apt. 11.

The last two years of his life Sochocki spent in a residential home for elderly scientists at 27 Millionnaya Street. He died on 14 December 1927 and was buried at Novodevichy Cemetery. His grave was recently restored at the initiative and expense of SPbGASU (Saint Petersburg State University of Architecture and Civil Engineering). The cross was replaced with a Catholic one, according to the rite by which Sochocki was buried. Flowers were laid on his grave from SPbGASU, where he worked for over 50 years, from the St. Petersburg Mathematical Society, of which he was chairman for about 20 years, and from the Consul General of Poland in St. Petersburg.

Galina Sinkevich

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Two results of Yulian Vasilyevich Sochocki in a historical perspective

In this note, we discuss two famous results due to Yu. V. Sochocki: the Casorati–Sochocki–Weierstrass theorem on the behavior of a holomorphic function near its essential singularity and Sochocki's formulas for boundary values of Cauchy-type integrals. We point out the prominent place where these results have been found in Complex Analysis and show how they influenced the development of this field of mathematics in the 20th century; we discuss new statements, problems, and results they led to in their turn.

Casorati–Sochocki–Weierstrass theorem

We start our acquaintance with the results of Sochocki stating a theorem that is familiar to everyone who has studied the basic university course in Complex Analysis. This is the theorem on the behavior of a holomorphic function near its essential singularity. There are several formulations of the corresponding result, and we will state it in a more general context of the theory of cluster sets.

Let D be a domain in the complex plane $\mathbb{C}$, let $a \in \overline{D}$, and let

$$f\colon D \setminus \{a\} \to \widehat{\mathbb{C}},$$

where $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the standard one-point compactification of $\mathbb{C}$.

The *cluster set of f at a* is the set $C_D(f, a)$ consisting of all points $w \in \widehat{\mathbb{C}}$ for which there exists a sequence $\{z_n\}_{n=1}^{\infty}$ of points $z_n \in D \setminus \{a\}$ such that $z_n \to a$ and $f(z_n) \to w$ as $n \to \infty$. Thus $C_D(f, a)$ is a non-empty closed subset of $\widehat{\mathbb{C}}$. The cluster set $C_D(f, a)$ is called *degenerate* if it is a singleton, the set $C_D(f, a)$ is called *total* if $C_D(f, a) = \widehat{\mathbb{C}}$. The cluster set $C_D(f, a)$ which is neither degenerate, nor total, is called *subtotal*.

The theory of cluster sets studies the properties of sets $C_D(f, a)$ for functions f from various classes of functions (for instance, for holomorphic or harmonic functions, for functions realizing various quasiregular mappings, or for functions that are solutions to various differential equations and systems). The following theorem contains an exact formulation of the aforementioned result of Sochocki and, moreover, it may be regarded as one of the very first results of the theory of cluster sets.

Theorem 1. *Let $a \in D$. If f is a meromorphic function in $D \setminus \{a\}$, then $C_D(f, a)$ is either total or degenerate.*

In Russian tradition, this theorem is known as Sochocki's theorem, because it was contained in the master dissertation by Yu. V. Sochocki presented and defended in St. Petersburg in 1868. At the same time, in the Western tradition, this result is usually called the Casorati–Weierstrass theorem, in connection with the works by F. Casorati (1868) and K. Weierstrass (1876), where it was independently published. There is also a mention of this theorem in the first edition of the book by C. Briot and C. Bouquet (1859), but it was omitted in the latter editions of this book. These historical remarks concerning the terminology may be found in such classical books as *Theory of functions of a Complex Variable* by A.I. Markushevich, *The Theory of Cluster Sets* by E.F. Collingwood and A.J. Lohwater. So the name Casorati–Sochocki–Weierstrass theorem seems to be more appropriate.

The proof of the Casorati–Sochocki–Weierstrass theorem is rather simple and can be obtained using the following arguments. Assume that the set $C_D(f, a)$ is subtotal. Then there exists $w \in \widehat{\mathbb{C}}$ such that $w \notin C_D(f, a)$. Therefore the function

$$g(z) = \begin{cases} \dfrac{1}{f(z) - w}, & w \in \mathbb{C}, \\ f(z), & w = \infty, \end{cases}$$

is holomorphic and bounded in a punctured neighborhood of a. This implies that a is a removable singularity for g, and hence the set $C_D(g, a)$ is degenerate. But the latter fact yields immediately that the set $C_D(f, a)$ is also degenerate.

A substantial strengthening of the Casorati–Sochocki–Weierstrass theorem was given by the great Picard theorem which says that if a holomorphic function f has an essential singularity at some point $a \in \widehat{\mathbb{C}}$, then in any punctured neighborhood of a the function f takes on all possible complex values, with at most a single exception, "infinitely often."

In the 20th century, various results in one and several complex variables were obtained inspired by or related to the Casorati–Sochocki–Weierstrass and Picard theorems. For instance, results of a similar nature were obtained for harmonic and polyanalytic functions in $\mathbb{C}$, for quasiregular mappings in $\mathbb{R}^n$, etc. However, the development of the theory of cluster sets took a different path: within the framework of this theory one begins to study the properties of the sets $C_D(f, a)$ in the most difficult case when a point a lies on the boundary of the domain D in which the function f is defined.

Sochocki's formulas for boundary values of Cauchy-type integrals

Another important contribution of Sochocki deals with the boundary behavior of Cauchy-type integrals. Let Γ be a rectifiable closed Jordan curve in $\mathbb{C}$, let

D be a Jordan domain bounded by Γ and D_∞ be the domain $\widehat{\mathbb{C}} \setminus \overline{D}$. Note that the standard Lebesgue spaces $L^p(\Gamma)$, $p \geqslant 1$, are well-defined with respect to the arc-length measure on Γ. Given a function $f \in L^1(\Gamma)$, consider the function

$$F(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)\, d\zeta}{\zeta - z},$$

which is well-defined and holomorphic on $D \cup D_\infty$. This function is traditionally called a Cauchy–type integral, or a Cauchy transform of the function f (or the measure $f\, d\zeta$).

An interesting, important, and deep question is to realize the behavior of $F(z)$ when z tends to Γ. Of course, we need to specify in what sense the boundary behavior of $F(z)$ is understood. To do that, let us consider the following settings.

a) Since Γ is rectifiable, it has a tangent almost everywhere (with respect to the arc-length measure on Γ). Therefore, it makes sense to pose the question of describing the set of points $\zeta \in \Gamma$ where there exist the non-tangential boundary values $F_i(\zeta)$ of F at ζ from D. By definition,

$$F_i(\zeta) = \lim F(z)$$

when z tends to ζ along a path lying in $D \cup \{\zeta\}$, ending at ζ, and running non-tangential to Γ, if this limit exists and has the same value along any such path.

b) One can ask the same question on the existence of non-tangential boundary values $F_e(\zeta)$ of F at ζ from D_∞.

c) One can also consider the principal value of the corresponding singular integral at the point $\zeta \in \Gamma$, that is

$$Cf(\zeta) = \lim_{\delta \to 0} \frac{1}{2\pi i} \int_{\Gamma \setminus B(\zeta,\delta)} \frac{f(w)\, dw}{w - \zeta}, \qquad \text{where} \quad B(\zeta, \delta) = \{w\colon |w - \zeta| < \delta\}.$$

To describe the Sochocki contribution to the topic let us consider the case when the initial function f is Hölder continuous on Γ, that is there exist $M > 0$ and $\alpha > 0$ such that $|f(\zeta) - f(\zeta')| \leqslant M|\zeta - \zeta'|^\alpha$ for all $\zeta, \zeta' \in \Gamma$. Then it can be readily proved that $Cf(\zeta)$ exists at every point $\zeta \in \Gamma$, where Γ has a tangent. Namely,

$$Cf(\zeta) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w) - f(\zeta)}{w - \zeta}\, dw + \frac{f(\zeta)}{2\pi i} \lim_{\delta \to 0} \int_{\Gamma \setminus B(\zeta,\delta)} \frac{dw}{w - \zeta}.$$

The first integral converges absolutely (due to Hölder continuity of f), while the second one can be computed explicitly and tends to $1/2$ since the internal angle of the domain D at the boundary point ζ is π (provided that Γ has a tangent at ζ), and, therefore,

$$Cf(\zeta) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w) - f(\zeta)}{w - \zeta}\, dw + \tfrac{1}{2} f(\zeta).$$

We omit all technical details that can be found, for instance, in the book by Markushevich, cited above.

Similarly, for a Hölder continuous function f and for every $z \in D$ and $\zeta \in \Gamma$ we have

$$F(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w) - f(\zeta)}{w - z}\, dw + f(\zeta)$$

because $\int_\Gamma (w-z)^{-1}\, dw = 2\pi i$. Assuming that Γ has a tangent at ζ and tending z to ζ non-tangentially we obtain

$$F_i(\zeta) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w) - f(\zeta)}{w - \zeta}\, dw + f(\zeta),$$

where the first integral converges absolutely as before. Thus, for almost all $\zeta \in \Gamma$ it holds

$$F_i(\zeta) = Cf(\zeta) + \tfrac{1}{2} f(\zeta).$$

Finally, for every $z \in D_\infty$ and $\zeta \in \Gamma$ we have

$$F(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w) - f(\zeta)}{w - z}\, dw,$$

which implies

$$F_e(\zeta) = Cf(\zeta) - \tfrac{1}{2} f(\zeta)$$

for every ζ where Γ has a tangent.

Summarizing the constructions above and observations, we are going to state the next result:

Theorem 2. *Let Γ be a rectifiable closed Jordan curve in $\mathbb{C}$, and let f be a Hölder–continuous function on Γ. Then the values $F_i(\zeta)$, $F_e(\zeta)$ and $Cf(\zeta)$ exist for almost all point $\zeta \in \Gamma$ and the following formulas hold*

$$\begin{aligned} F_i(\zeta) &= Cf(\zeta) + \tfrac{1}{2} f(\zeta), \\ F_e(\zeta) &= Cf(\zeta) - \tfrac{1}{2} f(\zeta), \\ F_i(\zeta) - F_e(\zeta) &= f(\zeta). \end{aligned}$$

The formulas linking $F_i(\zeta)$, $F_e(\zeta)$ and $Cf(\zeta)$ in the theorem stated above are called Sochocki's formulas; they were obtained for the first time in the doctorate dissertation by Yu. V. Sochocki (1873). Note, moreover, that this theorem gives the formulation of the corresponding Sochocki proposition in the modern language (it is very plausible that Sochocki himself dealt with sufficiently smooth curves rather than with rectifiable ones).

Traditionally, Sochocki's formulas are also called Sochocki–Plemelj formulas because they were obtained (most likely completely independently) by J. Plemelj in 1908 (*Monatshefte für Mathematik und Physik*, vol. 19).

This result opened a long line of research related to the boundary behavior of Cauchy-type integrals. The assumptions made above to obtain the Sochocki formulas turned out to be more restrictive than necessary, and it is interesting and important to obtain the most general conditions that ensure the existence of boundary values for Cauchy-type integrals.

The first famous result in this line was the celebrated Privalov lemma, which sounds as follows.

Theorem 3. *Let Γ be a rectifiable closed Jordan curve in $\mathbb{C}$, and let $f \in L^1(\Gamma)$. Then either the values $F_i(\zeta)$, $F_e(\zeta)$ and $Cf(\zeta)$ simultaneously exist or do not exist for almost all points $\zeta \in \Gamma$. Moreover, whenever these quantities do exist, they are connected by Sochocki's formulas in all points $\zeta \in \Gamma$ where Γ has a tangent.*

We emphasize that Privalov's lemma leaves open the question of how large the set of points can be where there are no boundary values of a Cauchy-type integral, and much time and effort were needed to solve it.

Note that Cauchy type integrals can be defined not only for functions, but for an arbitrary measure as well. Namely, for a measure μ on Γ let

$$F(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{d\mu(w)}{w - z}, \qquad z \in D \cup D_{\infty},$$

$$C\mu(\zeta) = \lim_{\delta \to 0} \frac{1}{2\pi i} \int_{\Gamma \setminus B(\zeta,\delta)} \frac{d\mu(w)}{w - \zeta}, \qquad \zeta \in \Gamma.$$

In this case, it can be verified that the Sochocki formulas and Privalov's lemma remain valid if we set $f = d\mu/dz$ (the Radon–Nikodym derivative).

Since the definitions of all quantities $F_i(\zeta)$, $F_e(\zeta)$ and $Cf(\zeta)$ are of local nature, one can define these objects in the case when Γ is an arbitrary rectifiable curve in $\mathbb{C}$, not necessarily closed and/or Jordan one. In this case we denote by $F_i(\zeta)$ and $F_e(\zeta)$ the limits of $F(z)$ taking when z approaching ζ from the one side of Γ (from the left-hand side or from the right-hand one with respect to the orientation on Γ, respectively).

A further natural question is as follows: Under what assumptions on a rectifiable curve Γ, do the quantities $F_i(\zeta)$, $F_e(\zeta)$ and $Cf(\zeta)$ exist at almost all points $\zeta \in \Gamma$ for all $f \in L^1(\Gamma)$, or for all finite complex valued Borel measures μ on Γ?

This question turned out to be very difficult, and its solution took a lot of time and many serious efforts. Actually, the deep theory of singular integral operators due to A. P. Calderón and A. Zygmund was created and highly developed because of investigations of this question. The following result was obtained at the end of the 1970s (a hundred years after Sochocki's formulas were obtained)

Theorem 4. *Let Γ be a rectifiable curve in $\mathbb{C}$. For every function $f \in L^1(\Gamma)$ the values $F_i(\zeta)$, $F_e(\zeta)$ and $Cf(\zeta)$ exist for almost all $\zeta \in \Gamma$. The same applies to every bounded complex valued Borel measure μ on Γ.*

This result is the consequence of two celebrated works. The first one is the work by St. Petersburg mathematician V. P. Havin who proved in 1965 (*Sbornik: Mathematics*, vol. 68, issue 4) that the boundary values $F_i(\zeta)$ exist for almost all $\zeta \in \Gamma$ for arbitrary measures provided that they exist for continuous functions. A suitable modifications of Havin's arguments allow us to reduce the case of an arbitrary rectifiable curve to the case of a Lipschitz curve with an arbitrarily small Lipschitz constant. The second work is the famous paper by A. P. Calderón, 1977 (*Proceedings of the National Academy of Science of the USA*, vol. 74). Calderón proved that if Γ is a Lipschitz curve with sufficiently small Lipschitz constant, then the operator C is of weak-type (1, 1) and bounded in $L^p(\Gamma)$ for $1 < p < \infty$, and $Cf(\zeta)$ exists for almost all $\zeta \in \Gamma$ for every $f \in L^p(\Gamma)$, $1 \leqslant p < \infty$.

Describing rectifiable curves Γ such that the operator C is bounded in $L^p(\Gamma)$, $1 < p < \infty$, requires a relatively long explanation to give its comprehensive survey. Nevertheless, we give a very brief outline of the history of this question, because it is closely related to our topic and, moreover, because its initial point is the Calderón work cited above. In 1981 R. R. Coifman, A. McIntosh and Y. Meyer (see *Annals of Mathematics*, vol. 116) showed that the operator C is bounded in $L^2(\Gamma)$ on an arbitrary Lipschitz curve Γ. Let us mention, that if C is bounded in $L^p(\Gamma)$ for some p, or if C is of weak-type (1, 1), then C is bounded in $L^p(\Gamma)$ for all p, $1 < p < \infty$, and it is of weak-type (1, 1). It also can be verified, that if C is bounded in $L^p(\Gamma)$ for some p, $1 < p < \infty$, then Γ is a Carleson curve, that is for every $\zeta \in \Gamma$ and for every $r > 0$ we have

$$\operatorname{Length}(\{w \in \Gamma\colon |w - \zeta| < r\}) \leqslant Ar,$$

where $A > 0$ is a constant that depends only on Γ. The final point of the story was set in 1984 by G. David who proved that C is bounded in $L^2(\Gamma)$ if and only if Γ is a Carleson curve (*Annales scientifiques de l'École Normale Supérieure*, sér. 4, vol. 17).

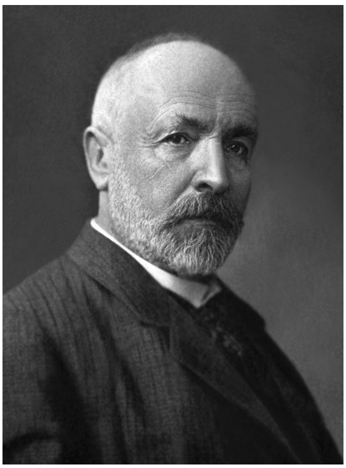
Speaking about the influence of the aforementioned results and constructions, one ought to mention the long-time story of the Painlevé problem about describing the sets of removable singularities for bounded holomorphic functions. We refer the reader to an excellent account of this problem given by M. S. Mel'nikov in 2001 (*Proceedings of the Steklov Institute of Mathematics*, vol. 235), where one can find the interesting explanation of the story and several important references to works by P. Mattila, M. S. Mel'nikov, J. Verdera, X. Tolsa concerning the matter.

Thus, the results obtained by Yu. V. Sochocki in 1873 have found their prominent place in Complex Analysis. Moreover, they can be linked to at least two interesting and important directions in Complex Analysis (namely to the theory of cluster sets and to the theory of boundary properties of Cauchy-type integrals), whose development had a noticeable impact on mathematics of 20th century.

Konstantin Fedorovskiy, Petr Paramonov

Georg Cantor (1845–1918)

Georg Cantor was the creator of the set theory. He was born and spent his childhood years in St. Petersburg. When he was 11, his family left for Germany, where he became a mathematician. Cantor wrote his main works in set theory from 1872 to 1884. In 1878, Cantor formulated the continuum hypothesis; in 1891, he invented the diagonal argument, and in 1895–1897, he created the theory of transfinite numbers.



Georg Cantor spent the first 11 years of his life in St. Petersburg. Three previous generations of his ancestors, who came from Hungary, Bohemia, Denmark, and Portugal, lived in St. Petersburg and worked for the benefit of St. Petersburg culture in the eighteenth and nineteenth centuries. Among his St. Petersburg kin were such bright talents as the violinists Joseph, Franz, and Ludwig Böhm, Maria Böhm-Moravek and Hartwig Meyer; Dimitri Meyer, the creator of Russian civil law; merchants and businessmen.

The future mathematician Georg Ferdinand Ludwig Philipp Cantor was born in St. Petersburg on March 3 (February 19, Old Style) 1845 into the family of Georg Voldemar Cantor and Maria Anna Cantor, née Böhm. His father, who came from Copenhagen Jews, with Portuguese ancestry, was baptized as a Lutheran. He was a stockbroker and trader who settled in St. Petersburg in 1834. Georg Cantor's mother was from a family of musicians of Austrian origin living in St. Petersburg. The mathematician was the first-born, and he was baptized in the same St. Catherine's Lutheran Church (Bolshoi Avenue 1, Vasilievsky Ostrov), where Leonhard Euler was a parishioner in his time.

A letter from Georg Woldemar Cantor to his cousin Dimitri Meyer, written in 1851, is kept in the Manuscript Department of the Russian National Library. In it, he described the "harsh turmoil of his life," the exhausting competition, his apathy, his children, house guests, cold winter evenings, boiled beef for dinner, and boiling samovar.[1] Here is what Georg Woldemar wrote of his eldest son: "The eldest, seven years old, is called Georg Ferdinand Ludwig Philipp; he is naturally gifted with a preference for order over everything else and is sanguine in character."

The family lived on the 11th Line of Vasilyevsky Island in the house of a merchant named Transchel. There is now a memorial plaque on the house. Pafnuty Chebyshev and Osip Somov, young University adjuncts then, lived in that house during the same years. On warm days, the windows were open, and they could hear the Cantor children learning to play the violin under their mother's guidance. Georg Cantor retained his love for the violin throughout his life, and as a student, he even organized a violin ensemble with his friends. Towards the end of his life, Cantor expressed his regret that he did not become a violinist.

In 1854, Georg began his studies at Petrischule (Saint Peter's School), the oldest school in St. Petersburg. Georg liked geography and mathematics best.

Georg Voldemar Cantor's illness (tuberculosis) forced him to ask for a leave to improve his health. His family was supposed to leave for a year. In the spring of 1856 the family left St. Petersburg, as it turned out, forever.

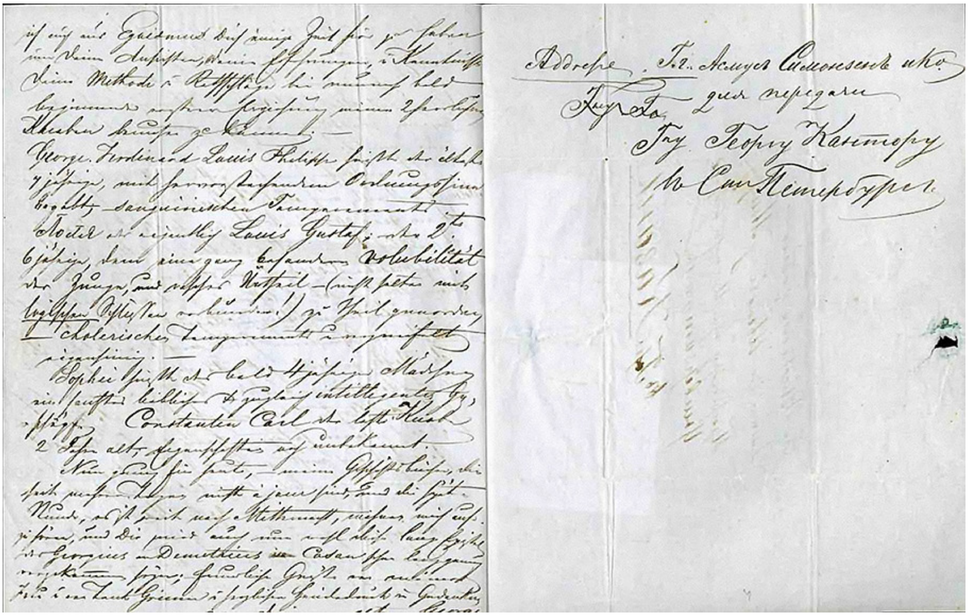
They first moved to Frankfurt, then to Halle. Cantor Sr wanted his son to become an engineer, but he chose mathematics, for which he enrolled at the Polytechnikum in Zürich (ETH Zürich).

After his father's death, Georg Cantor moved to Berlin and graduated from the University of Berlin. His teachers were Ernst Kummer, Leopold Kronecker, and Karl Weierstrass, who had a particular influence on Cantor. The generation of students who graduated in these years included many famous names. Among Cantor's fellow students whom he befriended were Wilhelm Thomé, Franz Mertens, and Hermann Schwarz. Every week, the friends would meet to drink wine and talk about mathematics. Their friendship and correspondence continued for many years.

The summer semester of 1866/67 Cantor spent at the University of Göttingen, where he attended lectures by the philosopher Hermann Lotze, the physicist Wilhelm Weber, and the mathematicians Bernhard von Minnigerode and Ernst Schering. At first, Cantor took an interest in number theory. In 1867, in Berlin, Cantor defended his thesis on number theory *De aequationibus secundi gradus indeterminatis*[2] under the supervision of Kummer, for which Cantor was awarded a degree. For a while, Cantor worked at the Friedrich-Wilhelm Gymnasium for Girls in Berlin. In 1869, Georg Cantor submitted his work on number theory *On the Transformation of Ternary Quadratic Forms*, which he wrote under the guidance of Eduard Heine, to earn the right to

[1] Samovar is a kettle of sorts, which has a distinctive feature: a metal pipe running vertically through the middle, which is filled with solid fuel, which is ignited to heat the water in the surrounding container.

[2] Indeterminate equations of the second-degree.



Fragment of G.W. Cantor's letter.

lecture in Halle. Cantor was interested in problems of analysis, particularly the convergence of trigonometric series. Cantor wrote to his sister:

> The more I look at my mathematics, the more I see that it in my heart and mind leads me to happiness and good fortune. Work has been and will be for me the true meaning of my life and my desire, filled with a physical sensation and a sense of satisfaction, in it I feel that I am free in my activity with regard to the benefit to society, which is a pleasurable opportunity as well. I believe that this hope is above all linked to Halle, a real and holistic field awaits me there, which corresponds to my work, perhaps there I will gain recognition and my aspirations will find application.

In 1869 Cantor was made a *privatdozent* at the University of Halle and became a lecturer in the mathematical seminar at the Department of Philosophy. He remained at that university for the rest of his life. From 1872 to 1877, he was professor extraordinaire at the Faculty of Natural and Social History, and from 1877 to 1913, he was professor *ordinarius*.

The need for a new understanding of real numbers and continuity was part of the mathematical *zeitgeist* of the nineteenth century. From 1822 onwards, the problem of the convergence of trigonometric series and the uniqueness of the expansion of a function in a series arose in the works by J.-B. Fourier. In 1829, G. Lejeune Dirichlet formulated a sufficient condition for the convergence of trigonometric series. But applications demanded an expansion of the class of functions presentable as convergent Fourier series and, consequently, a deep analysis of the notions of number and continuity.

Eduard Heine began his work in Berlin with Dirichlet, whom he considered his teacher, and continued research of the latter conducted on the convergence of trigonometric series. In 1870 he published the paper *On the Convergence of Trigonometric Series*, in which he stated another sufficient condition for convergence. In pursuit of extending the generality of his results he introduced in 1872 the notion of uniform continuity. Note that the terms "continuity" and "convergence" were then vague and intuitive and were understood by each mathematician in his own way. For example, there were more than seven kinds of uniform convergence. There were no other familiar analysis terms then either: Heine would not have recognized his famous Heine–Borel theorem in its modern form.

In 1870 Cantor proved that for a function continuous on an interval, its representation by a trigonometric series is unique. He extended this result to functions with a finite number of discontinuities. But Cantor wanted to find out whether this uniqueness would remain if the set of discontinuity points is infinite, and it was in connection with this problem that he started the study of subsets of the *number line*(real line), and then of sets in general.

The year 1872 was the birth year of the set theory. Heine, Georg Cantor, and Richard Dedekind published their works on the subject. Cantor's paper was entitled *Über die Ausdehnung eines Satzs aus der Theorie der trigonometrischen Reihen.*[3] This was his first work on trigonometric series, but in it Cantor also considered numbers as limits of what is now called Cauchy sequences and introduced the notion of limit points (accumulation points in modern terminology). At the same time, in 1872, Richard Dedekind introduced his "Dedekind cuts."

Georg Cantor and Richard Dedekind met during a summer holiday in Interlaken in the early 1870s. This acquaintance started a long-standing friendship and correspondence. Cantor continued to reflect on the structure of number sets. He has not yet formed his own terminology ("set", "countability", "power"), but he considered one-to-one correspondence to be the central concept.

In 1873, he wrote to Dedekind about his attempt to establish such a correspondence between positive integers and real numbers, although he noted that the question was of no practical value.

Cantor's ideas and his correspondence with Dedekind led to a construction that was highly appreciated by Weierstrass, who visited Cantor in Berlin on 23 December 1873 and recommended the results be published. This work by Cantor, *Über eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen*[4] was published in 1873. In it, Cantor considers the set of roots of algebraic equations with integer coefficients and establishes a one-to-one

[3] On the extension of a theorem from the theory of trigonometric series.
[4] On a Property of the Collection of All Real Algebraic Numbers.

St. Catherine Lutheran Church, where Cantor was baptized and where Leonhard Euler was a parishioner in his time, Bolshoi Avenue 1, Vasilievsky Ostrov.

correspondence between them and positive integers. In 1878, Cantor published his first major work *Ein Beitrag zur Mannigfaltigkeitslehre*,[5] in which he introduced the concept of equivalence or equinumerosity, proving the existence of a one-to-one correspondence between one-dimensional and multidimensional continuous objects. The continuum hypothesis was formulated for the first time in this paper. In 1879 Cantor was elected a corresponding member of the Society of Sciences in Göttingen. His work *Über einen Satz aus der Theorie der stetigen Mannigfaltigkeiten*[6] was published, in which he continued to establish the correspondence between objects of different dimensions.

In 1879, Cantor published *Über unnendliche lineare Punktmannigfaltigkeiten*[7] in *Mathematische Annalen*, the first in a series of papers. The following parts appeared in 1880, 1882, and 1883. The famous "Fifth Memoir" entitled *Grundlagen einer allgemeinen Mannigfaltigkeitslehre: Ein mathematisch-philosophischer Versuch in der Lehre des Unendlichen*,[8] as well as the last in the series *Über unendliche lineare Puktmannigfaltigkeiten*[9] was also published in 1883.

Ernst Zermelo wrote that the series of works

[5] On the Study of Manifolds.

[6] On a theorem from the theory of continuous manifolds.

[7] On infinite linear point manifolds.

[8] Foundations of a General Theory of Manifolds: A Mathematical-Philosophical Study in the Doctrine of the Infinite.

[9] *On infinite linear point manifolds.* "Linear point manifold" is a certain subset of a real line.

> ...published in several volumes of Annalen in 1879–1884 goes far beyond the limits indicated by its title and actually includes all the results obtained by Cantor in the field of both abstract and applied set theory, in particular the theory of equivalence and cardinality, as well as ordering and ordinal numbers. It also provides a more detailed account of Cantor's theory of irrational numbers (no. 5, § 9) as well as a philosophical polemic with opponents of the actual infinite. In fact, the works in this cycle contain a statement of Cantor's whole set theory; the works written in subsequent years comment on and supplement the theory.

The diagonal argument appeared in 1892 in *Über eine elementare Frage der Mannigfaltigkeitslehre.*[10]

Cantor's subsequent papers were extracts from his letters to Gösta Mittag-Leffler, the editor of *Acta Mathematica.* These are *Sur divers théorèmes de la théorie des ensembles de points situés dans un espace continu a n dimensions,*[11] *De la puissance des ensembles parfaits de points,*[12] and *Über verschiedene Theoreme aus der Theorie der Punktmengen in einem n-fach ausgedehnten stetigen Raume Gn.*[13] Here we find for the first time the statement that any cardinality is an aleph number.

The year 1884 was a difficult one for Cantor. His awareness of his work as a coherent theory clashed with his colleagues' misunderstanding. Cantor dreamed of working at the University of Berlin, but this was opposed by his former teacher, Leopold Kronecker, who chaired the Department of Mathematics at the university. He denounced Cantor's theories, calling him a "corrupter of youth." Kronecker was the founder of constructive mathematics, and Cantor's set theory explored the properties of sets without any concrete representation of them. Cantor took the criticism hard. Mistrustful and prone to depression, he would shut himself away for long periods of time and lose his efficiency after situations that traumatized his psyche.

Cantor tried to conceptualize and defend his investigations of the infinity and wrote a series of philosophical works. The first of them was *Principien einer Theorie der Ordnungstypen. Erste Mitteilung: Auszug eines Schreibens an den Herausgeber.*[14] Mittag-Leffler refused to print this work, and it was first published in 1970.

From 1884 Cantor had to be treated in a psychiatric hospital. Each exacerbation of his illness entailed a change in the direction of his interests. In 1884 he refused to lecture in mathematics and expressed a desire to read philosophy, while at the same time, on the advice of his sister Sophie, he turned

[10] On one elementary question of the doctrine of manifolds.
[11] On various theorems in the theory of sets of points situated in a continuous space of n dimensions, 1883.
[12] On the power of perfect sets of points, 1884.
[13] On various theorems in the theory of point sets in an n-times extended space Gn, 1885.
[14] Principles of a theory of order types. First message: Extract of a letter to the editor.

to the literature of the Elizabethan era, wishing to justify the hypothesis that the real author of the works of Shakespeare was Francis Bacon. He published the results of his research in two articles in 1897. He also drew on the work of the German theologian and visionary Jacob Boehme (1575–1624) and the English thinker John Dee (1527–1609), who was interested in magic. Cantor's lectures in philosophy were not popular, and he abandoned the endeavor.

From this time onwards, his interests shifted to the philosophical grounding of set theory, and he entered into correspondence with philosophers and theologians. Interestingly, although Cantor was a Protestant, he turned to Catholic theologians. This may have been due to the rich literature on infinity, and above all to the works of such a classic of Catholic theology as Thomas Aquinas. Cantor's correspondence on philosophical issues was published in 1886–1888.

In fact, Cantor's entire set theory was expounded in 24 of his papers. In 1889 Cantor was accepted into the German Academy of Natural Sciences (Deutsche Akademie der Naturforscher Leopoldina), which had been based in Halle since 1878. Cantor's authority among mathematicians gradually increased. He was always very kindly treated and appreciated by Karl Weierstrass. In 1890 Cantor initiated the formation of the German Mathematical Society. Cantor became the first chairman of the Society and published the first two volumes of its yearbook. In September 1891 the first meeting of the Society was held in Halle. Meanwhile, the set theory grew in popularity and acceptance among European mathematicians.

In 1890/91 Cantor published *On an Elementary Question of the Theory of Manifolds*, outlining the diagonal argument. In 1895–97 he wrote his last (and incomplete) work *Beiträge zur Begründung der transfiniten Mengenlehre.*[15] In it, he summed up the statement of his theory, finalized and substantiated it. The works in which the theory of transfinite numbers emerged date from 1895–97. In the late 1890s Cantor realized the paradoxes of set theory. In 1896 Cantor discussed them with Hilbert, and in 1897 a paper by Burali-Forti was published, on the paradox that Cantor himself had discovered.

In 1897, Cantor initiated and participated in the First International Congress of Mathematicians in Zürich. At this congress, Adolf Hurwitz expressed his deep admiration for Cantor and his contributions to function theory. The French mathematician Jacques Hadamard also praised the set theory as a necessary research tool.

In September 1903 at the second meeting of the Mathematical Society in Munich Cantor gave a talk on the paradoxes of set theory. However, due to illness, he withdrew as the chairman. In the same year Cantor was again admitted to a mental institution.

[15] Contributions to the Founding of the Theory of Transfinite Numbers.

A year later, in August 1904 Cantor attended the International Mathematical Congress in Heidelberg. At this congress, there were criticisms of set theory, which Cantor took somewhat hard.

In 1913 Cantor retired. World War I broke out. Living conditions became hard, and food shortages followed. Cantor suffered from poverty and malnutrition. An all-European celebration of his 70th birthday was planned for 1915, but the war limited the festivities to a small party at home and the congratulations from his German colleagues. In June 1917, Cantor was for the last time admitted to the Hospital for Nervous Diseases in Halle, where he died of a heart attack on 6 January 1918.

Cantor's ideas gradually spread around the world and gained wide acceptance. His set theory began to play the role of the language of mathematics, but it also aroused wariness and controversy among mathematicians with its paradoxes, open questions, and hypotheses, which subsequently gave rise to various logical and philosophical theories, as well as new fields of mathematics.

From 1909, the Polish mathematician Waclaw Sierpinski was teaching a full course in set theory at the University of Lvov. Russian mathematicians who had been to the universities of Berlin and Göttingen and read *Crelle's Journal* (all universities received it) were introduced to the ideas of set theory.

In 1894 in Odessa Samuil Shatunovsky published a translation of Dedekind's work *Continuity and Irrational Numbers*, and in 1896 *Proof of the existence of transcendental numbers (as per Cantor).*

From 1900 to 1901, at Moscow University, Boleslaw Mlodzeewsky taught a course in the theory of functions of a real variable involving set theory.

In 1904, a student at Moscow University, Pavel Florensky published *On the Symbols of Infinity*, a good exposition of Cantor's theory.

Between 1904 and 1908, Kazan University Press published in several parts lectures by Alexander Vasiliev (1853–1829), a propagandist of set theory, entitled Introduction to Analysis.

In 1905 Samuil Shatunovsky read mathematical analysis at Novorossiysk University in Odessa, using concepts and methods of set theory. The course was lithographed in 1906–1907; it influenced Grigory Fichtenholz, Dmitry Kryzhanovsky, and Igor Arnold. In 1910 the seminar on the theory of functions organized by Dmitri Egorov at the Moscow University began, and with it started the history of the Moscow school of the theory of functions, which was headed by Egorov and Nikolai Luzin. In 1914 three works by Cantor from the *Foundations of a General Theory of Manifolds*, translated by the Russian philosopher and public figure Pavel Yushkevich (1873–1945), were published in St. Petersburg in the 6th issue of New Ideas in Mathematics edited by Vasiliev. It is worth mentioning that Alexander Vasiliev, the Kazan mathematician and popularizer of Cantor's work, also highly appreciated the work of Cantor's uncle, Dimitri Meyer, a professor of law at the universities of

Kazan and St. Petersburg. Vasiliev had two portraits in his house: of Nikolai Lobachevsky and Dimitri Meyer.

Despite the incredible popularity of Cantor's theory, no one was rushing to translate his works into Russian. Suffice it to say that the publication of Cantor's works in Russian was stopped at the insistence of Lev Pontryagin. The translation was done by the well-known mathematician Abram Fet (1924–2007), who lived in Novosibirsk. He translated Cantor's biography written by Adolf Frenkel, as well as all his works.

In 1985, *Works in Set Theory by Georg Cantor* was published by Nauka Publishing House in translations by Fedor Medvedev and Pavel Yushkevich, edited by Adolph Yushkevich and Andrei Kolmogorov.

Galina Sinkevich

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Cantor and the Founding of Set Theory

Georg Cantor (1845–1918) was a pivotal mathematician who, with his seminal work on sets and numbers, ushered in a new way of proceeding in mathematics, one at base infinitary and combinatorial, and brought forth a new field of inquiry, set theory. Steadily driven by mathematical problems, first in ongoing analysis and then in his developing context of transfinite cardinals, Cantor struggled in a classical intensional milieu to articulate and secure an expanding conceptual terrain of infinite sets and limits. That Cantor's conceptualizations and arguments may now be rendered succinctly, as here, is a testament to how his ways of thinking have become commonplace in modern mathematics.

After completing a *Habilitation* in number theory, in 1870, Cantor began working in real analysis, specifically on the uniqueness of trigonometric series. In this, Cantor was soon driven by the necessity (as the mother of invention) to articulate proofs clearly. He *defined* the real numbers — an insurgent move at the time — in terms of Cauchy sequences; he defined for a set P of reals the set P' of its limit points, introducing for the first time an *operation on infinite sets*; and he already considered the possibility of its *transfinite iteration*, taking intersections at limits and indexing with "symbols of infinity": $P^{(\infty)} = \bigcap P^{(n)}$, $P^{(\infty+1)}$, $P^{(\infty+2)}$, $\ldots P^{(\infty\cdot 2)}$, $\ldots P^{(\infty^2)}$, $\ldots P^{(\infty^\infty)}$, $\ldots$

Uncountability and Continuum Hypothesis

Spurred to consider enumerations of real numbers for their own sake, Cantor realized that, while the algebraic real numbers are *countable*, i.e., can be put into one-to-one correspondence with the positive integers, there is a counting reason why there must be transcendental numbers. Set theory was born on 7 December 1873 when Cantor established: *The real numbers are uncountable.*

Cantor's proof was structured with limits à la Cauchy sequences, and can be given a quick modern gloss: Suppose that $\{a_i\}_i$ is a sequence of reals indexed by positive integers i. Define a sequence $\{I_i\}_i$ of closed intervals of reals recursively so that: $a_1 \notin I_1$, and I_{k+1} is a subinterval of I_k such that $a_{k+1} \notin I_{k+1}$. Then, any real in the intersection of the I_i's is not in the sequence.

With this, Cantor began to investigate cardinality for infinite totalities, two such said to have the same *power* if there is a one-to-one correspondence between them. In 1877, he carefully established that $[0, 1]^n$, the n-tuples of real numbers in the unit interval [0,1], has the same power as [0, 1] itself. This seemed to compromise the continuous manifolds studied then, but Dedekind pointed out that the "dimension-number of a continuous manifold remains its first and most important invariant." Cantor agreed, and he and others soon worked on articulating why there are no "continuous" one-to-one correspondences. Eventually, the invariance of dimension was definitively established by the young Brouwer in 1911, thus leading to the field of algebraic topology.

Continuity aside, Cantor had reduced considerations of power to linear manifolds, and in 1878, he contended that "the linear manifolds would consist of two classes" one consisting of manifolds having the same power as the positive integers and the other consisting of manifolds having the same power as the unit interval. This was the Continuum Hypothesis in nascent form, and Cantor, having made prodigious progress to arrive at this *continuum problem* to be resolved, would grapple with it in increasing arithmetical and combinatorial terms raising basic questions of set existence—as we continue to do today.

The Transfinite Numbers

Abstracting enumerations to handle power, Cantor, in his magisterial 1883 *Grundlagen* set out the *transfinite numbers* and the key concept of *well-ordering*. Well-orderings carry the sense of sequential enumeration; the transfinite numbers, no longer the contrivance of "symbols of infinity", became numerals for gauging well-orderings; and in a new notation, the numbers themselves could be sequentially cast: $0, 1, 2, \ldots, \omega, \omega+1, \omega+2, \ldots, \omega+\omega(=\omega\cdot 2), \ldots, \omega^{\omega}, \ldots \omega^{\omega^{\omega}}, \ldots$

The transfinite numbers were to provide the framework for Cantor's two approaches to the continuum problem, one through power and the other through definable sets of reals. As for the first, Cantor established, with (II) the class of those transfinite numbers gauging *countable* well-orderings, that: (II) is uncountable, yet any subset of (II) is either countable or else has the same power as (II) itself. Thus, Cantor had reduced the continuum problem to showing that (II) and the unit interval have the same power. As for the second, this evolved from Cantor's early engagement with the P' operation. Cantor defined the concept of *perfect* set of reals (non-empty, closed, and containing no isolated points), and established through transfinite iteration of the P' operation that any *closed* uncountable set of reals is the union of a perfect set and a countable set. Since he had shown that a perfect set has the same power as the unit interval, he had reduced the continuum problem to showing that there is a closed set of reals having the same power as (II).

Cantor's 1895–7 *Beiträge* presented his mature theory of the transfinite. He now had *cardinal numbers* and their arithmetic; *ordertypes* of linear orderings; and *ordinal numbers* and their order comparability. He generalized his correlation of tuples of reals with "a few strokes of the pen": $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0\cdot\aleph_0} = 2^{\aleph_0}$.

He characterized the ordertype of the rationals as the countable dense linear orderings without endpoints. He provided a normal form for the ordinal numbers. Although he could now state the Continuum Hypothesis as $2^{\aleph_0} = \aleph_1$, it was evident that he had not made any further progress toward it. Be that as it may, it was the continuum problem that had inspired enumeration structures that would become basic in the further development of set theory. And a specific argument arrived at by Cantor in context would become pivotal.

The Diagonal Argument

Almost two decades after establishing that the reals are uncountable, Cantor in a short 1891 note subsumed the result and moreover affirmed "the general theorem, that the powers of well-defined sets have no maximum", by establishing: *For any set L, the totality of all functions from L into a fixed two-element set is of a higher power than L.* We can, by following his own words, witness the first appearance of the *diagonal argument*:

Taking M to be the totality of all functions: $L \to \{0, 1\}$, assume that there *is* a one-to-one correspondence between L and M, so that "M could be ... thought of in the form of a single-valued function of the two variables x and z [ranging over L], $\phi(x, z)$, such that ... to every element $f(x) \in M$ there corresponds a single determinate value of z such that $f(x) = \phi(x, z)$. But this leads to a contradiction. For if one understands by $g(x)$ the single-valued function of x which takes on only the values 0 and 1 and is different from $\phi(x, x)$ for every value of x, then on the one hand $g(x)$ is an element of M, and on the other hand $g(x)$ cannot arise from any value $z = z_0$ of $\phi(x, z)$, because $\phi(z_0, z_0)$ is different from $g(z_0)$."

With this argument, a new simplicity was achieved, made possible by the assumption of universality, the positing beforehand of the totality of *all* functions of a certain sort. Once appreciated, the argument would become technique, and detached from its initial moorings it would undergird the development of mathematical logic itself.

Lebesgue in 1905 deployed the diagonal argument to develop a transfinite hierarchy for the Borel sets and the Baire functions. This diagonal approach to hierarchy would be deployed into the 1930s by Luzin and his Moscow school of descriptive set theory in their investigation of "regularity" properties like Cantor's perfect set property. In 1901, Bertrand Russell analyzed Cantor's diagonal argument applied to the class of all classes and came up with the famous Russell's paradox. In reaction, he built a complex logical structure, the ramified theory of types, in which mathematics was developed in 1910–13 *Principia Mathematica*. The young Gödel, after reading in *Principia* about a hierarchy of truth and a paradox, Richard's, based on the diagonal argument, established in 1930 his celebrated Incompleteness Theorems, the crux being a positive use of Richard's paradox. In the 1930s, recursion theory would be developed out of Gödel's work with the diagonal argument ever in play, to a truly remarkable fixed-point application, Kleene's 1938 Second Recursion Theorem. By then, Gödel had developed the inner model of constructible sets as a transfinite extension of Russell's ramified theory of types, and showed that in the model the Continuum Hypothesis holds. So much, then, can be seen as having been driven forth by Cantor's diagonal argument, all the way to the consistency of his Continuum Hypothesis.

Even though Cantor's continuum problem remains unresolved, its continuing investigation is central to Set Theory today. Of course, there is no possible formal solution by the results of Gödel and Cohen, but Cantor would be the first to claim that this does not mean the problem has no answer. There have been notable successes; for example, in the case of the natural extension of the Borel sets to the field of sets generated by Borel sets under continuous images and complements, the continuum problem has been resolved by the unexpected influence of the existence of very large infinite cardinalities. This, in turn, has completed the axiomatization of Second Order Number Theory.

Akihiro Kanamori and W. Hugh Woodin

Egor Ivanovich Zolotarev (1847–1878)

The scientific career of Egor Ivanovich Zolotarev lasted for only ten years, but his record of achievements is impressive. He made essential contributions to the divisibility theory of the Gaussian integers, to the theory of quadratic forms with integral coefficients, and to approximation theory.

Zolotarev was born on March 31, 1847, in St. Petersburg into the family of a watchmaker. In 1863, he graduated from Gymnasium No 5 in St. Petersburg with a silver medal and began attending the Faculty of Physics and Mathematics of St. Petersburg University as an auditor (he did not become a full-time student until 1864 because of his age).

Zolotarev had developed an interest in mathematics as a gymnasium student under the influence of his teachers, the renowned Russian pedagogues Konstantin Krayevich and Alexander Belyaev. The lectures and advice of Alexander Korkin and Pafnuty Chebyshev helped him develop his talent further.

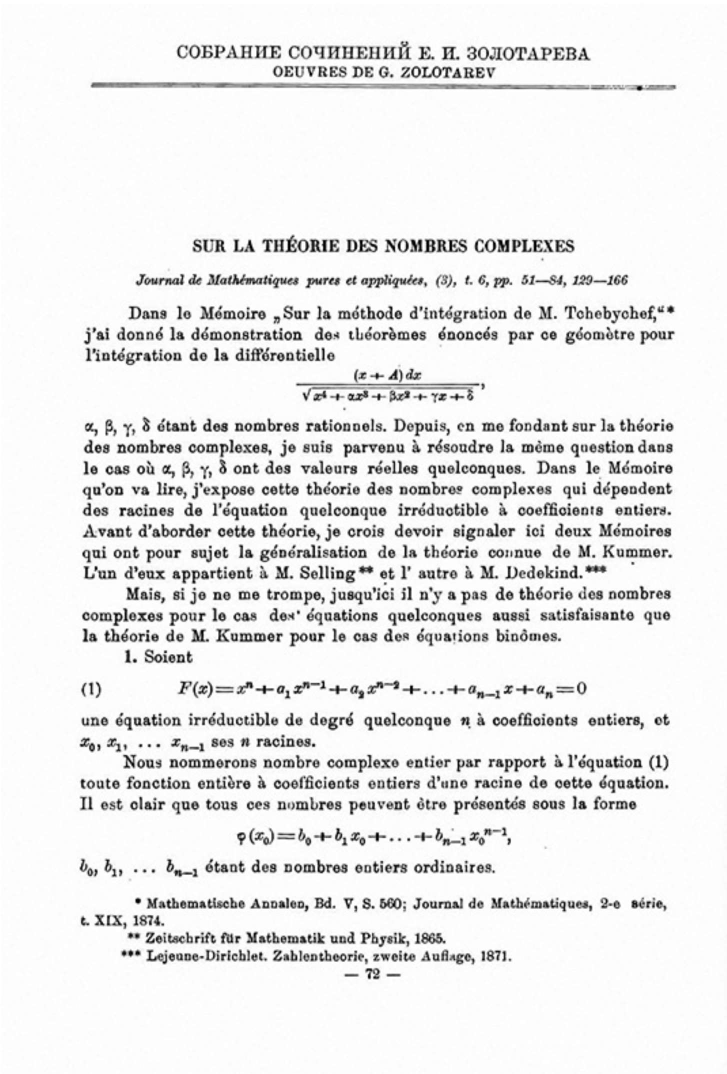
In 1867, upon graduation with a candidate degree[1] (his thesis was *About the Integration of Gyroscope Equations*) he was invited to continue his studies at the University in preparation for the examination for the master's degree. At the age of 21, he was given permission to lecture on mathematics as a Privatdozent after the public defense of his *pro venia legendi* thesis *On One Question of Minima*.

Chebyshev suggested the core problem of the thesis. Zolotarev generalized the results obtained by his famous teacher and presented a solution to a broader problem using elliptic functions. Zolotarev defended his doctoral thesis *Theory of Complex Integers with Application to Integral Calculus* at St. Petersburg University in 1874; his opponents were Chebyshev and Korkin. In his thesis, Zolotarev solved, as Korkin put it, "... two problems at which the analysis had

[1] At the time, the candidate degree was awarded to students graduating with honors.

stopped": he expanded Kummer's theory of ideal numbers to complex numbers and their application to integral calculus.

Zolotarev built, parallel to Dedekind and independently of him, the theory of divisibility (and a proper definition!) for algebraic integers and their factorization into ideal numbers (prime ideals), see the illustration below. The


СОБРАНИЕ СОЧИНЕНИЙ Е. И. ЗОЛОТАРЕВА
OEUVRES DE G. ZOLOTAREV

SUR LA THÉORIE DES NOMBRES COMPLEXES

Journal de Mathématiques pures et appliquées, (3), t. 6, pp. 51—84, 129—166

Dans le Mémoire „Sur la méthode d'intégration de M. Tchebychef,"* j'ai donné la démonstration des théorèmes énoncés par ce géomètre pour l'intégration de la différentielle

$$\frac{(x+A)\,dx}{\sqrt{x^4+\alpha x^3+\beta x^2+\gamma x+\delta}},$$

$\alpha, \beta, \gamma, \delta$ étant des nombres rationnels. Depuis, en me fondant sur la théorie des nombres complexes, je suis parvenu à résoudre la même question dans le cas où $\alpha, \beta, \gamma, \delta$ ont des valeurs réelles quelconques. Dans le Mémoire qu'on va lire, j'expose cette théorie des nombres complexes qui dépendent des racines de l'équation quelconque irréductible à coefficients entiers. Avant d'aborder cette théorie, je crois devoir signaler ici deux Mémoires qui ont pour sujet la généralisation de la théorie connue de M. Kummer. L'un d'eux appartient à M. Selling** et l' autre à M. Dedekind.***

Mais, si je ne me trompe, jusqu'ici il n'y a pas de théorie des nombres complexes pour le cas des équations quelconques aussi satisfaisante que la théorie de M. Kummer pour le cas des équations binômes.

1. Soient

$$(1) \qquad F(x)=x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n=0$$

une équation irréductible de degré quelconque n à coefficients entiers, et $x_0, x_1, \ldots x_{n-1}$ ses n racines.

Nous nommerons nombre complexe entier par rapport à l'équation (1) toute fonction entière à coefficients entiers d'une racine de cette équation. Il est clair que tous ces nombres peuvent être présentés sous la forme

$$\varphi(x_0)=b_0+b_1x_0+\ldots+b_{n-1}x_0^{n-1},$$

$b_0, b_1, \ldots b_{n-1}$ étant des nombres entiers ordinaires.

* Mathematische Annalen, Bd. V, S. 560; Journal de Mathématiques, 2-e série, t. XIX, 1874.

** Zeitschrift für Mathematik und Physik, 1865.

*** Lejeune-Dirichlet. Zahlentheorie, zweite Auflage, 1871.

— 72 —

Zolotarev had sent this paper to Henri Résal (the editor of the *Liouville journal* after the death of Liouville) in 1876 but it was published only in 1880. Meanwhile, the works of Dedekind were published (1879–1881) and mathematicians paid little attention to the works of Zolotarev, however containing cases not covered by Dedekind's works.

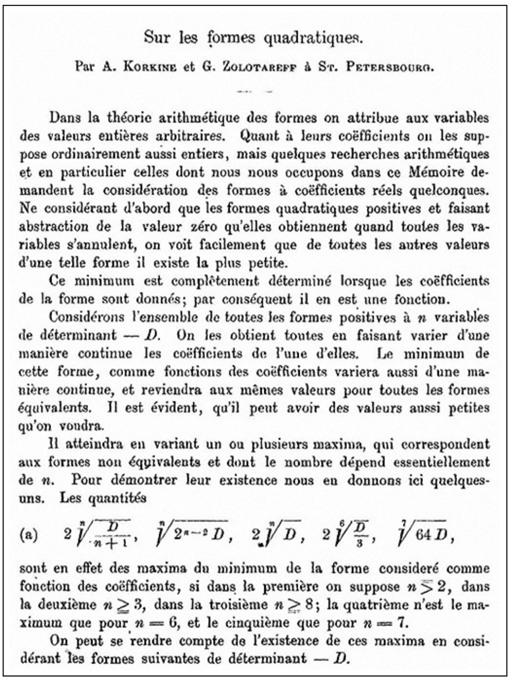
starting point was the question of integrability in logarithms of the elliptic differential

$$\int \frac{(x+A)dx}{\sqrt{x(x-1)(x-\alpha)(x-\beta)}}$$

which is integrable as long as α and β are algebraic integers. This theory extended Kummer's results for cyclotomic polynomials to arbitrary polynomials.

Korkin and Zolotarev found the minimal value of a quadratic form in four variables as a function of the form determinant and identified in the process a quadratic form associated with the famous E_8 lattice.

While dealing with the function approximation problem, Zolotarev found polynomials that are the least deviations from zero, with the first two of their coefficients fixed.



Sur les formes quadratiques.

Par A. KORKINE et G. ZOLOTAREFF à ST. PETERSBOURG.

Dans la théorie arithmétique des formes on attribue aux variables des valeurs entières arbitraires. Quant à leurs coëfficients on les suppose ordinairement aussi entiers, mais quelques recherches arithmétiques et en particulier celles dont nous nous occupons dans ce Mémoire demandent la considération des formes à coëfficients réels quelconques. Ne considérant d'abord que les formes quadratiques positives et faisant abstraction de la valeur zéro qu'elles obtiennent quand toutes les variables s'annulent, on voit facilement que de toutes les autres valeurs d'une telle forme il existe la plus petite.

Ce minimum est complètement déterminé lorsque les coëfficients de la forme sont donnés; par conséquent il en est une fonction.

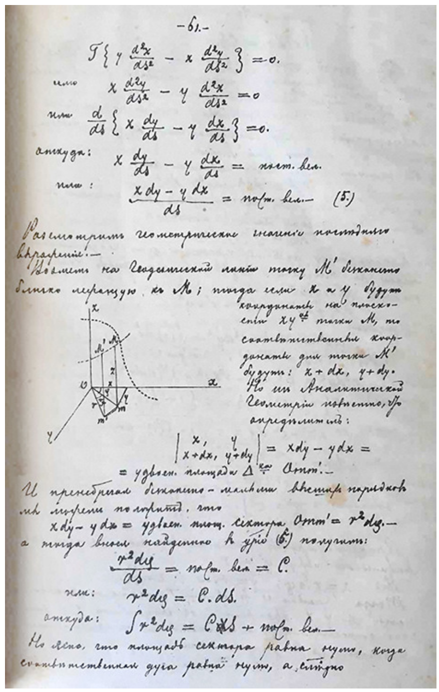
Considérons l'ensemble de toutes les formes positives à n variables de déterminant — D. On les obtient toutes en faisant varier d'une manière continue les coëfficients de l'une d'elles. Le minimum de cette forme, comme fonctions des coëfficients variera aussi d'une manière continue, et reviendra aux mêmes valeurs pour toutes les formes équivalents. Il est évident, qu'il peut avoir des valeurs aussi petites qu'on voudra.

Il atteindra en variant un ou plusieurs maxima, qui correspondent aux formes non équivalents et dont le nombre dépend essentiellement de n. Pour démontrer leur existence nous en donnons ici quelques-uns. Les quantités

$$\text{(a)}\qquad 2\sqrt[n]{\frac{D}{n+1}},\quad \sqrt[n]{2^{n-2}D},\quad 2\sqrt[n]{D},\quad 2\sqrt[6]{\frac{D}{3}},\quad \sqrt[7]{64D},$$

sont en effet des maxima du minimum de la forme consideré comme fonction des coëfficients, si dans la première on suppose $n \gtreqless 2$, dans la deuxième $n \geqq 3$, dans la troisième $n \geqq 8$; la quatrième n'est le maximum que pour $n = 6$, et le cinquième que pour $n = 7$.

On peut se rendre compte de l'existence de ces maxima en considérant les formes suivantes de déterminant — D.



Left: The first page of the paper about E_8. Right: A page in a lithographic Zolotarev's lectures on mechanics.

It is worth mentioning that Zolotarev came up with one of the shortest and most elegant proofs of the law of quadratic reciprocity. Zolotarev began teaching at the age of 20, first at the St. Petersburg Construction School (1867–1871), then as a Privatdozent (1869) and as a professor (1870–1878) at the St. Petersburg Institute of Railway Engineers. At the same time, he also taught at St. Petersburg University as a Privatdozent (1868–1874, until he was awarded a doctorate), then as a docent for two years (1874–1876), and for further two years as a professor (1876–1878). He published textbooks on analytic geometry, differential calculus, and analytical dynamics. Being an excellent lecturer, Zolotarev was able to arouse interest in science among his students and unerringly chose and strongly supported the most gifted of them.

Many of Zolotarev's scientific writings opened new research directions for his students and disciples. His findings in the theory of algebraic numbers were further explored in the work of Yulian Sochocki, Dmitry Grave, and Nikolai Chebotarev. Zolotarev's work on the theory of quadratic forms (written both in collaboration with Alexander Korkin and independently) led to research by his talented disciples Vladimir and Andrei Markov, Ivan Ptashitsky, Georgy Voronoy and, subsequently, by their students. Zolotarev's approach to the integrability of algebraic functions via ideal numbers was crucial for the consolidation of the results obtained by Abel and Chebyshev in the field of integrability of algebraic functions in a finite form.

The Russian academic community highly esteemed Zolotarev's scientific. At the age of 29, he was elected an adjunct member of the Academy of Sciences in Applied Mathematics (1876) and, in two years, was named an extraordinary academician. The works of the young scientist became known in Europe in the mid-1870s after his trips to Berlin and Paris, where he met Weierstrass, Neumann, Hermite, and Kummer and published several articles in Western academic journals such as *Mathematische Annalen*, *Nouvelles Annales*, and *Journal de Mathématiques pures et appliquées*. In addition, Zolotarev sought to establish ties between scientists from different schools of thought both in Russia and abroad. Alexander Korkin and Egor Zolotarev were the first Russians to become members of the editorial staff of the review journal *Jahrbuch über die Fortschritte der Mathematik* (from 1873), and their publications made the achievements of Russian mathematicians known to their Western colleagues.

An accident led to the death of the young scientist. On the way to a family country house he fell under a train and in a few days, on July 7, 1878, died of septicemia (blood poisoning) at the age of 31. Zolotarev was buried in the Mitrofanievskoye Cemetery in St. Petersburg (it was demolished in the XXth century). He was an extraordinary person; his life was immersed in science and, at the same time, burdened by family obligations, as he had to support his mother and four younger siblings, three sisters and a brother. After his father's death, he became the breadwinner in the family, and every year, he would rent a country house for them. When he died, his family was left with no means of subsistence, so St. Petersburg University paid them an allowance of 1000 roubles.

His talent, the depth of his mathematical discoveries, his wholehearted devotion to science, and his tragic death resemble the life of Niels Henrik Abel, who also passed away as a young man. Egor Zolotarev, according to Charles Hermite, was "a great mathematician whose work will remain in science."

Natalia Lokot

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Egor Zolotarev and the Quadratic Reciprocity Law

The quadratic reciprocity law belongs to the class of theorems that continue attracting mathematicians' attention for centuries. Its particular cases, obtained as experimental observations, were mentioned in the middle of the 17th century in Pierre Fermat notes and his correspondence with Frénicle. In 1775 Lagrange, and in 1783 Euler formulated the reciprocity law as a series of propositions. The first incomplete proof was given by Legendre in 1788. He also introduced the symbol that now bears his name[1], and stated the theorem in its modern form (1798). Actually, even the name "quadratic reciprocity law" was also proposed by Legendre. The first complete proof was published by Carl Friedrich Gauss in 1801 [2]. The mathematical heritage of Gauss contains at least eight different proofs.

The quadratic reciprocity law served as the starting point for many generalizations. In modern mathematics one can find Eisenstein's and Weyl's reciprocity laws, power reciprocity laws, and many others. A whole branch of number theory — class field theory — grown from the same root. Over the past two centuries, numerous proofs of the classical law have continued to emerge.

The book [3], which seems to be the most complete subject guide, mentions more than three hundred proofs. The last of them had appeared just a year before the book was published. Certainly, not all of these proofs present fundamentally new ideas. Many are merely kind of variations on a theme.

In 1872, Egor Zolotarev published a proof [4], in which the connection between the quadratic residue symbol and permutations was explicitly used for the first time. In the list of proofs from [3, pp. 132–139] it is placed at number 49. Since that time, as can be seen from the same list, at least twelve proofs using this idea of Zolotarev have been published.

As we will see below, Zolotarev's principal lemma is equivalent to Gauss' lemma from his third proof (1808) [5]. However, Zolotarev's approach makes the proof more intuitive. Another its advantage is that the proof can be generalized without any trouble to the case of finite fields.

Let us recall to the reader what, actually, we are talking about. Let us first fix some notation. All mentioned numbers are assumed to be integers.

[1] This historical fact breaks Stigler's eponymy law [1], also well known as Arnold's principle.

Letters p and q always denote odd primes. We will also frequently denote $\frac{p-1}{2}$ by $\bar{p}$, and so on. The complete modulo p *non-zero* residue system is assumed to consist of the numbers $\{1, \ldots, p-1\}$, the half residue system — of the numbers $\{1, 2, \ldots, \bar{p}\}$. Below, speaking of residue systems, we will usually omit the word "non-zero", since the class 0 barely participates in our reasoning.

The two-term quadratic congruence $x^2 \equiv a (\text{mod } p)$ for $(a, p) = 1$ plays the key role in modular arithmetic. The Legendre symbol $\left(\frac{a}{p}\right)$ takes values ± 1 depending on the solvability of this congruence. In these cases, we will call the number a a quadratic *residue* or *nonresidue* modulo p. Formally extending the symbol at zero by the relation $\left(\frac{0}{p}\right) = 0$, we obtain a character on the complete modulo p residue system.

Strictly speaking, the quadratic reciprocity law is the identity

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}.$$

However, in books, next to this statement one can usually find the calculation of symbols $\left(\frac{-1}{p}\right)$ and $\left(\frac{2}{p}\right)$ as well as the proof of the symbol multiplicativity property $\left(\frac{a}{p}\right)\left(\frac{b}{p}\right) = \left(\frac{ab}{p}\right)$. The practical significance of these statements is that they enable us to calculate the Legendre symbol easily using an algorithm similar in structure and efficiency to the Euclidian algorithm. We will discuss these properties altogether, calling them collectively *the quadratic reciprocity law.*

The path of proof followed by Zolotarev naturally breaks down into several steps. We give the reader different variants of proofs for the key statements. Thus, by choosing the preferred method at each stage, one can get many proofs of the reciprocity law, including the original proof of Zolotarev.

Let us start with the well-known Euler criterion.

Proposition. *For a such that $(a, p) = 1$, the relation* $\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} (\text{mod } p)$ *holds.*

Proof. Every non-zero modulo p residue satisfies either the congruence $a^{\bar{p}} \equiv 1 (\text{mod } p)$ or $a^{\bar{p}} \equiv -1 (\text{mod } p)$. This is obvious from Fermat's little theorem and the factorization $a^{p-1} - 1 = (a^{\bar{p}} - 1)(a^{\bar{p}} + 1)$. Since each of these congruences has at most $\bar{p}$ solutions (actually, exactly $\bar{p}$), it suffices to verify that every quadratic residue a satisfies the congruence $a^{\bar{p}} \equiv 1 (\text{mod } p)$. Namely, let $a \equiv b^2$. Then, $a^{\bar{p}} \equiv b^{p-1} \equiv 1 (\text{mod } p)$. □

The Euler statement also easily follows from the fact that the multiplicative group of p residues is cyclic.

Proof 2. Let us fix a primitive root g. Obviously, every quadratic residue is equal to an even, and every quadratic non-residue to an odd power of g. If a is a quadratic residue, the number $A = a^{\bar{p}} = g^{2k\bar{p}} = g^{k(p-1)}$ satisfies the congruence $A \equiv 1$. Assume now that a is a quadratic non-residue. The congruence $A \equiv 1$ would immediately imply that $p-1$ is a divisor of $k\bar{p}$ for odd k, leading to contradiction. Since $A^2 \equiv 1$, the only case left is $A \equiv -1$. □

Since the numbers a and p are coprime, the transformation $x \mapsto ax$ makes a permutation (bijection) σ_a on the complete modulo p residue system. □

Lemma (Zolotarev)**.** *The Legendre symbol* $\left(\frac{a}{p}\right)$ *equals to the sign*[2] *of the permutation* σ_a*.*

Proof. Consider the product

$$P(a) = \prod_{1 \le i < j \le p-1} (aj - ai) \bmod p.$$

The expressions $P(1)$ and $P(a)$ are different only by the order of factors and, possibly, by their signs. One can easily see that every transposition changes the product sign. This implies the equality $\operatorname{sign} \sigma_a = P(a)/P(1)$. On the other hand, the products $P(1)$ and $P(a)$ are different by the factor

$$a^{\frac{(p-2)(p-1)}{2}} \equiv a^{p-1} a^{\frac{(p-2)(p-1)}{2}} = a^{\frac{p(p-1)}{2}} \equiv a^{\bar{p}} \equiv \left(\frac{a}{p}\right)$$

that completes the proof. □

Proof 2. Again, the alternative proof of the lemma is based on the existence of a primitive root g modulo p. Indeed, multiplying all terms of the sequence of powers $(g^0, g^1, \ldots, g^{p-2})$ by g^k leads to its cyclic shift by k. Since $p-1$ is even, the sign of the obtained permutation[3] coincides with $(-1)^k$. □

Remark. For an arbitrary (not necessarily prime) integer m and a coprime number a, one can define the symbol $\left(\frac{a}{m}\right)$ as a sign of the permutation $x \mapsto ax$ of the complete modulo m residue system. Checking, with the help of Zolotarev's lemma, the necessary list of properties, one can show [6] that this new symbol coincides with the classical Jacobi symbol. In a similar way one obtains an analogue of Zolotarev's lemma for finite fields.

The following properties are immediate consequences of the lemma:

- $\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}}$;

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[2] Let us recall that an inversion in a permutation σ is a pair $\sigma(i) > \sigma(j)$ such that $i < j$. Minus one to the power of the number of inversions in the permutation σ is called its sign. Permutations bearing sign $+/-$ are usually called even/odd.

[3] This permutation is conjugated in the symmetric group to the permutation from Zolotarev's lemma. Hence, they have the same signs.

- $\left(\dfrac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$.

The first equality and the multiplicativity property of the Legendre symbol directly follow from the Euler formula. Let us look at the second one. Compute the sign of corresponding Zolotarev's permutation $(2, 4, \ldots, p-1, 1, 3, \ldots, p-2)$.

All inversions in this permutation are only between odd-even pairs of numbers. The number $2k+1$ totally participates in $\bar{p}-k$ inversions. Adding all of them, we get the total number of inversions $\frac{1}{2}\bar{p}(\bar{p}+1) = \dfrac{p^2-1}{8}$ that calculates the desired symbol. □

Now, we look at the transformation $x \mapsto ax$ from a slightly different point of view.

Let a act by multiplication on the modulo p half residue system $\{1, 2, \ldots, \bar{p}\}$. Every number ax can be then written as $\pm y$, where $1 \le y \le \bar{p}$. As a result, we again obtain the half-system supplied with plus/minus signs. Denote the number of minus signs by μ_a. The following statement holds.

Lemma (Gauss's lemma). $\left(\dfrac{a}{p}\right) = (-1)^{\mu_a}$, *provided that* $(a, p) = 1$.

Proof. Let us consider the products $\prod_{x=1}^{\bar{p}} x$ and $\prod_{x=1}^{\bar{p}} ax$. As we have already seen, these products (modulo p) are different only by the sign $(-1)^{\mu_a}$. On the other hand, they differ by the factor $a^{\bar{p}} \equiv \left(\dfrac{a}{p}\right) (\mathrm{mod}\ p)$. □

It is easy to show the equivalence of Gauss' and Zolotarev's lemmas. We call a $(p-1)$-permutation σ *mirror* if for any k satisfying the inequality $1 \le k \le \bar{p}$ the conditions $\sigma(k) \le \bar{p}$ and $\sigma(k) + \sigma(p-k) = p$ are fulfilled. Such a permutation is always even, since the numbers lying in its different halves make no inversions, and inversions of $\sigma(i)$ and $\sigma(j)$ correspond one-to-one to the inversions of $\sigma(p-i)$ and $\sigma(p-j)$. Consider the Zolotarev permutation $(a, 2a, \ldots, (p-1)a)$. Note that since $ka + (p-k)a \equiv 0 (\mathrm{mod}\ p)$, to make this permutation mirror, it is necessary to transpose all the elements in the pairs $(ka, (p-k)a)$ where the inversion appears. But we have exactly μ_a such pairs. Therefore, the sign of the permutation $x \mapsto ax$ equals to $(-1)^{\mu_a}$.

Now, we pass to the proof of the quadratic reciprocity law.

Theorem. *Let* p *and* q *be odd primes. Then, the following equality holds.*

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}.$$

Proof. Consider the set $M = \mathbb{Z}_p \times \mathbb{Z}_q$ and construct three bijections between M and the interval set $\{0, 1, \ldots, pq-1\}$.

One can visualize these bijections as filling the $p \times q$ matrix[4] with numbers from 0 to $pq-1$ in three different ways

- by rows: $R_{ij} = qi + j$;
- by columns: $C_{ij} = i + pj$;
- and diagonally: $\begin{cases} D_{ij} \equiv i (\text{mod } p), \\ D_{ij} \equiv j (\text{mod } q). \end{cases}$

The last method gives us a bijection by the Chinese Remainder theorem.

We will denote by τ_{RD} the permutation taking the arrangement R to D, and so on. Obviously,

$$\tau_{RC} = \tau_{DC} \circ \tau_{RD}. \tag{$*$}$$

Surprisingly, this naive equality already encodes the proof of the reciprocity law. Really, let us compute the signs of all the permutations involved. The transformation τ_{RD} moves the entry (i, j) to $(qi + j, j)$ and, therefore, makes the column index invariant.

The permutation in the column number k is the cyclic k-shift of the 0th column permutation $i \mapsto qi$. By the oddness of p, signs of all the column permutations are the same. By Zolotarev's lemma, they equal to $\left(\frac{q}{p}\right)$.

Hence, $\operatorname{sign} \tau_{RD} = \left(\frac{q}{p}\right)^q = \left(\frac{q}{p}\right)$. Similarly, $\operatorname{sign} \tau_{DC} = \left(\frac{p}{q}\right)$.

Applying the function sign to ($*$), we obtain the equality:

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = \operatorname{sign} \tau_{RC}.$$

To complete the proof, it remains to calculate the sign of the permutation τ_{RC}. The general principle shall be clear from the example below.[5] Consider the transformation of the 3×5-matrix from row to column arrangement.

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 & 9 \\ \hline 10 & 11 & 12 & 13 & 14 \end{pmatrix} \xrightarrow{\tau_{RC}} \begin{pmatrix} 0 & 3 & 6 & 9 & 12 \\ 1 & 4 & 7 & 10 & 13 \\ 2 & 5 & 8 & 11 & 14 \end{pmatrix}$$

Clearly, under such a transformation, each entry is transposed with ones, which are below and to the left of it in the original matrix. (In the matrix above, we marked the inversions for the element 8.) Thus, the total number of inversions for the $p \times q$ matrix is

$$\sum_{m=0}^{q-1} \sum_{n=0}^{p-1} mn = \left(\sum_{m=1}^{q-1} m\right)\left(\sum_{n=1}^{p-1} n\right) \equiv \frac{p-1}{2}\frac{q-1}{2} (\text{mod } 2).$$

[4] We number the rows and columns of the matrix starting from zero.

[5] In [7] one can find a nice interpretation of this calculation as a card trick.

The resulting equality $\operatorname{sign}\tau_{CR} = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}$ proves the quadratic reciprocity law. □

Proof 2. The following argument is due to Rousseau [8] and is a deep modification of Gauss' fifth proof. Denote by $\mathbb{Z}_m^*$ the multiplicative group of modulo m reduced residues. Compute the product π of elements in the factor group[6] $(\mathbb{Z}_p^* \times \mathbb{Z}_q^*)/\mathbb{Z}_2$ in two different ways. As a set of coset representatives, one can take the set of pairs $\{(i, j)|\ i = 1, \ldots, \bar{p},\ \ j = 1, \ldots, q-1\}$. Obviously, the product of all elements in the factor group is equal to $((\bar{p}!)^{q-1}, (q-1)!^{\bar{p}})$. Taking into account the identity $(\bar{p}!)^2 = (-1)^{\bar{p}}(p-1)!$, one gets:

$$\pi = ((-1)^{\bar{p}\bar{q}}(p-1)!^{\bar{q}}, (q-1)!^{\bar{p}}). \qquad (**)$$

On the other hand, using the isomorphism $\mathbb{Z}_{pq}^* \simeq \mathbb{Z}_p^* \times \mathbb{Z}_q^*$ (Chinese Remainder Theorem), one can choose the complete set of coset representatives as the image in the group $\mathbb{Z}_{pq}^*$ of integers from 1 to $\frac{1}{2}(pq-1)$ coprime to pq. Let us first calculate the product of such integers. It is obviously equal $\frac{(\frac{pq-1}{2})!}{p^{\bar{q}}\bar{q}!q^{\bar{p}}\bar{p}!}$. Reducing the fraction by modules p and q, correspondingly, we get:

$$\pi = \left(\frac{(p-1)!^{\bar{q}}}{q^{\bar{p}}}, \frac{(q-1)!^{\bar{p}}}{p^{\bar{q}}}\right).$$

Thus, taking into account the Euler formula:

$$\pi = \left((p-1)!^{\bar{q}}\left(\frac{q}{p}\right), (q-1)!^{\bar{p}}\left(\frac{p}{q}\right)\right).$$

Comparing the obtained expression with (**), we immediately get the reciprocity law. □

Remark. Applying Wilson's theorem to the latter formula, it is not difficult to prove the following beautiful, although absolutely useless formula:

$$\pi = \left(\left(\frac{-q}{p}\right), \left(\frac{-p}{q}\right)\right).$$

So far, all the considered arguments derived the reciprocity law either from Zolotarev's lemma, or directly from the Euler criterion. In conclusion, we present, for comparison, a proof based on Gauss' lemma. The following beautiful argument is due, with some modifications, to Eisenstein [9].

We start from the statement of the lemma: $\left(\frac{a}{p}\right) = (-1)^{\mu_a}$. Let us note that the residue ax arising in the course of the proof gets a minus sign if and only if $\left\{\frac{ax}{p}\right\} > \frac{1}{2}$. If this inequality is satisfied, the integer $\left[\frac{2ax}{p}\right] - 2\left[\frac{ax}{p}\right]$ is equal

[6] Here the subgroup $\mathbb{Z}_2$ is $\{(1, 1), (-1, -1)\}$.

to one, otherwise it is zero. Therefore,

$$\mu_a = \sum_{x=1}^{\bar{p}} \left(\left[\frac{2ax}{p}\right] - 2\left[\frac{ax}{p}\right] \right) \equiv \sum_{x=1}^{\bar{p}} \left[\frac{2ax}{p}\right] \pmod 2. \qquad (***)$$

Assume that a is odd and compute the Legendre symbol

$$\left(\frac{2a}{p}\right) = \left(\frac{2a+2p}{p}\right) = \left(\frac{4\frac{a+p}{2}}{p}\right) = \left(\frac{\frac{a+p}{2}}{p}\right).$$

By Gauss' lemma: $\left(\frac{\frac{a+p}{2}}{p}\right) = (-1)^{\mu_{\frac{a+p}{2}}}$, and by $(***)$, we have:

$$\mu_{\frac{a+p}{2}} \equiv \sum_{x=1}^{\bar{p}} \left[\frac{(a+p)x}{p}\right] = \sum_{x=1}^{\bar{p}} \left[\frac{ax}{p}\right] + \frac{p^2-1}{8} \pmod 2.$$

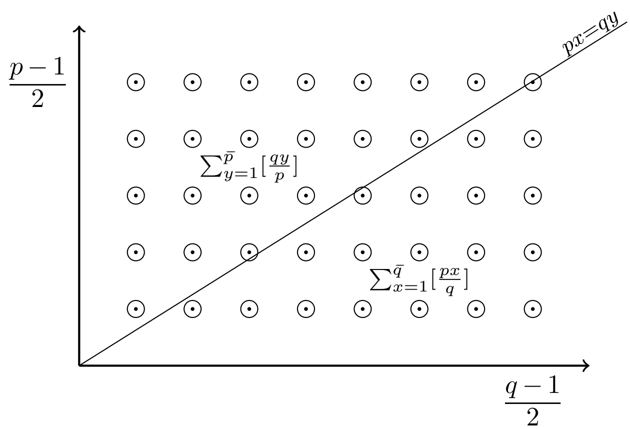
Taking $a=1$ into the last equalities, we are reproving that $\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$. Using this fact and the multiplicativity of the Legendre symbol, we get:

$$\mu_a \equiv \sum_{x=1}^{\bar{p}} \left[\frac{ax}{p}\right] \pmod 2. \qquad (1)$$

Thus, to prove the quadratic reciprocity law, it suffices to verify that for odd primes p and q we have the equality:

$$\sum_{x=1}^{\bar{p}} \left[\frac{qx}{p}\right] + \sum_{y=1}^{\bar{q}} \left[\frac{py}{q}\right] = \frac{p-1}{2}\frac{q-1}{2}.$$

The proof of the latter is obvious from the picture below ($p=11$, $q=17$):



Unfortunately, the limited volume of the text does not allow us to tell more about the various brilliant ideas used in the proofs of the quadratic reciprocity law. For more details and references, we refer the reader to the book [3].

Serge Yagunov

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Nikolay Yakovlevich Sonin (1849–1915)

Nikolay Sonin is known for his achievements in the theory of orthogonal polynomials and the theory of cylinder functions. He devoted much of his time to teaching mathematics and was a member of the International Commission on Mathematical Education organized by Felix Klein.

Nikolay Sonin was the descendant of an old noble family from the Tula province. He was born in Tula, but was moved to Moscow at an early age, graduated from the 4th Moscow gymnasium (1865) and then from the physics and mathematics faculty of Moscow University with a candidate's degree[1] (1869) and was left for two years at university "preparation for professorship."[2]

Both his master's and doctor's theses, *On the Infinite Series Expansion of Functions* (1871) and *On the Integration of Second Order Partial Differential Equations* (1874), respectively, were defended at Moscow University. Sonin started his teaching career in 1871 with secondary-school-level courses for women that were taught at the Second Moscow male gymnasium and based on the curriculum for men's secondary schools.[3] In 1872, he moved to the University of Warsaw, where he worked for 20 years, first as Associate Professor (1872–1877), then as Extraordinary[4] (1877) and Ordinary[5] (1879) Professor and was Dean of the Physics and Mathematics Faculty for six years.

After he was elected a corresponding member (1891) and then a full member of the St. Petersburg Academy of Sciences, Nikolay Sonin lived and worked

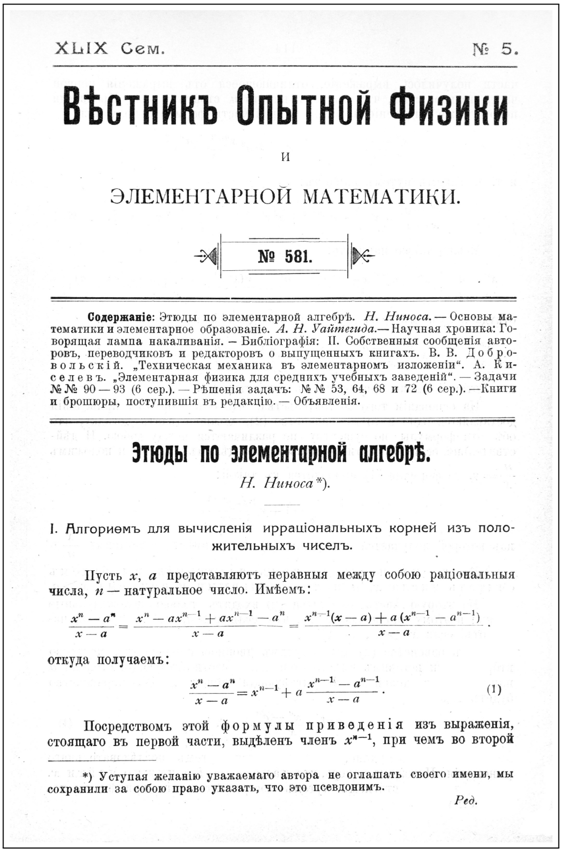
[1] At the time, the candidate's degree was awarded to students who graduated with honors.

[2] The students chosen to stay at the department for a "preparation for a professorship," roughly equivalent to a Ph.D., were those chosen by a professor for a merit scholarship. It was considered a mark of distinction.

[3] The Lyubyansky secondary school courses for women were operational from 1869 to 1886 in Moscow.

[4] Extraordinary professorship was a salaried position, but not as a departmental chair.

[5] Ordinary professorship was a salaried position as the departmental chair.



XLIX Сем. № 5.

Вѣстникъ Опытной Физики

и

ЭЛЕМЕНТАРНОЙ МАТЕМАТИКИ.

№ 581.

Содержанiе: Этюды по элементарной алгебрѣ. *Н. Ниноса*. — Основы математики и элементарное образованiе. *А. Н. Уайтегида*. — Научная хроника: Говорящая лампа накаливанiя. – Библiографiя: II. Собственныя сообщенiя авторовъ, переводчиковъ и редакторовъ о выпущенныхъ книгахъ. В. В. Добровольскiй. „Техническая механика въ элементарномъ изложенiи“. А. Киселевъ. „Элементарная физика для среднихъ учебныхъ заведенiй“. — Задачи №№ 90 — 93 (6 сер.). — Рѣшенiя задачъ: №№ 53, 64, 68 и 72 (6 сер.). — Книги и брошюры, поступившiя въ редакцiю. — Объявленiя.

Этюды по элементарной алгебрѣ.

Н. Ниноса *).

I. Алгориѳмъ для вычисленiя ирраціональныхъ корней изъ положительныхъ чиселъ.

Пусть x, a представляютъ неравныя между собою рацiональныя числа, n — натуральное число. Имѣемъ:

$$\frac{x^n - a^n}{x - a} = \frac{x^n - ax^{n-1} + ax^{n-1} - a^n}{x - a} = \frac{x^{n-1}(x-a) + a(x^{n-1} - a^{n-1})}{x - a}.$$

откуда получаемъ:

$$\frac{x^n - a^n}{x - a} = x^{n-1} + a\frac{x^{n-1} - a^{n-1}}{x - a}. \qquad (1)$$

Посредствомъ этой формулы приведенiя изъ выраженiя, стоящаго въ первой части, выдѣленъ членъ x^{n-1}, при чемъ во второй

*) Уступая желанiю уважаемаго автора не оглашать своего имени, мы сохранили за собою право указать, что это псевдонимъ.

Ред.

The first page of *Etudes in Elementary Mathematics* (1913), Sonin (under the pseudonym of N. Ninos, the anadrome of “Sonin”), [5].

in St. Petersburg. He taught higher education courses for women[6] and at St. Petersburg University, and took an active part in the daily work of the Academy. Together with Andrei Markov, he edited the collected works of Pafnuty Chebyshev and prepared them for publication. From 1899 onwards, he was engaged in administrative activities as a trustee of the St. Petersburg school district, and from 1901 he was the chairman of the Academic Committee of the Ministry of Public Education. Sonin was characterized as a supporter of strict legality and an opponent of the student movement.

Sonin’s scientific interests were influenced by the St. Petersburg school of mathematics and were concentrated on the study of various special functions and their application to problems in mathematical analysis. For example, his master’s thesis was about the joint generalization of the results obtained

[6] The Bestuzhev Courses in St. Petersburg were the most important providers of higher education for women in Imperial Russia.

by Eduard Heine and Carl Neumann concerning the expansion of fractions $1/(a - z)$ by spherical harmonics and cylinder functions. For his doctoral dissertation, which was translated into German 23 years later by Friedrich Engel, Sonin first solved the problem of the existence of the general integral of the first order, and put in final form the method of integration proposed by Jean Gaston Darboux. Thanks to the recommendations of Pafnuty Chebyshev, Vasily Imshenetsky, and Andrei Markov, Prof. Sonin was awarded the Viktor Bunyakovsky prize for seven works written in 1886–1889.

In total, Sonin wrote about 50 papers, a feat that made him famous in Russia and Western Europe. His main results were in the theory of orthogonal polynomials, the Bernoulli polynomials, and the theory of cylinder functions. His memoir *On Some Inequalities Concerning Definite Integrals* (1898) is of particular importance. In it, Sonin proposed a new orthogonal basis (Sonin–Laguerre) with a parameter a, different from the bases already known (Legendre, Laguerre, Dirichlet, Jacobi, where $a = 1$), and developed a method for the orthogonalization of a system of functions.

Nikolay Sonin's studies have not lost their scientific importance. The main ones have been republished in *Studies on Cylindrical Functions and Special Polynomials* (1954), prepared by Naum Akhiezer. In addition to the articles published by Sonin in Russian, the collection includes hard-to-find translations of his notes published in foreign journals, as well as detailed comments and an article by Akhiezer entitled *Sonin's Works on Approximation of Definite Integrals.*

Sonin's article on the history of mathematics, "Johann Bernoulli's Series" ("Рядъ Ивана Бернулли", [4]), clarifying the role of Johann Bernoulli in the development of the series named after Taylor, attracts attention. Sonin presented arguments for the precedence of Johann Bernoulli's research over that of Taylor and McLaren. In addition, he wrote several biographical essays, notes, and speeches on Viktor Bunyakovsky, Vasily Imchenetsky, Pafnuty Chebyshev, Andrei Markov, Karl Weierstrass, Charles Hermite, Alexander Lyapunov, Ernst Leonard Lindelöf, Vito Volterra, Vladimir Steklov, and others.

As chairman of the Scientific Committee of the Ministry of Public Education and a member of the Council to the Minister of Education, Sonin organized the review of textbooks and auxiliary books for schools, recruiting venerable scholars and the best teachers, and he wrote reviews of many textbooks himself. It was to help schools that he published, under the pseudonym of N. Ninosa, his last work *Etudes in Elementary Mathematics* (1913), which was comprehensible even to high school students, see Figure 24. It contains an algorithm developed by the author for calculating irrational roots of positive numbers, a new way to derive Newton's Binomial, and a simple method for calculating natural logarithms.

Felix Klein chose to send a letter to Sonin, on 19 January 1909, requesting to take part in the work of the International Commission on Mathematical

Education. Sonin initiated the creation of such a commission in Russia, which was composed of, besides Sonin (who was an academician), of Professor Boris Koyalovich, and the director of the 2nd St. Petersburg Scientific College[7] Karl Vogt. The commission's first work was a translation from French of the book *The International Commission for the Teaching of Mathematics: A Preliminary Report on the Organization of the Commission and Its General Plan of Work*, it was published in St. Petersburg in 1909. Later, the commission was joined by Professors Konstantin Posse and Dmitry Sintsov. Here is what is said about the activities of this commission in [3]:

> The National Commission met regularly; leading mathematicians were involved, who represented different stages of education and different types of educational institutions of the largest cities in the country (universities, scientific colleges,[8] cadet corps, etc.), such as K.A. Posse, S.P. Glazenap, M.G. Popruzhenko, V.G. Alekseev and others. Reports on the state of mathematics teaching in Russia were collected and published in European languages abroad. The reports of this commission, and the reports of all the countries involved, were presented at the V International Congress of Mathematicians in Cambridge in 1912.

Nikolay Sonin died of stomach cancer at the age of 66 after a long illness and was buried in the Smolensk cemetery in St. Petersburg. He was not a citizen of St. Petersburg by birth or education, but, having become one of the outstanding mathematicians of the late XIXth–early XXth centuries, he was very close to the St. Petersburg school of mathematics in the subject matter and character of his research.

Natalia Lokot

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[5] https://www.vofem.ru/ru/articles/58601/

[7] This type of scientific college, called a реальное училище in Russian, has no exact equivalent outside of Russia but is similar to the German Realschule, a technical secondary school, only at a basic college level. It could be the equivalent of an Associate's degree in a scientific field.

[8] See footnote above.

Sofia Vasilyevna Kovalevskaya (1850–1891)

Kovalevskaya is known for the Cauchy–Kovalevskaya theorem on the existence of solutions to differential equations. She wrote important and well-known works on Saturn's rings, on Abelian integrals, and on the rotation of a heavy body around a fixed point. The latter work, the last integrable case[1] (Euler and Lagrange found the first and second), won the Prix Bordin of the French Academy of Sciences in 1888 and used the recently developed theory of theta functions to solve hyperelliptic integrals.

Kovalevskaya, being deeply involved in the feminist currents of late nineteenth-century Russian nihilism, wrote a semi-autobiographical novel, *Nihilist Girl*, as well as a memoir, *A Russian Childhood*.

Sofia Kovalevskaya was the first woman to obtain a doctorate in mathematics and the first woman to be appointed to a full professorship in Northern Europe. She was the first woman to be an editor of a mathematical journal (*Acta Mathematica*).

Sofia Vasilyevna Kovalevskaya was born in Moscow, the second daughter of Vasily Vasilyevich Korvin–Krukovsky, a general in the Russian army, and his wife Yelizaveta Fedorovna Schubert, who was twenty years his junior.

As was usual in well-to-do families of the nobility, the children's education began with an English governess and continued with private tutors. Sofia became fascinated with mathematics at an early age. She listened eagerly to the conversations between her father and uncle on mathematical subjects, never mind that she could not understand what they were discussing in detail. It has been reported that her curiosity was piqued by the temporary wallpaper in the children's nursery (in the absence of ordinary wallpaper), with pages from the manuscript of calculus lectures given by Mikhail Ostrogradsky.

Eventually, when one of her tutors initiated systematic instruction in mathematics, Sofia neglected all her other subjects, which led her father to

[1] Strictly speaking, the Goryachev–Chaplygin top (with nonholonomic constraints) is also integrable. (Ed. note)

forbid her from studying mathematics. But Sofia got hold of an algebra book, which she read secretly at night. One day, their neighbor, a physics professor named Nikolai Nikanorovich Tyrtov, gave the family a copy of his new physics textbook. The twelve-year-old Sofia began to read it. She had trouble with the formulas involving trigonometric functions. However, she persisted and figured out their meaning from the context. Her neighbor was stunned when she asked him questions about his book that suggested that she had actually understood what she had read. Tyrtow attempted to convince the girl's father that his daughter must have further instruction in mathematics. But several years passed before the father allowed his daughter to study the subject again. If she were to obtain instruction in "higher" mathematics, she would have to leave home. But an unmarried woman could not do so without her father's consent. Since he was unwilling to give such consent, there was only one way out: marriage[2].

Shortly before her eighteenth birthday, Sofia married Vladimir Kovalevski. It was to be a "fictitious marriage" though, just for the sake of appearances, so that Sofia could pursue her interests. The groom was one of the so-called nihilists, who were active in agitating for the rights of women to obtain an education and saw it a point of honor to "free Russia's daughters." The unsuspecting parents accepted the budding paleontologist as their daughter's husband.

The following year, the newlyweds traveled to Heidelberg, so that Sofia Kovalevskaya could take up studies in mathematics and the natural sciences. However, women could not officially enroll in German universities at the time. After many futile attempts, she was finally allowed to petition individual lecturers to "audit" their lectures. The professors, including Leo Königsberger (a student of Karl Weierstrass), Hermann Helmholtz, and Gustav Robert Kirchhoff, quickly recognized the young woman's exceptional talent. After three semesters, she moved to Berlin, following the recommendation of Königsberger, in order to continue her studies

[2] To travel abroad, a passport and money were required. Young women could obtain a passport with the consent of their parents or husband. It is possible that SofiaXs father might not have refused to issue her a passport even without marriage, as SofiaXs unmarried sister was allowed to go study in Europe together with Sofia and her husband. Sofia also persuaded the parents of her unmarried friend, Y.V. Lermontova, to let her go abroad with her. Additionally, another friend of theirs, A.M. Evreinova, illegally crossed the border without a passport, and her parents, accepting their daughterXs decision, sent her money and a passport to Heidelberg. (Ed. note)

under the supervision of Karl Weierstrass himself, whose lectures on analysis had become renowned for their intellectual rigor.

At first, Weierstrass ignored the letters of recommendation that she produced, but instead gave the "supplicant" a problem that, to his great surprise, she was quickly able to solve. Since the university, despite Weierstrass's petition, would not give Sofia permission even just to audit the lectures, Weierstrass saw only one way to help: he taught her three times a week in private sessions.

In 1874, Sofia Vasilyevna Kovalevskaya completed three papers that Weierstrass judged sufficient to grant a doctoral degree. The first paper brought her research on the solvability of partial differential equations to a provisional conclusion. In the second paper, Kovalevskaya dealt with the so-called Abelian integrals (named in honor of Niels Henrik Abel) and provided methods for reducing these integrals to simpler ones. In the third paper, she improved upon a theory of Pierre Simon de Laplace on the physics of the rings of Saturn.

Weierstrass had a hard time finding a university in Germany that would recognize these works as the basis for granting a doctoral degree. Finally, the University of Göttingen expressed its willingness to do so and granted Kovalevskaya the title of *doctor in absentia*, with the addition of summa cum laude. In support of his application, Weierstrass went so far as to cite the great Carl Friedrich Gauss, who in 1837 expressed his regret that German universities had failed to grant the mathematician Sophie Germain a doctorate during her lifetime. It was many years later before women were granted the right and opportunity to pursue scientific work. For example, the physician and neurologist Paul Möbius insisted that there was no originality in the ideas and scientific work of Sofia Kovalevskaya (this judgment appeared in the chapter "On Women in Mathematics" in a book that Möbius published in 1900 with the title *On the Natural Aptitude for Mathematics*).

When an error was found in one of Kovalevskaya's later articles on the refraction of light, many voices chimed in to claim that such an error could never have been made by a man. In fact, the cause of the error in that article was that she had adopted one of the experimental conditions of another scientist (a man) without having verified the specifics of his work.

Despite her academic title and a number of letters of recommendation from her sponsor, Weierstrass was unable to find a university position for her. This series of rejections and disappointments plunged her into a six-year emotional crisis, in which she ceased work in mathematics. She returned to Russia,

Bazaar in the Stockholm Stock Exchange in 1885. Calla Curmann, Ann-Charlotte Edgren Leffler, Sofia Kovalevskaya, Signe Mittag-Leffler, Anna Scholander, Ellen Key and Alfhild Agrell.

where her academic title, obtained in Germany, was not recognized. She was considered to be qualified at most as a teacher of young girls. What had begun as a fictitious marriage eventually became a real one. She gave birth to a daughter, but then later separated from her husband. His suicide in 1883 first came out as a shock, but it also represented a liberation for her. She returned to the study of mathematics with great intensity to displace her feelings of guilt. As a widow, she was allowed to travel without restrictions, which earlier would have been possible only with the express permission of her father, and later her husband — even after their separation.

A breakthrough arrived for Kovalevskaya when Magnus Gösta Mittag-Leffler, the first professor of mathematics appointed at the newly founded University of Stockholm, a student of Charles Hermite (Paris) and Karl Weierstrass (Berlin), created a five-year position for her as a lecturer in mathematics in Stockholm.

During her first year, she lectured in German but eventually learned enough Swedish to lecture in that language. In 1889, she finally obtained a professorship in mathematics, being the first woman to do so. She gave lectures on mathematical analysis, was an editor of a mathematical journal, and organized international mathematical conferences.

In 1886, she was awarded the Prix Bordin of the French Academy of Sciences, having won a competition to which she had submitted her contribution anonymously. Judging the quality of her work to be unusually high, the jury raised the awarded prize money from 3000 to 5000 francs. In 1889, she won the prize of the Swedish Academy, and following the personal intervention of Pafnuty Lvovich Chebyshev, she was named a corresponding member of the Russian Academy of Sciences, which in conservative tsarist Russia was possible only after a modification in the university statutes. Amid a new and intensive creative period, Sofia Kovalevskaya died of a lung infection that had not been treated in a timely manner.

Heinz Klaus Strick

First published in 2010 by Spektrum der Wissenschaft Verlagsgesellschaft Heidelberg
https://www.spektrum.de/wissen/sofia-kowalewskaja-1850-1891/1017473

English version first published by the European Mathematical Society 2013 (Translation by David Kramer), now linked to MacTutor History of Mathematics Archive
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Reduction of Abelian integrals and Kowalevski top

The level surfaces defined by the four constants of the Kowalevski[1] top motion can be completed into a reducible Abelian variety. We suppose that Kowalevski used this fact in her original calculations.

As early as 1832 Legendre had shown that the two hyperelliptic integrals

$$\int \frac{dx}{\sqrt{x(1-x^2)(1-\varkappa^2 x^2)}} \qquad \text{and} \qquad \int \frac{x\,dx}{\sqrt{x(1-x^2)(1-\varkappa^2 x^2)}}$$

are each expressible in terms of two elliptic integrals of the 1-st kind by means of a quadratic transformation. Abel singled out the first kind integrals in his Paris mémoire by the condition that the corresponding Abelian sum is a constant; this is equivalent to the Abelian integral being a locally bounded function of the upper limit of integration.

Immediately after, Jacobi pointed out that this property holds for integrals on the hyperelliptic curve defined by the equation

$$C: \qquad y^2 = R(x) = x(1-x)(1-\kappa\lambda x)(1+\kappa x)(1+\lambda x)\,,$$

which is isomorphic to a curve with an affine equation

$$C: \qquad y^2 = x^6 - s_1 x^4 + s_2 x^2 - 1\,, \tag{1}$$

having elliptic involutions $\sigma_{1,2}$, see [1]. The quotients $E_i = C/ < \sigma_i >$ are the elliptic curves

$$E_1 : y^2 = x^3 - s_1 x^2 + s_2 x - 1 \quad \text{and} \quad E_2 : y^2 = x(x^3 - s_1 x^2 + s_2 x - 1) \tag{2}$$

and the Jacobian of C decomposes up to isogeny as $Jac(C) \simeq E_1 \times E_2$. Now, such elliptic fibrations of reducible abelian varieties are studied extensively due to the promising post-quantum cryptography applications [8].

In a more general case, the reduction of abelian integrals had been investigated by Picard, Weierstrass and Poincaré, while the genus $g = 3$ hyperelliptic curves have been treated in detail by Kowalevski [5] in 1884 concerning the unpublished Weierstrass lectures, which were of great value in those years, see Pokrovsky's book [7]:

[1] Kovalevskaya preferred the spelling “Kowalevski” in her publications and letters.

> The printed memoirs of Weierstrass, due to their brevity and incomplete proof, give only a vague idea of his theory, which, however, is perfectly developed. Last winter in Berlin, we managed to get acquainted with one of the Professor's handwritten courses.

As a result, both then and now, references to these Weierstrass theorems lead only to the Kowalevski paper [5], for instance see Poincaré's corresponcence:

> Mon attention fut de nouveau attirée sur cette question par un Mémoire de M^{me} Kowalevski, où se trouvaient cités deux théorèmes de M. Weierstraß, sur la réduction des intégrales abéliennes aux intégrales elliptiques. [2]

In today's geometric language, S. Kowalevski studied hyperelliptic curves of genus $\mathrm{g}=3$ whose Jacobians are isogenic to a product of elliptic curves. Such genus $\mathrm{g}=3$ hyperelliptic curves together with the corresponding two elliptic curves always appear in the modern treatment of the Kowalevski rigid body motion [2, 3, 9].

The rotation of a rigid body about a fixed point in a gravitational field is described in the moving frame by the Euler–Poisson equations

$$\dot{\ell}=\ell\times\frac{\partial H}{\partial \ell}+g\times\frac{\partial H}{\partial g},\qquad \dot{g}=g\times\frac{\partial H}{\partial \ell}\,, \tag{3}$$

where $\ell=(\ell_1,\ell_2,\ell_3)$ is the angular momentum vector, $g=(g_1,g_2,g_3)$ is the unit vector in the direction of gravity, H denotes the Hamilton function and $x\times y$ means the cross product of two vectors.

In 1889, S. Kowalevski [6] solved the following problem: find all rigid bodies, rotating about a fixed point in the presence of gravity, such that the equations of motion (3) are integrable in the sense of the Cauchy–Kowalevski theorem [4]. The latter means that the system admits solutions, expressible as the Laurent series in time, which contain a number of free parameters equal to the number of degrees of freedom minus one. This condition leads to the Euler and Lagrange tops, and to the Kowalevski top, which is a solid body rotating about a fixed point with the Hamiltonian

$$H=\ell_1^2+\ell_2^2+2\ell_3^2-2bg_1,\qquad b\in\mathbb{R}$$

and second integral of motion

$$K=(\ell_1^2+\ell_2^2)^2+4b\big(g_1(\ell_1^2-\ell_2^2)+2g_2\ell_1\ell_2\big)+4b^2(g_1^2+g_2^2)\,,$$

which are in the involution, i.e., their Poisson brackets vanish identically

$$\{H,K\}=0.$$

The Poisson brackets on the Euclidean group algebra $e(3)^*$ are defined by

$$\{\ell_i\,,\ell_j\,\}=\varepsilon_{ijk}\ell_k\,,\qquad\{\ell_i\,,g_j\,\}=\varepsilon_{ijk}g_k\,.\qquad\{g_i\,,g_j\,\}=0\,,$$

[2] "My attention was attracted anew to this problem by a memoir of M^{me} Kowalevski, where two theorems of M. Weierstraß are mentioned."

where ε_{ijk} is the skew-symmetric tensor. These brackets have two Casimir functions

$$c_1 = g_1^2 + g_2^2 + g_3^2\,, \qquad c_2 = g_1\ell_1 + g_2\ell_2 + g_3\ell_3\,.$$

Fixing their values, one gets a generic symplectic leaf of $e(3)^*$ which is a four-dimensional symplectic manifold.

Solving the Euler–Poisson equations S. Kowalevski introduces variables

$$z_{1,2} = \ell_1 \pm \mathrm{i}\ell_2$$

such that

$$H = -\frac{\dot z_1 \dot z_2 + R(z_1, z_2)}{(z_1 - z_2)^2}\,, \qquad K = \frac{\big(\dot z_1^2 - R(z_1, z_1)\big)\big(\dot z_2^2 - R(z_2, z_2)\big)}{(z_1 - z_2)^4}\,,$$

where $R(z_1, z_2)$ is equal to

$$R(z_1, z_2) = z_1^2 z_2^2 - (z_1^2 + z_2^2)H - 4b(z_1 + z_2)c_2 - 4b^2 c_1 + K\,.$$

To remove cross-terms $\dot z_1 \dot z_2$ she uses group operations for the divisor $D = (z_1, Z_1) + (z_2, Z_2)$

$$\frac{dz_1}{Z_1} + \frac{dz_2}{Z_2} = \frac{dw_1}{W_1}\,, \qquad \frac{dz_1}{Z_1} - \frac{dz_2}{Z_2} = \frac{dw_2}{W_2}$$

on the elliptic curve

$$E: \quad Z^2 = R(z, z) = z^4 - 2z^2 H - 8bc_2 z - 4b^2 c_1 + K\,.$$

Simultaneously she reduces this equation for E to the short Weierstrass form

$$E: \quad W^2 = 4w^3 - g_2 w - g_3$$

and obtains standard Abel's differential equations on a hyperelliptic curve C. The meaning of these calculations has been discussed by many authors and is more or less clear, see [3, 10] and references within.

However, instead of solving these Abel's equations, S. Kowalevski does an additional transformation of variables

$$w_{1,2} \to s_{1,2} + H/3 \tag{4}$$

and obtains her famous variables of separation

$$s_{1,2} = \frac{R(z_1, z_2) \pm \sqrt{R(z_1, z_1)}\sqrt{R(z_2, z_2)}}{2(z_1 - z_2)^2}\,,$$

which also satisfy standard Abel's equations

$$\frac{\dot s_1}{\sqrt{\mathcal{P}_5(s_1)}} + \frac{\dot s_2}{\sqrt{\mathcal{P}_5(s_2)}} = 0\,, \qquad \frac{s_1 \dot s_1}{\sqrt{\mathcal{P}_5(s_1)}} + \frac{s_2 \dot s_2}{\sqrt{\mathcal{P}_5(s_2)}} = 1,$$

where

$$\mathcal{P}_5(s) = (4s^2 + 4Hs + H^2 - K)\,(4s^3 + 4Hs^2 + (4b^2 c_1 + H^2 - K)s + 4b^2 c_2^2)$$

and which are in the involution $\{s_1, s_2\} = 0$ with respect to the Lie–Poisson brackets on $e^*(3)$.

Of course, S. Kowalevski never computed the Poisson brackets between the variables $s_{1,2}$, so her reason for the additional transformation of the variables (4) is not known for sure. Moreover, we do not find any suitable discussion of this transformation in the extensive list of papers devoted to the Kowalevski top.

We can only suppose that S. Kowalevski reduces the elliptic curve E to the form (2)

$$E: \quad S^2 = \mathcal{P}_3(s), \quad \mathcal{P}_3(s) = 4s^3 + 4Hs^2 + (4b^2c_1 + H^2 - K)s + 4b^2c_2^2\,,$$

which appeared in her investigation of reducible abelian integrals and reducible abelian varieties, see [5].

Andrey Tsiganov

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Evgraf Stepanovich Fedorov (1853–1919)

Boris Delone, the renowned mathematician, placed Fedorov in the ranks of the legendary Plato and Archimedes. Vladimir Vernadsky, a founder of geochemistry, placed him in the same category as his great contemporaries Mendeleev and Pavlov. Yet, unlike these household names, Fedorov is little known outside his field and country. Who was he, you ask? I will try, in this brief space, to sketch an answer.

Evgraf Stepanovich Fedorov was born in Orenburg, Russia in 1853; his father was a military engineer. Soon after Evgraf's birth, the family moved to St. Petersburg, and he grew up there. It is said that from his father he inherited a "sharp disposition" and that his mother gave him a love of reading and the habits of hard work.[1]

She also gave him a musical education and taught him to knit lace-like tablecloths. The tablecloths may have sparked his life-long interest in patterns and geometry: at the age of ten, he worked through his older brother's geometry textbook in two days. At sixteen, fascinated by crystals and their geometry, he began writing the first of his famous works, *An Introduction to the Study of Figures*.

During the ten years that Evgraf worked on the book, he explored first medicine and then engineering as possible careers before deciding on mineralogy. In 1873, he met his future wife, Ludmila Vasilievna Panyutina, a medical student from the Urals. Her memoir, *Our Everyday Life, Our Joys and Sorrows*, is the primary source for this and other biographical sketches of her husband.[2] Evgraf's other extracurricular activities during that decade included revolutionary politics. The "Land and Freedom" Party sent him abroad to forge connections with revolutionaries in Belgium, France, and Germany; this no doubt enhanced his foreign language skills. He also played a part in the dramatic escape of the

[1] https://www.encyclopedia.com/science/dictionaries-thesauruses-pictures-and-press-release/fyodorov-or-fedorove-evgraf-stepanovich/

[2] L. V. Fedorova, Our Everyday Life, Our Joys and Sorrows (in Russian), Science Legacies vol. 20, Nauka, 1992.

anarchist Prince Peter Kropotkin from the prison of a St. Petersburg military hospital. According to Fedorov's family legend, he was the one who gave Kropotkin the agreed-upon all-clear signal: a Schubert serenade on his violin. Evgraf and Lyudmilla were married in 1880 and had two daughters and a son.

Medal, honoring the 100th anniversary of Fedorov's result on space groups.

The other side of the medal.

Also, in 1880, Fedorov, as we will call him from now on, enrolled in the prestigious St. Petersburg Mining Institute. Founded in 1873, the Mining Institute was Russia's first technical college. Its attractions for students included a library, a dining hall, and the right to hold meetings. Another was the Institute's splendid collection of minerals from all over Russia, initially assembled on the order of Catherine the Great. Expanded since then by donations from all over the world, the St. Petersburg Mining Museum became (and is today) a hands-on resource for Mining Institute students, a world-class scientific/industrial resource, and a leading St. Petersburg tourist destination.

In 1883, Fedorov completed his studies at the Institute, ranking first in his class. That summer, he joined a geological mapping expedition in the northern Urals, the first of many arduous expeditions he would conduct in that and other regions of Russia.

Fedorov's achievements during the decade of 1885 to 1895 would put him on the world map of science. Beyond his complex, painstaking enumeration of the 230 crystallographic groups, and the long-delayed publication of *An Introduction to the Study of Figures*, he transformed the practice of mineralogy and petrography. Fedorov believed that these crystallographic groups determined the positions of molecules in crystals, though this was not widely accepted at the time. He also believed that the inner structures of crystals could be deduced from their macroscopic properties, and devised tools for measuring them. As Daniel Kile explains,[3]

> The optical behavior in crystals varies with the direction of light travel within the crystal, and measurement of optical properties has to be carried out in specific orientations... Therefore, a great variety of rotating devices have been designed to align crystals into a proper orientation. The universal stage is certainly the most important in such devices, and for more than half a century [i.e., in the pre-computer era] it was considered essential for petrographic work. E.S. Fedorov developed a four-axis universal stage in 1892 in response to the need to characterize feldspar-group minerals, a major group of rock-forming minerals. His universal stage, with independently tilting and rotating axes, facilitated these measurements by allowing complete freedom to orient crystallographic and optical planes in the sample.

The Fedorov stage received a prize at the 1893 International Exposition in Chicago.

In 1895 Fedorov was appointed professor of geology at the Moscow Agricultural Institute (now the Timiryazev Agricultural Academy); he continued to lecture at the St. Petersburg Mining Institute as well. He also continued leading summer expeditions, lecturing widely, and writing articles and books. In 1896, he was elected to the Bavarian Academy of Sciences, an indication of his growing international prestige and the first of several such honors. The times were tumultuous, and Russia's institutions of higher education were not untouched by the unrest. Faced with demands for self-governance from university and institute faculty and students, the tsar decreed that faculty councils could elect their own rectors and assistant rectors, pending the approval of relevant ministries, and that student discipline would be handled by faculty disciplinary courts.[4] In 1905, Fedorov became the Mining Institute's first elected rector.

Fedorov was re-elected rector in 1908, but his second term was terminated by the Russian Minister of Internal Affairs. At issue was student financial aid. The Mining Institute students had declared a strike in response to cuts to the Institute's financial aid budget. When the cuts were eliminated and the budget restored, the Minister dismissed Fedorov from the rectorship.

A sad ending to a brilliant career? No, the story does not end there. Fedorov's love of fine literature lasted all his life; he would have disdained last-minute plot twists devised to save the hero. Nevertheless, he lived to see his crystal structure hypothesis corroborated, in principle if not in exact detail,

[3] Daniel Kile, *The Universal Stage: the past, present, and future of a mineralogical research instrument*, Geochemical News 140, June 2009.

[4] Samuel D. Kassow, Students, Professors, and the State in Tsarist Russia, the University of California Press, 1989.

by just such a *deus ex machina*:[5] the discovery of X-ray diffraction. (See the next section "Fedorov and Geometry") Fedorov was elected to the Soviet Academy of Sciences in 1919, a few months before his death from pneumonia during the privations of the Russian Civil War.

Although it is true that, a century later, Fedorov is not yet a household name, Delone and Vernadsky did not exaggerate.

Marjorie Wikler Senechal

Additional materials to read: https://www.mdpi.com/2075-163X/10/2/181

[5] *Deus ex machina*: A power, event, person, or thing that comes in the nick of time — Oxford English Dictionary.

E.S. Fedorov and Geometry

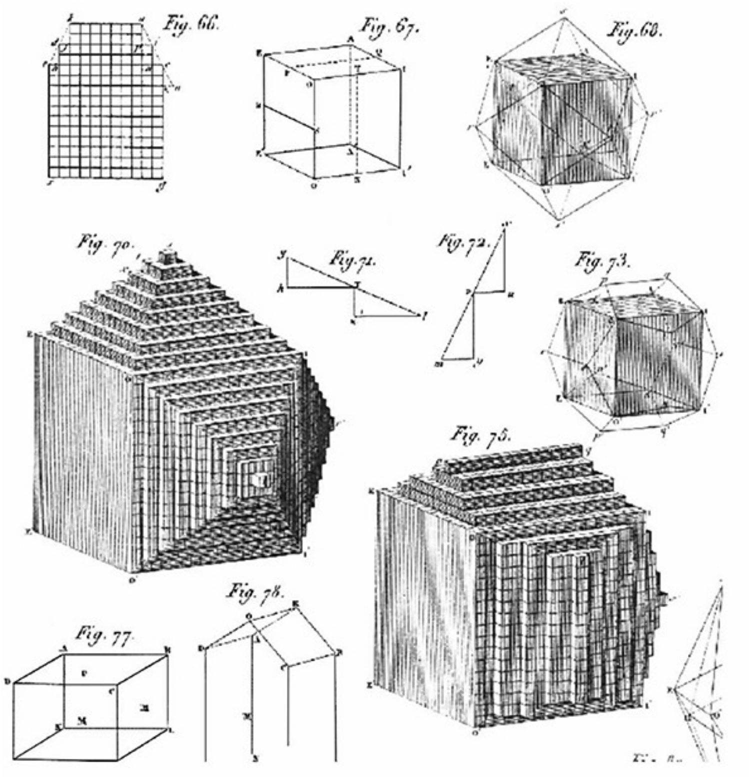
The mathematician B.N. Delone ranked Fedorov with the great geometers:

> Tradition ascribes to Plato the discovery of the five regular convex polyhedra, to Archimedes the thirteen convex semi-regular polyhedra, to Kepler and Poinsot the four regular nonconvex solids, and Fedorov found the five parallelohedra.

For crystallographers, the name “Fedorov” evokes the 230 crystallographic (or space) groups and the Fedorov stage, a sophisticated device for measuring crystals. But in Fedorov’s own eyes, these three pantheonic achievements were stepping stones in his broader but quixotic quest to deduce the atomic structures of crystals from their external shapes and properties.

An exhibit of mineral crystals and mining apparatus in the Mining Museum, St. Petersburg, Russia.

How does the arrangement of their chemical subunits give rise to crystals' symmetric, polyhedral, forms? Until the discovery of the X-ray diffraction in 1912, no one could say. But one could guess. The turn-of-the-19th-century French mineralogist R.J. Haüy proposed that crystals are stacks of sub-visible congruent bricks (see the illustration below).



R.J. Haüy's drawing (1801) shows how crystal forms can be approximated by cube-like bricks.

Haüy's theory accounted for crystal shapes and symmetries, and more. Why can crystals of the same species have different geometric forms? Because the stacks of bricks can be completed in different ways! Why is the pentagonal dodecahedral form of pyrite never quite regular? Because the vertices of crystals have rational coordinates!

But what were these bricks supposed to be? Haüy called them *molécules intégrantes*, but these were not our modern molecules. Indeed, not even Haüy could say just what they were. So his critics changed the question from *what* the bricks are to *where* they are, marking their positions with the points at their centers. Sets of translation-equivalent points in $\mathbb{R}^3$ are now called Bravais lattices, named for the French polymath who enumerated them in 1850. The

Bravais lattices are still a (if no longer the) foundation stone of mathematical crystallography.

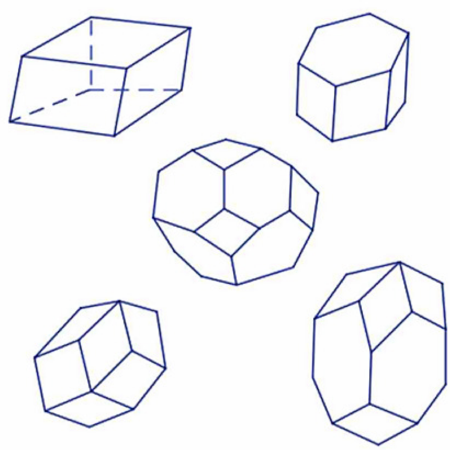
But Fedorov was interested in actual crystals and thus in the bricks themselves. What did they look like? More precisely, *Which convex polyhedra fill the space when placed face to face in a parallel position?*

Defining a parallelohedron P to be such a polyhedron, he showed that it is completely characterized by three properties:

1. P is centrosymmetric;
2. Each face of P is centrosymmetric.

   (1) and (2) together imply that each edge of P belongs to a set of parallel edges that span around P like a belt. The length of a belt is the number of edges in it.
3. Each belt of P has a length of four or six.

With these three conditions and Euler's formula relating the faces, edges, and vertices of polyhedra, Fedorov proved that every convex parallelohedron is one of five combinatorial types (see the illustration below).



The five combinatorial types of parallelohedra. Top row: the rhombic hexahedron and the hexagonal prism; Middle: the truncated octahedron; Bottom row: the rhombic dodecahedron and the elongated dodecahedron.

Fedorov presented the parallelohedra in a book he called *An Introduction to the Theory of Figures*, published in 1885. It had taken him six years to find a publisher. Contemporary science, the mathematician P.L. Chebyshev told him, was not interested in this subject. Undiscouraged, Fedorov persisted. He knew that Camille Jordan, in his 1868 *Memoire sur les groupes des mouvements*, had generalized lattices to orbits of groups of orientation-preserving motions (rotations, translations, and screw-rotations). Fedorov added orientation-reversing motions (reflections, glide-reflections, and rotatory reflections), thus quadrupling the list of groups of motions.

Independently, at the suggestion of Felix Klein, Arthur Schoenflies in Germany had begun working on the same problem. He and Fedorov joined forces. “Ich freue mich besonders, dass ich nun mit meiner Theorie nicht mehr allein stehe,”[1] Schoenflies wrote to Fedorov. Each found errors in the other’s list; together they established the correct number, 230, in 1891.[2]

Schoenflies was interested in the space groups as groups, Fedorov in their use in crystallography. To each crystal, he believed, corresponded a tiling of $\mathbb{R}^3$ by parallelohedra. The parallelohedra, in turn, could be partitioned into congruent “stereohedra,” each containing a crystal molecule. “Fedorov passively accepted the concept of the molecule as the final stage of matter,” his biographers explain, “and considered, in principle, that the ultimate aim of crystallography should be the classification of all possible “receptacles” for this finite unit.”

Fedorov knew that not all 230 space groups afforded such receptacles. In modern terminology: each space group is a product of a translation group and the subgroup of $O(3)$ that describes the crystal’s macroscopic symmetry. Only when the product is semi-direct can their orbits be partitioned as Fedorov’s crystal-structure theory required. He declared the other groups to be “imaginary” and continued his classification project with the “real” ones.

Contemporary science was not interested in this problem either; in 1891, even crystallographers thought lattices were at best a useful fiction and left the space groups on the library shelf. That changed, suddenly, in 1912.

> You [said] that the human eye shall never see atoms... approximately at the time when people saw atoms with their own eyes; if not the atoms themselves, then the photographic images caused by them,

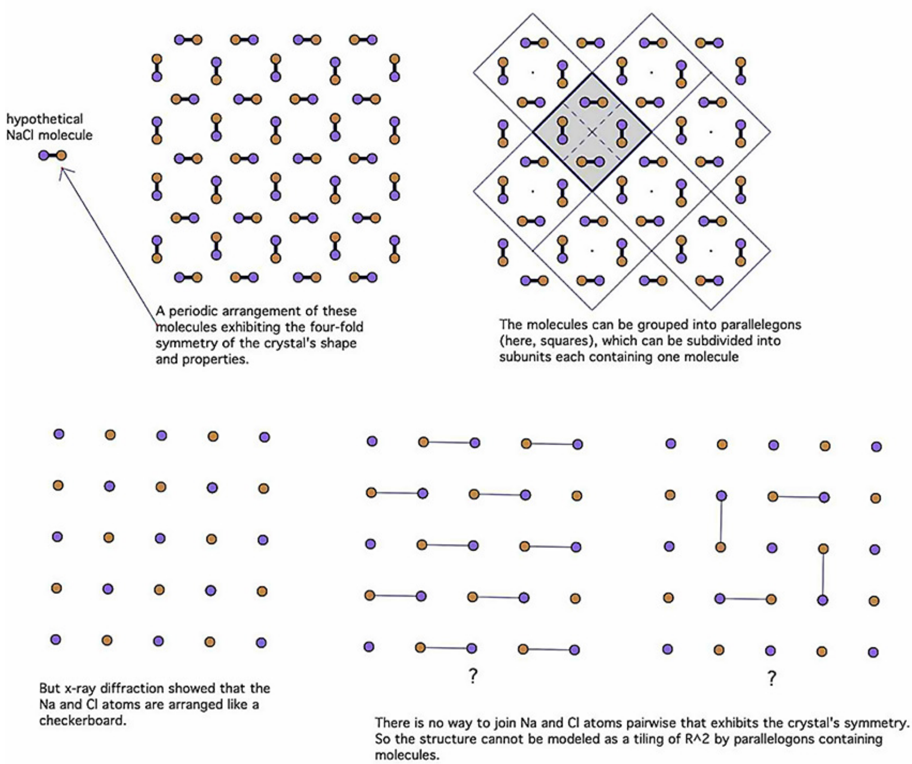
Fedorov wrote to a scientist friend, referring to an experiment Max von Laue had conducted in Munich “several weeks ago.” Von Laue had solved the X in X-rays. Scientists had debated whether X-rays were waves or particles. If they were waves, they would behave like waves: they would be diffracted by a grating of suitable dimensions. And if the space theory of crystal structure was correct, crystals would be just such gratings. Testing both hypotheses in a single experiment, Von Laue passed a beam of X-rays through a crystal and captured them on a photographic plate on the crystal’s opposite side. The photographs showed sharp bright spots, confirming both.

[1] “I am especially pleased that I’m no longer alone with my theory.”

[2] Schoenflies granted Fedorov the priority (“Die Prioiritat gebe ich Ihnen gern zu”), but his version has been the more influential. German was the international scientific language of the time, and even without the language barrier Fedorov, an autodidact in mathematics, was difficult to understand.

On hearing this news, young W.L. Bragg, a student at Cambridge University, had a Eureka moment: he realized that the diffraction patterns could be read backward to deduce the atomic positions that gave rise to them. "For us crystallographers this discovery is of prime importance," Fedorov continued his letter, "because now, for the first time, we can have a clear picture of that on which we have but theoretically placed the structure of crystals and on which the analysis of crystals is based."

One of the first crystals Bragg "solved" was ordinary salt. Today his three-dimensional checkerboards of Na and Cl atoms are found in every chemistry classroom. But in 1912 his model shocked the establishment, much as the discovery of quasicrystals seventy years later. Everyone had expected to find $NaCl$ molecules arranged symmetrically in the crystal. But, Bragg showed, there are no molecules in salt (see the illustration below). Fedorov acknowledged with good grace that



X-ray diffraction showed that the crystal structure of rocksalt — known to be cubic — is a checkerboard arrangement of Na and Cl atoms. Top Row: Fedorov had expected to see an arrangement of $NaCl$ molecules like this. Bottom Row: A plane in the actual structure. There is no way to group Na and Cl atoms into "molecules" in a pattern with cubic symmetry.

> Some of these conclusions are unexpected, at least in the sense that in the points of real systems one expected to find centers of chemical particles, while the experiments of this scientist permit one to draw the conclusion that these are the centers of atoms.

In short, Fedorov's imaginary groups were real.

Overnight, contemporary science became interested in crystal structure and the space groups were taken off the shelf. But now that one could "see" inside them, crystal shapes weren't needed anymore. Morphology was just for museums. *The Crystal Kingdom*, the massive tome that Fedorov thought would be his legacy, was obsolete before its posthumous publication in 1920.

But Fedorov's contributions to geometry are undimmed. He would be pleased to know that space-filling polyhedra are of great interest in contemporary mathematics, chemistry, condensed-matter science, and even industrial design, in spaces of every dimension. And that the space groups remain a basic tool in crystallography, a science that has grown far beyond crystals.

Marjorie Wikler Senechal

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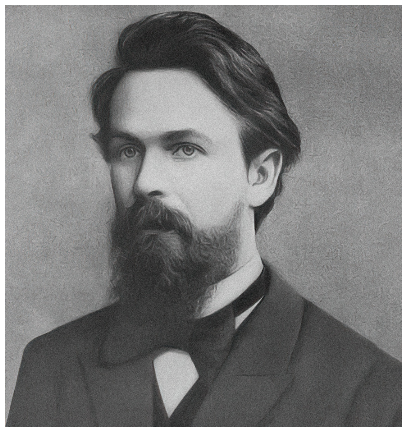
Andrei Andreyevich Markov, Sr. (1856–1922)

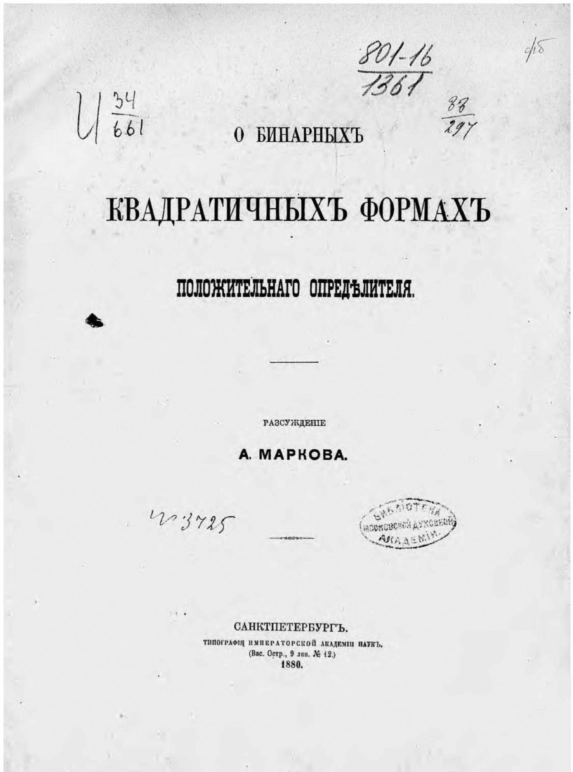
Andrei Markov Sr. was an outstanding representative of the St. Petersburg school of mathematics, the author of fundamental works in number theory, real analysis and, especially, probability theory. The scheme he proposed for studying texts by mathematical methods, later called Markov chains, became the basis for a new section of probability theory, the theory of stochastic processes, whose applied importance in modern science can hardly be overestimated.



Andrei Markov was born on the 14th of June 1856 (New Style) in Ryazan in the family of Andrei Grigoryevich Markov, an official of the Forest Department. The family moved to St. Petersburg (in the early 1860s), and when Andrei graduated from St. Petersburg Gymnasium 5 in 1874 he entered St. Petersburg University, with which his entire scientific biography was connected. In 1886 he became a professor and a member of the Academy of Sciences.

His passion for mathematics, which emerged in his gymnasium years, grew into an independent study of higher mathematics, and, according to his father, he did not want to do anything but mathematics, which, naturally, affected his graduation grades. There was also a curious incident: Markov thought he had invented a new method of integrating linear differential equations with constant coefficients! The results of his appeal to famous mathematicians from St. Petersburg (Viktor Bunyakovsky, Egor Zolotarev, and Alexander Korkin) concerning his imaginary discovery were, on the one hand, sad, for they explained to him that his method was not new and pointed out the errors. On the other hand, the gymnasium boy Markov became acquainted with professors of St. Petersburg University and entered the house of Korkin whom Zolotarev often visited. This certainly strengthened Markov's desire to link his future life with science.

During his years at the university (1874–1878) Markov had a reputation as one of the ablest students; his teachers were Pafnuty Chebyshev, Egor Zolotarev, Alexander Korkin, Julian Sochocki, and others, and conversations with Professor Korkin had a strong influence on the choice of the subject of Markov's independent research. It was at Korkin's request that in 1878 Andrei Markov was kept at the university to prepare for professorship, which started in 1880 after the defense of his master's thesis.



О БИНАРНЫХЪ

КВАДРАТИЧНЫХЪ ФОРМАХЪ

ПОЛОЖИТЕЛЬНАГО ОПРЕДѢЛИТЕЛЯ.

РАЗСУЖДЕНІЕ

А. МАРКОВА.

САНКТПЕТЕРБУРГЪ.

ТИПОГРАФІЯ ИМПЕРАТОРСКОЙ АКАДЕМІИ НАУКЪ.

(Вас. Остр., 9 лин. № 12.)

1880.

The main areas of Markov's research pertain to number theory, differential equation theory, function theory, approximation theory, and most prominently to probability theory. His master's thesis *On binary quadratic forms of the positive determinant* was a worthy continuation of the research by Egor Zolotarev and Alexander Korkin. According to Boris Delone, this work

> ...belongs to the sharpest achievements of the St. Petersburg school of number theory, and perhaps of all Russian mathematics. [2, p. 144].

Another major event in Russian science was Markov's doctoral dissertation *On some applications of algebraic continuous fractions* (1885), which continued the research by Pafnuty Chebyshev and developed the latter's doctrine on the limit values of integrals.

This was the first in a series of papers by Markov on moment theory, a research tool in both interpolation and function approximation, as well as in probability theory. Markov developed a rigorous proof of the central limit theorem of probability theory under sufficiently general conditions, and a substantial extension of the scope of the law of large numbers. His series of papers (1906–1912) gave rise to the modern theory of Markovian processes and Markov chains, which is widely used in science and engineering.

Markov had taught since 1880 until his death (with few interruptions). Even after his retirement, he did not sever ties with his alma mater, reading his favorite course on probability theory in the Physics and Mathematics Department. His two widely known textbooks *Finite-difference calculus* (1886, lithographed edition) and *Probability Calculus* (1900) were in demand not only in Russia, but also abroad (foreign translations were published). In Russia, for example, his *Probability Theory* was already reprinted four times in its lifetime. Besides, we are indebted to Markov for the emergence in Russian mathematics of such scientists as Nikolai Günther, Georgy Voronoy, Boris Koyalovich, Andrei Markov Jr., and others.

Andrei Markov was always notable for his proactive attitude, keen sense of justice, and unconditional rejection of everything evil. The uncompromising, straightforward, and open-minded Markov fought all his life against what he considered unjust and an obstacle to progress. His objections against the cancellation of the election of Maxim Gorky at the meeting of the Department of Language and Literature of the Academy of Sciences on February 25 1905 as an honorary academician are widely known.

In 1908, following the release of an instruction from the Ministry of Public Education administering police functions to teachers, Markov sent a letter to the minister informing him of his refusal "to be a government agent at the university." After Leo Tolstoy was excommunicated by the Holy Synod, Markov sent a petition, which created quite a stir, asking that he be excommunicated as well...

As a counterbalance to the celebration of the 300th anniversary of the House of Romanov organized in 1913, the Academy of Sciences on the proposal of Academician Markov celebrated the 200th anniversary of the publication of the outstanding Swiss mathematician Jakob Bernoulli's *The Art of Conjecturing* (Ars Conjectandi). But sometimes, unfortunately, his aversion to diplomacy which so often is found in academia and the rejection of authority complicated his relations with his colleagues. In addition, Markov underestimated the works of Sofia Kovalevskaya, Viktor Imshenetsky, Karl Pearson, and Pavel Nekrasov.

Few people know that Andrei Markov was a passionate chess player since his gymnasium days and was friends with Mikhail Chigorin, the strongest Russian chess player and a contender for the title of world champion. After Markov took first place (with 6 points out of 6 possible) at the First All-Russian Tournament by Correspondence, Chigorin chose him as his sparring partner to prepare for a correspondence match against Wilhelm Steinitz (1890). Markov kept this passion for chess throughout his life. In fact, a year before his death Markov, having deteriorated eyesight due to glaucoma, took part in a tournament held

АКАДЕМИКЪ МАРКОВЪ—ВНѢ РЕЛИГІИ

Заслуженный профессоръ с.-петербургскаго университета академикъ А. А. Марковъ отлученъ отъ церкви по его желанію

Небывалый переполохъ вызвало въ синодѣ ходатайство заслуженнаго профессора спб. университета, ординарнаго академика академіи наукъ, извѣстнаго математика д. с. с. Андрея Андреевича Маркова объ отлученіи его отъ церкви.

Въ своемъ прошеніи, поданномъ въ синодъ, академ. А. А. Марковъ въ корректной формѣ заявляетъ, что онъ въ теченіе 20 лѣтъ на основаніи выводовъ науки и собственныхъ наблюденій пришелъ къ полному атеизму и выработалъ опредѣленный критическій взглядъ на догматы и обязанности всѣхъ церквей и религій.

Поэтому А. А. Марковъ проситъ св. синодъ отлучить его отъ церкви.

Онъ утверждаетъ, что это желаніе не мимолетная прихоть, а продуманная мысль.

Для большей убѣдительности А. А. Марковъ ссылается на рядъ своихъ сочиненій, въ которыхъ онъ проводилъ мысль, безусловно отрицающую церковь и религію.

Св. синодъ переслалъ прошеніе А. А. Маркова въ с.-петербургскую духовную консисторію на усмотрѣніе митрополита с.-петербургскаго и ладожскаго Антонія, по принадлежности епархіи.

Бесѣда съ архіепископомъ Антоніемъ Волынскимъ

По поводу необычайнаго ходатайства академика А. А. Маркова мы бесѣдовали съ виднымъ членомъ св. синода архіепископомъ Антоніемъ Волынскимъ.

— Да, дѣйствительно, ходатайство А. А. Маркова объ отлученіи его отъ церкви въ синодѣ получено и переслано по принадлежности митрополиту с.-петербургскому и ладожскому Антонію— сказалъ намъ владыко Антоній.

По правиламъ церкви, къ А. А. Маркову для увѣщанія былъ посланъ протоіерей Орнатскій.

А. А. Марковъ остался при своемъ мнѣніи. Увѣщанія остались безплодными.

Послѣ этого А. А. Марковъ былъ отлученъ отъ церкви.

Но почему-то все это носило слишкомъ деликатную форму и отпавшему отъ православія не была провозглашена анаѳема.

Вообще подобные случаи отпаденія отъ церкви безъ перехода въ какую-либо другую религію весьма рѣдки.

У насъ на Волыни бываютъ отпаденія въ католичество или штунду. Тамъ отпадавшимъ провозглашается анаѳема при торжественной обстановкѣ.

А. Невскій

Newspapers about Markov's demand to be excommunicated. Петербургская газета, 8 мая, 1912; Peterburgskaia gazeta, 1912.05.08.

at the natural science station in Novy Peterhof, playing with professor Nikolai Günther without looking at the board.

Andrei Markov died in 1922 in Petrograd, was buried in Mitrofanievsky cemetery, and re-buried in 1954 at Literator Bridges (Literatorskiye Mostki) in the Volkov cemetery.

Natalia Lokot

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Markov numbers in arithmetic and geometry

Andrei Andreevich Markov (1856–1922) was an outstanding Russian mathematician. His works on probability theory and mathematical analysis are widely known and generally recognized. He developed a theory of an extensive class of stochastic processes with discrete and continuous time components, named after him. This theory has countless applications in modern theoretical and applied research, its influence is difficult to overestimate. A. A. Markov made a huge contribution to the theory of continued fractions and the calculus of finite differences. In the theory of pattern recognition and artificial intelligence tasks most of the algorithms use the concept of a hidden Markov model, which originates in Markov's works.

However, A.A. Markov is no less well-known as a specialist in number theory. He received the first significant result in his master thesis *On binary quadratic forms of a positive determinant* [11, 13] (see also [10, 12]). One of the central objects of the dissertation is a certain Diophantine equation that subsequently arose in many areas of mathematics, quite far from the original problem of minimizing of quadratic forms. In this note we will discuss this aspect of A. A. Markov's extensive mathematical heritage.

The Markov equation

The *Markov equation* is a Diophantine equation of the form

$$x_1^2 + x_2^2 + x_3^2 = 3x_1x_2x_3. \tag{1}$$

Solutions of this equation are now known as *Markov triples*. The *Markov numbers* are all natural numbers appearing in these triples. Let (x_1, x_2, x_3) be a Markov triple. Consider the following three transformations

$$\begin{array}{ccccc} & & (x_1, x'_2, x_3) & & \\ & & \uparrow{\scriptstyle t_2} & & \\ & & (x_1, x_2, x_3) & & \\ & \swarrow{\scriptstyle t_1} & & \searrow{\scriptstyle t_3} & \\ (x'_1, x_2, x_3) & & & & (x_1, x_2, x'_3) \end{array} \tag{2}$$

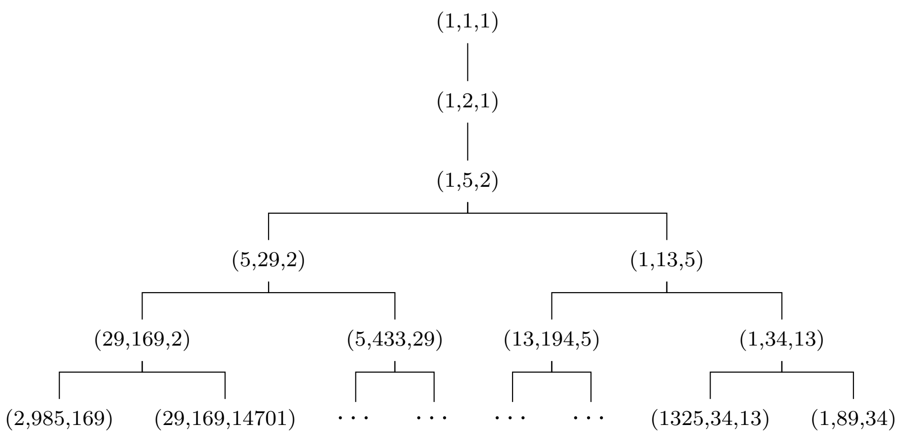
where

$$x'_i := \frac{3x_1x_2x_3}{x_i} - x_i.$$

According to Vieta's formulas, new triples

$$(x_1', x_2, x_3), \quad (x_1, x_2', x_3), \quad (x_1, x_2, x_3') \tag{3}$$

are also solutions of the Markov equation, moreover $x_i' \neq x_i$. Such a procedure t_i is called an *elementary transformation* or *mutation* in the element x_i, and the corresponding triples are called *neighboring*. It can be shown that if all three entries x_1, x_2, x_3 are different, then all triples (3) are also different. Moreover, a mutation in the maximal element of the triple reduces this element. For example, if $x_1 = \max(x_1,\ x_2,\ x_3)$, then $x_1' < \max(x_2,\ x_3) < x_1$. It follows that any solution of the Markov equation is obtained from $(1, 1, 1)$ by successive application of mutations. All the Markov triples can be written as a graph in which the neighboring ones are connected by an edge. The graph has the form of an infinite trivalent tree:



It is easy to see that any Markov number is maximal in some triple. In 1913, Frobenius proposed the following conjecture.

Conjecture (uniqueness conjecture)**.** *A Markov triple is uniquely determined by its maximal element.*

Despite numerous attempts, the conjecture has not yet been proven, see [1] for a very good introduction and historical overview.

The geometry of the Markov surface

Consider the surface X defined in the affine space $\mathbb{A}^3$ by the equation (1). Its projective closure $\bar{X} \subset \mathbb{P}^3$ is a nodal cubic with a unique singular point so that the boundary divisor is the union of three lines forming a "triangle."

The maps t_i are automorphisms of the surface X as an affine variety. One can check that they generate a subgroup $\Gamma_0 \subset \mathrm{Aut}(X)$ isomorphic to the free product

$$(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}).$$

The complete automorphism group $\mathrm{Aut}(X)$ is generated by Γ_0, permutations, and sign changes of pairs of coordinates [3]. In this presentation, $\mathrm{Aut}(X)$ acts transitively on the set of integer points of the surface X and its subgroup of index 4, isomorphic to $\mathrm{PGL}_2(\mathbb{Z})$, acts transitively on the set of Markov triples.

The projection

$$\Psi : X \dashrightarrow \mathbb{P}^2$$

from the origin is a birational map, i.e., it is one-to-one on nonempty Zariski-open subsets of $U \subset X$ and $V \subset \mathbb{P}^2$. Moreover, Ψ induces an embedding of $\mathrm{Aut}(X)$ into the group of birational transformations of the plane so that all the elements preserve, up to sign, the symplectic form

$$\frac{\mathrm{d}u \wedge \mathrm{d}v}{uv}.$$

Thus, the subgroup of the index 2 in $\mathrm{Aut}(X)$ can be embedded to *symplectic Cremona group* [15].

Markov numbers in approximation theory and quadratic form theory

In Markov's original work, equation (1) arose in connection with the problem of finding the arithmetic minimum of binary quadratic forms.

Consider a binary quadratic form

$$f(x, y) = \alpha x^2 + \beta xy + \gamma y^2, \qquad \alpha, \beta, \gamma \in \mathbb{R}.$$

We assume that the form is *indefinite*, i.e. its discriminant

$$D := \beta^2 - 4\alpha\gamma$$

is positive. The *Markov constant of the form* f is the number

$$\mu(f) := \frac{\sqrt{D}}{\min'(f)},$$

where $\min'(f)$ is the arithmetic minimum:

$$\min'(f) := \min\left\{\ |f(x, y)|\ \middle|\ x, y \in \mathbb{Z}, \quad (x, y) \neq (0, 0)\ \right\}.$$

The *Markov spectrum* is the set of all Markov constants:

$$\mathbb{M} := \left\{\mu(f) \mid f \text{ is a binary quadratic form with } D > 0\right\}.$$

The forms f and f' are called *equivalent* if they are obtained from each other by integer coordinate changes. It is clear that the equivalent forms have the same minimum.

It turns out that the problem of computing the arithmetic minimum of quadratic forms is closely related to the theory of Diophantine approximations. The well-known theorem of A. Hurwitz states that for any irrational number θ there are infinitely many rational fractions $\frac{p}{q} \in \mathbb{Q}$ satisfying the inequality

$$\left|\theta - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}$$

and the constant $\sqrt{5}$ in the denominator cannot be increased. In this regard, the following natural definition arises: the *Lagrange number* for $\theta\in\mathbb{R}$ is the supremum $\lambda(\theta)$ of the set of all real numbers λ such that the inequality

$$\left|\theta-\frac{p}{q}\right|<\frac{1}{\lambda q^2} \tag{4}$$

holds for infinitely many rational fractions $\frac{p}{q}\in\mathbb{Q}$. Thus, by Hurwitz's theorem for the irrational θ we have $\lambda(\theta)\geq\frac{1}{\sqrt{5}}$. The *Lagrange spectrum* is the set

$$\mathbb{L}:=\{\lambda(\theta)\mid\theta\in\mathbb{R}\}$$

of all possible values of Lagrange numbers. The numbers θ, $\theta'\in\mathbb{R}$ are called *equivalent* if they are contained in the same orbit of the action group $\mathrm{GL}_2(\mathbb{Z})$ on $\mathbb{R}$ by Möbius transformations. It is clear that the Lagrange numbers of equivalent real numbers are equal.

Note that the exponent 2 for q on the right side of the inequality (4) cannot be increased: as was shown by K. Roth (1955), for any irrational *algebraic* number and for any $\epsilon>0$ inequality

$$\left|\theta-\frac{p}{q}\right|<\frac{1}{q^{2+\epsilon}}$$

has only a finite number of solutions for coprime p and q.

The results of Markov

Let $m_1=1$, $m_2=2$, $m_3=5$,... be an ordered sequence of all Markov numbers. Denote

$$\lambda_m=\sqrt{9-4/m}.$$

Also, to each ordered Markov triple (m,m',m''), $m>m'>m''$ one can associate, by a certain explicit rule, an indefinite quadratic form

$$F_{m,m',m''}(x,y)$$

which is called the *Markov form*. Assuming the Frobenius conjecture we can think that $F_{m,m',m''}$ depends only on the maximal element: $F_{m,m',m''}=F_m$.

Theorem 1 (Markov)**.** *For an indefinite binary quadratic form $f(x,y)$ the inequality $\mu(f)<3$ is satisfied if and only if f is equivalent to a multiple of the form $F_{m,m',m''}$ for some Markov triple (m,m',m'').*

Hurwitz noticed that the methods of the proof of this theorem allows to obtain immediately a similar result for Diophantine approximations.

Theorem 2. *For an irrational real number θ, the inequality $\lambda(\theta)<3$ holds if and only if $\lambda(\theta)=\lambda_m$, where m is a Markov number. In this case, the number θ is equivalent to a root of the equation $F_{m,m',m''}(x,1)=0$.*

In particular, it follows that on the interval $[0, 3)$ the Lagrange and Markov spectra are discrete and coincide:

$$\mathbb{L} \cap [0, 3) = \mathbb{M} \cap [0, 3) = \{ \lambda_n \} .$$

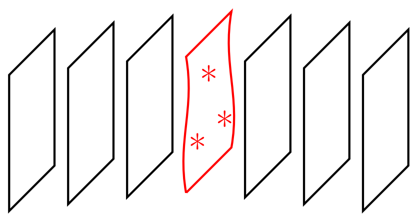
On the contrary, on the right hand side of the real line, these spectra are continuous: G. A. Freiman in 1975 proved that the Lagrange and Markov spectra contain the interval $[\lambda_{\mathrm{F}}, +\infty]$ (Hall ray), where

$$\lambda_{\mathrm{F}} := \frac{2221564096 + 283748\sqrt{462}}{491993569} \approx 4.52.$$

On the other hand, the behavior of Lagrange and Markov spectra on the interval $[3, \lambda_{\mathrm{F}}]$ is quite complicated and still not fully understood.

Markov numbers in geometry

Degenerations of the projective plane. Consider an analytic family $\{S_t\}_{t\in\Delta}$ of compact complex surfaces over a disk $\Delta \subset \mathbb{C}$ such that for $t \neq 0$ the fiber S_t is isomorphic the projective plane $\mathbb{P}^2$. In this situation, the central fiber of S_0 is called *degeneration* of $\mathbb{P}^2$.



In general, the structure of degenerations of $\mathbb{P}^2$ can be quite complicated. M. Manetti [9] posed a problem of classification of degenerations of $\mathbb{P}^2$ admitting only *quotient singularities*, i.e., those degenerations whose singularities are analytically equivalent to quotients $\mathbb{C}^2/G$, where $G \subset \mathrm{GL}_2(\mathbb{C})$. This problem is interesting, important, and motivated by its applications in the theory of modules of curves and surfaces, as well as in the Minimal Model Program.

Recall that the *weighted projective plane* $\mathbb{P}(d_1, d_2, d_3)$ is the set of triples of numbers $(x_1, x_2, x_3) \neq (0, 0, 0)$ with identification:

$$(x_1,\ x_2,\ x_3) = (t^{d_1}x_1,\ t^{d_2}x_2,\ t^{d_3}x_3), \qquad t \in \mathbb{C}^*.$$

Here d_1, d_2, d_3 are natural numbers called *weights*. We will assume that the weights are pairwise coprime. For $d_1 = d_2 = d_3 = 1$ we get the usual projective plane. Otherwise, $\mathbb{P}(d_1, d_2, d_3)$ has quotient singularities.

Theorem 3 ([7]). *If the weighted projective plane is a degeneration of* $\mathbb{P}^2$, *then it has the form*

$$\mathbb{P}(m_1^2, m_2^2, m_3^2),$$

where (m_1, m_2, m_3) *is a Markov triple. Conversely, each weighted projective plane* $\mathbb{P}(m_1^2, m_2^2, m_3^2)$ *is a degeneration of* $\mathbb{P}^2$.

A complete classification of degenerations of $\mathbb{P}^2$ was obtained in [7], as well as similar results for degenerations of the two-dimensional quadric and other del Pezzo surfaces.

Exceptional vector bundles on $\mathbb{P}^2$. A vector bundle $\mathcal{E}$ on a nonsingular complex projective algebraic variety X is called *exceptional* if

$$\mathrm{Hom}(\mathcal{E},\mathcal{E}) = \mathbb{C} \quad \text{and} \quad \mathrm{Ext}^q(\mathcal{E},\mathcal{E}) = 0 \quad \text{when } q > 0.$$

An ordered collection of vector bundles $\mathcal{E}_1, \dots, \mathcal{E}_n$ is called *exceptional* if all $\mathcal{E}_i$ are exceptional and

$$\mathrm{Ext}^q(\mathcal{E}_i,\mathcal{E}_j) = 0 \quad \text{for } i > j \text{ and } q \geq 0.$$

An exceptional collection is said to be *complete* if it generates a bounded derived category $\mathcal{D}^{\mathrm{b}}(X)$ of coherent sheaves on X. The presence of a complete exceptional collections imposes very strong restrictions on the variety X. We will consider only the case of the projective plane $X = \mathbb{P}^2$. In this case, any line bundle is exceptional and the triple

$$\big(\mathcal{O}_{\mathbb{P}^2},\ \mathcal{O}_{\mathbb{P}^2}(1),\ \mathcal{O}_{\mathbb{P}^2}(2)\big)$$

is a complete exceptional collection. Moreover, an exceptional collection on $\mathbb{P}^2$ is complete if and only if it consists of three elements.

In the works of A.N. Rudakov [14] and A.L. Gorodentsev and Rudakov [4] a surprising fact was established: one can define certain transformations (mutations) of the complete exceptional collections of vector bundles on $\mathbb{P}^2$, similar to the mutations of Markov triples (2). In particular, the ranks of bundles in complete exceptional collections are exactly Markov triples. These results have generalizations to arbitrary del Pezzo surfaces [8].

Markov numbers in Lobachevsky geometry. The classical Fricke identity states that for any matrices A, B, $C = AB \in \mathrm{SL}_2(\mathbb{R})$ the following equality holds

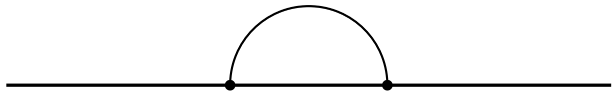
$$\mathrm{tr}(A)^2 + \mathrm{tr}(B)^2 + \mathrm{tr}(C)^2 = \mathrm{tr}(A)\,\mathrm{tr}(B)\,\mathrm{tr}(C) + \mathrm{tr}(ABA^{-1}B^{-1}) + 2.$$

If the matrices are integer and the commutator $ABA^{-1}B^{-1}$ is a parabolic matrix, then $\mathrm{tr}(ABA^{-1}B^{-1}) = -2$ and the numbers

$$\mathrm{tr}(A)/3, \quad \mathrm{tr}(B)/3, \quad \mathrm{tr}(C)/3$$

form a Markov triple. This observation allows us to reformulate many questions about Markov numbers in terms of the action of the modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ and its congruence subgroup $\Gamma(3)$ on the Lobachevsky plane.

Consider the Poincare model $\mathbb{H}$ (the upper half-plane in $\mathbb{C}$) of the Lobachevsky plane. The action of a hyperbolic transformation $A \in \Gamma(3)$ on the closure $\overline{\mathbb{H}}$ has two real fixed points θ and θ'. The circle passing through these points and perpendicular to the real axis is a straight line in the Lobachevsky geometry,



and its image on the quotient $\mathbb{H}/\Gamma(3)$ is a geodesic γ_A. It turns out that γ_A has no self-intersections if and only if $\lambda(\theta), \lambda(\theta') < 3$ and its length can be expressed in terms of Markov numbers. The uniqueness conjecture also has an interpretation in these terms [1]. This approach, using Lobachevsky geometry was applied by D.S. Gorshkov [5, 6] in order to reprove Markov's results in purely geometric methods.

Markov numbers in symplectic geometry. One of the interesting and important problems in symplectic geometry is the question of the classification of Lagrangian tori in the complex projective plane with a symplectic form equal to the Kähler form of the standard Fubini-Study metric. In the recent works of R. Viano [16], significant progress has been made in this direction. In particular, an infinite family of nonequivalent Lagrangian tori parametrized by Markov triples was constructed.

In conclusion, we note that our brief overview is not complete. Unexpected applications of Markov triples continue to appear in various parts of mathematics. We hope that there will be many more other appearances, as well as interesting connections between them will be found. Here is what the outstanding Soviet mathematician B.N. Delone wrote about the master's thesis of A.A. Markov [2]:

> This work, highly appreciated by Chebyshev, is one of the most insightful achievements of the St. Petersburg school of number theory and, perhaps, of all Russian mathematics.

Yuri Prokhorov

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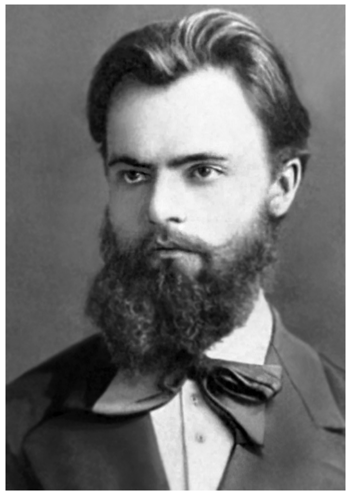
Alexander Mikhailovich Lyapunov (1857–1918)

Alexander Lyapunov created the modern theory of equilibrium stability. Before him, stability problems were typically solved in the first approximation, i.e., by discarding all the non-linear terms of the equations without ascertaining the validity of such a linearization. The foundational work in this area is Lyapunov's doctoral dissertation *The General Problem of Stability of Motion.*



In 1900, as he was preparing for a series of lectures on the theory of probability, Lyapunov suggested a new method: the method of characteristic functions. Having generalized the research by Chebyshev and Markov (Sr.), he proved the central limit theorem in the probability theory under much weaker assumptions than his predecessors. A large series of research papers by Lyapunov concerned the theory of figures of equilibrium of a uniformly rotating fluid whose particles are mutually attracted according to the law of universal gravitation. Before his research, it had been proved that the equilibrium figures for homogeneous fluids are ellipsoidal,[1] and Lyapunov was the first to rigorously prove the existence of close to ellipsoidal equilibrium figures for homogeneous and weakly inhomogeneous fluids.

The great mathematician's grandfather, Vasily Alexandrovich Lyapunov (1778–1847), was a syndic of the Imperial Kazan University Board, i.e., an official representative of the university authorized to conduct its affairs. Vasily Alexandrovich and his wife Anastasia Evseevna had nine children: three sons and six daughters. All the children received a good education.

Mikhail Vasilievich Lyapunov (1820–1868), Alexander's father, graduated from the Mathematics Department of Kazan University with a silver medal in 1839. His main fields of study were mathematics and astronomy. In 1840, he was appointed as the observer-astronomer at the university observatory.

[1] The ellipsoids of Maclaurin (1742) and Jacobi (1831).

After the Kazan fire of 1842, Mikhail Vasilyevich was sent to Pulkovo to oversee the repair of equipment from the university observatory that was damaged in the fire. There, he studied astronomy under Vasily Struve (Friedrich Georg Wilhelm von Struve), Otto Struve (Otto Wilhelm von Struve), Yegor Sabler (Georg Thomas Sabler), and others. In 1845, Lyapunov returned to Kazan to his former position as an astronomical observer, and in 1850, he was appointed director of the Kazan observatory.

In 1853, Mikhail Lyapunov married Sofia Alexandrovna Shipilova, who was then aged 28.

Akexander Lyapunov with his wife, Bolobonovo, 1904. From archives of A.N. Lyapunov.

In 1855, Mikhail Lyapunov was elected a corresponding member of the university, and soon retired. The following year, he was appointed principal of the Yaroslavl Demidov Lyceum, and he and his wife moved to Yaroslavl. They had a son there, Alexander, born on June 6th (May 25th Old Style[2]) 1857. Alexander spent his early childhood with his mother and brothers in the village of Bolobonovo, Kurmyshsky county, Simbirsk province. His mother's letters, sent during this period, show what a troublesome time it was: serfdom had just been abolished, and it was very difficult for her to build a new relationship with the peasants. She did not know how to deal with this unprecedented situation, and no one around her had any advice for her or experience around such issues.

Mikhail Vasilyevich resigned from his position as headmaster of the Demidov Lyceum in 1864 due to health reasons; the family settled in Bolobonovo in a wooden (one-story, five-room) house, which Mikhail Vasilyevich had just built on the estate received by Sophia Alexandrovna from her parents.

Here, Mikhail Lyapunov had a wonderful "library, replete with works in Russian, German, and French, and ranging across not only subjects like mathematics, astronomy, and the natural sciences, but also philosophy, history, ethnography, political economy, and literature."

Initially, the children's education was supervised by their mother, but from the age of seven, they were taught by their father, who devoted himself entirely to this task after his retirement.

When his father died in 1868, Alexander was 11 and a half years old. He continued his studies in the family of Raphail Mikhailovich Sechenov, an artist, married to his father's sister Ekaterina Vasilievna. Here, he and his cousin, his future wife Natalia Rafailovna, a year younger than him, were trained in gymnasium[3] subjects and learned new languages under the guidance of hired teachers and his aunt Glafira Vasilievna Lyapunova.

Two years later, Alexander, his mother, and his brothers moved to Nizhny Novgorod where he was admitted to the third class of Nizhny Novgorod gymnasium. In the autumn of 1876, having graduated with a gold medal, Alexander entered the physics and mathematics department of St. Petersburg University. At first, he studied at the department of natural sciences, being particularly zealously engaged in chemistry with Dmitri Mendeleev. Just a month later, however, he transferred to the department of mathematics, where Pafnuty Lvovich Chebyshev taught at that time, who, in Alexander's own words, "had a significant influence on the character of [my] subsequent scientific activity with his lectures and advice."

In 1880, Alexander Lyapunov received a gold medal for an essay written on a particular topic in analytical mechanics that was suggested by the faculty, and published two papers on hydrostatics in the *Journal of the Physical and Chemical Society* after making a presentation at a meeting of the Society.

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[2] 'Old Style' refers to the Julian calendar, used by the Russian Orthodox Church. For example, Orthodox Christmas Day occurs on January 7th, whereas the use of the Gregorian calendar means Christmas Day is celebrated on December 25th.

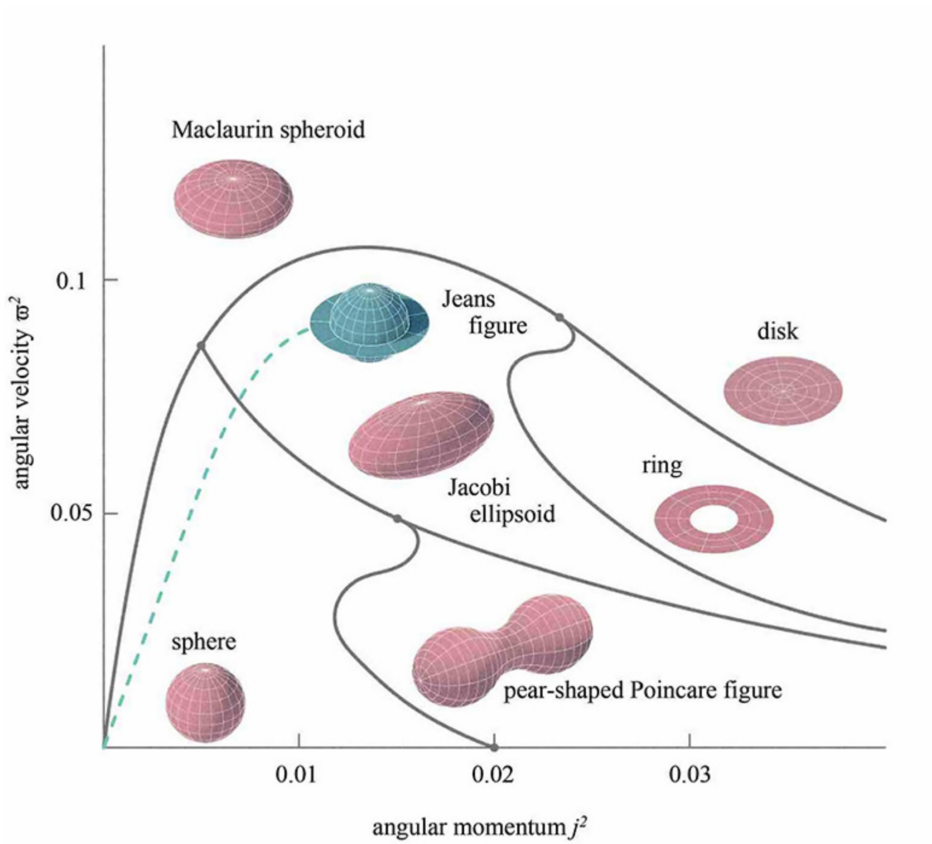
[3] *Gymnasium* is a grammar school.

After his mother died suddenly in 1879, Alexander took over the care of his younger brother Boris. From 1881 to 1885, he and his brother shared a room in the flat of the sister of Ivan Mikhailovich Sechenov,[4] Professor of Physiology. During that time, Alexander liked to work at night. Once a week, the family, including Ivan Sechenov, liked to relax in the company of young students, who gathered on Sundays at Sechenov's sister's flat. In those days, Alexander gave Sechenov lessons in branches of mathematics that he considered especially important for a physiologist. Sechenov delighted in and wholeheartedly supported all of Alexander's scientific successes. For two years, Lyapunov had been diligently working on a problem proposed by Pafnuty Chebyshev. Although he failed to solve Chebyshev's problem, he obtained a closely related result on the stability of the ellipsoids of Maclaurin and Jacobi. In 1885, Alexander Lyapunov defended this work as his thesis for

[4] Ivan Mikhaylovich Sechenov (1829–1905), a Russian physiologist, was one of the originators of objective psychology and the author of the classic Reflexes of the Brain, in which he introduced electrophysiology and neurophysiology into laboratories and the teaching of medicine.

The house in Bolobonovo where Lyapunovs lived after the retirement of Mikhail Vasilyevich. 1904. On the left: Boris Lyapunov, future academician, and philologist. From archives of A.N. Lyapunov.

a master's degree and, in the autumn of the same year, moved to a position as a privatdozent at Kharkov University.



Types of stability figures (ellipsoids of Maclaurin and Jacobi, etc.) in Chebyshev's problem, [6].

Being in Kharkov, Lyapunov taught extensively and strenuously and later considered this period a complete loss to science, although he did obtain a number of remarkable results during this time. His famous version of the central limit theorem in probability theory is also linked to his teaching activities during this period. For the winter holidays, Alexander came to St. Petersburg, where on the 17th of January 1886, he married his cousin Natalia Sechenova, Ivan Sechenov's niece, to whom he was deeply attached from his early childhood.

According to his closest pupil, Vladimir Steklov, Lyapunov worked every day until 4 or 5 AM. He allowed himself almost no entertainment, and when he occasionally appeared (once or twice a year) at a theatre or concert, it was only on the most exceptional of occasions, such as the rare concerts given by his brother, the renowned composer Sergei Lyapunov. Steklov wrote: "He sometimes gave off the impression of being a silent, frowning, and reserved person because he was so absorbed in his scientific speculations that he looked — but didn't see, listened — but didn't hear, as his father-in-law Raphael Sechenov sometimes kindly laughed at Lyapunov's absent-minded behaviors amongst close friends."

The results obtained by Lyapunov on stability formed the subject of his doctoral dissertation, *The General Problem of the Stability of Motion*, defended at Moscow University in 1892. In 1900, Alexander Lyapunov was elected a corresponding member of the Imperial Academy of Sciences, and in 1901 he was elected an ordinary member of the Applied Mathematics Division. A year later, Lyapunov moved to St. Petersburg. He now had the opportunity to devote himself wholly to science and to work on Chebyshev's problem.

In 1908, Alexander Lyapunov was sent to the Fourth International Mathematical Congress in Rome. He planned to meet Henri Poincaré, with whom he had common scientific interests and a scientific correspondence that began during the last decade of the XIXth century. Unfortunately, due to the poor organization of the congress, these plans were not fulfilled, and he never had the chance to meet Poincaré.

However, Lyapunov did meet some other colleagues in Rome: the French mathematicians Émile Picard, Jacques Hadamard, and Édouard Goursat, and the Italian mathematicians Vito Volterra, Giuseppe Veronese, and Tullio Levi-Civita, to name a few.

Lyapunov published four grand memoirs containing a complete solution to the Chebyshev problem. After Lyapunov's tragic death, he left behind a completed extensive manuscript in which he developed his results.

The work accomplished by Lyapunov during the last 15 years of his life is nothing short of a remarkable feat.

In the summer of 1917, Alexander left for Odessa with the hope that its southern climate would benefit his wife's health, which was severely damaged by tuberculosis. They never returned to Petrograd, where they had left their apartment with all their possessions.

The last year of his life was a tragic one. Alexander Mikhailovich struggled to deliver a series of lectures at Novorossiysk University due to complete exhaustion, his impending blindness (from a cataract), and the increasingly deteriorating condition of his wife.

The news he and his brother Boris received from relatives and colleagues on rare occasions was not cheerful. One day they received word that the beloved house built by their father, where they had spent their childhood, had been burned down, along with their library, by peasants in the area.

On the 31st of October 1918, the tragic ending came. His wife Natalia died; Alexander shot himself and was taken to a surgical clinic with a gunshot wound, where he died on the 3rd of November 1918 without regaining consciousness.

In the note he left behind, he wrote that he wished to be buried in the same grave as his wife.

Lyapunov's work was widely recognized. During his lifetime, he was elected an honorary member of the Universities of St. Petersburg, Kharkov, and Kazan, a foreign member of the Accademia dei Lincei in Rome, a corresponding member of the Academy of Sciences of Paris, a foreign member of the Circolo Matematico di Palermo, an honorary member of the Kharkov Mathematical Society, a full member of the Moscow Mathematical Society, etc. Lyapunov Stability Theory is now studied in universities around the world. In 1969, the USSR Academy of Sciences established the Lyapunov Gold Medal, and after the collapse of the USSR in 1995, the Russian Academy of Sciences established the Lyapunov Prize. In Moscow and Kharkov, there are Lyapunov streets that are named after him.

Askold Khovansky and Tatiana Belokrinitskaya

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Lyapunov stability theory

Alexander Mikhailovich Lyapunov (1857–1918) was a great Russian mathematician. His classic works on the qualitative theory of differential equations, mechanics, mathematical physics, and probability theory are internationally recognized. His theory of stability has countless applications in theoretical and applied research; its influence can hardly be overestimated.

Lyapunov's Stability Theorem. *Recall a fundamental and very elegant theorem on the stability of a stationary solution, to one degree or another familiar to every student of mathematics.*

The motion of a point $y \in \mathbb{R}^n$ with velocity $F(y,t)$, depending on the position of the point and on the time t, is determined by the differential equation

$$y' = F(y,t) \tag{1}$$

and by the position y_0 of the point y at the initial moment t_0.

A solution $y_0(t)$ of equation (1) with the initial data $y_0(t_0) = y_0$ is called *Lyapunov stable* if any solution $y_1(t)$ with sufficiently close initial data y_1 at any moment of time is sufficiently close to $y_0(t)$, that is, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $|y_1 - y_0| < \delta$ implies $|y_1(t) - y_0(t)| < \varepsilon$ for $t > t_0$, where $y_1(t)$ denotes a solution with the initial data $y_1(t_0) = y_1$.

A solution $y_0(t)$ is called *asymptotically stable* if it is Lyapunov stable and there exists a $\delta > 0$ such that $|y_1 - y_0| < \delta$ implies $\lim_{t\to\infty}(y_1(t) - y_0(t)) = 0$.

Below we consider autonomous differential equations $y' = F(y)$ with $F(0) = 0$, and we discuss the stability problem for the stationary solution $y_0(t) \equiv 0$ of such equations.

Let $\tilde{y}' = A\tilde{y}$ be the linearization of the equation $y' = F(y)$ at the fixed point $y = 0$, where A denotes the differential of the vector function F at 0. Let $\Lambda(A)$ denote the maximum of the real parts for the eigenvalues of the differential A.

Lyapunov's Theorem. *If $\Lambda(A) < 0$, then the stationary solution $y(t) \equiv 0$ to the equation $y' = F(y)$ is asymptotically stable. If $\Lambda(A) > 0$, then the stationary solution is not Lyapunov stable.*

This theorem almost always solves the stability problem for the stationary solution: it fails to apply only in the exceptional case where $\Lambda(A) = 0$.

Lyapunov Theorem splits the spaces of the first degree Taylor polynomials for vector functions F, $F(0) = 0$, into the following three sets:

1) a *stable* set defined by the condition $\Lambda(A) < 0$, for which, regardless of the other coefficients of the Taylor series for the vector function F, the stationary solution is stable;
2) an *unstable set* defined by the condition $\Lambda(A) > 0$, for which, regardless of the other coefficients of the Taylor series for the vector function F, the stationary solution is unstable;
3) a *neutral set* defined by the condition $\Lambda(A) = 0$, for which the stability of the stationary solution depends on the remaining coefficients of the Taylor series for the vector function F.

The partition of the space of 1-jets and vector functions F into stable, unstable, and neutral sets is semi-algebraic: the sign of $\Lambda(A)$ can be determined by calculating the values of a finite number of special polynomials in the coefficients of the differential matrix A.

The relative simplicity of this partition does not mean at all that the question of stability is simple. A similar partition of the space of k-jets of a vector function F into stable, unstable, and neutral sets for a large k is extremely sophisticated, and definitely not semi-algebraic.

Therefore, the very general and relatively simple stability criterion found by Lyapunov should be regarded as a remarkable result and a rare piece of luck. In 1892, he published his fundamental work on the general problem of motion stability. Lyapunov's theory of stability has become classical and is included in the compulsory mathematics program at universities worldwide.

Lyapunov function. Lyapunov has found a surprisingly simple and flexible proof of his stability criterion.

A function $G(y)$ is called a *Lyapunov function* for the dynamical system $y' = F(y)$ if G does not increase as y moves along the trajectory of the dynamical system, that is, if $\frac{dG(y(t))}{dt} = \langle \operatorname{grad} G, F \rangle \le 0$, where $\operatorname{grad} G$ is the gradient of G and $\langle v, w \rangle$ denotes the scalar product of v and w.

For any constant C, the domain $G \le C$ is invariant with respect to the dynamical system. Therefore, if the Lyapunov function G has a strict local minimum at y_0, then $y(t) \equiv y_0$ is a Lyapunov stable stationary trajectory of the dynamical system. If, in addition, the strict inequality $\langle \operatorname{grad} G, F \rangle < 0$ is satisfied at the non-critical points of G, then this stationary solution is asymptotically stable.

Indeed, given a differential A with $\Lambda(A) < 0$, it is easy to construct a positive definite quadratic form, which in a neighborhood of the origin is a Lyapunov function of the system under consideration. The existence of such a quadratic form immediately implies the Lyapunov stability of the stationary solution. The rest of the theorem can be verified just as easily.

Imagine a mechanical system, the energy of which is conserved or reduced over time, for example, due to friction. Energy is a Lyapunov function of this system. Damped small oscillations of such a mechanical system near the equilibrium position give a visual representation of the dynamics of the system around the stable equilibrium position described in the Lyapunov Theorem.

Chebyshev's problem. In this part of the note, we use one article by V.A. Steklov, the closest student of A.M. Lyapunov, dedicated to the work of his teacher. As an aspiring mathematician, Lyapunov began solving a problem of P.L. Chebyshev, which he was engaged in until the last days of his life. Chebyshev formulated his problem as follows:

> It is known that a liquid homogeneous mass, whose particles are attracted by Newton's law and which rotates uniformly around a certain axis, can maintain the shape of an ellipsoid as long as the angular velocity ω does not exceed a certain limit.
>
> For values of ω greater than this limit, ellipsoidal figures of equilibrium become impossible.
>
> Let ω be a value of the angular velocity with an equilibrium ellipsoid E. Let us give the angular velocity a sufficiently small increment ε. The question is, are there other equilibrium figures for the angular velocity $\omega+\varepsilon$, not ellipsoidal ones, continuously depending on ε and coinciding with the ellipsoid E for $\varepsilon=0$?

This extremely sophisticated question, connected with the problem of possible forms of celestial bodies, interested many scientists.

Lyapunov obtained the first partial result related to the Chebyshev problem in his master's thesis in 1885.

The great French mathematician A. Poincaré also dealt with this problem. He investigated the first approximation and, on the basis of this approximation to the solution (without rigorous proof and without estimating the error), he came to a conclusion about the existence of an infinite number of different forms of equilibrium close to ellipsoids. He did not know that Lyapunov had reached similar conclusions three years earlier. Poincaré's results were viewed by contemporaries as an outstanding achievement.

In 1901, Lyapunov was elected a full member of the Imperial Academy of Sciences and could devote himself entirely to scientific activities. After that, Lyapunov published a series of memoirs devoted to the Chebyshev problem, which, even from a purely external side, make a strong impression: their volume is over 1000 large-format pages. Using a completely original approach, Lyapunov constructed successive approximations of any order, proved the convergence of the corresponding series and thus obtained a complete solution to the problem.

After the tragic death of A. M. Lyapunov, a completed manuscript of 489 pages remained, containing deep generalizations of his results.

Askold Khovanskii

Georgy Feodosevich Voronoy (1868–1908)

Georgy Voronoy[1] published a total of 12 papers, all of which became classics. The works belong mainly to three major areas: algebraic number theory, theory of quadratic forms, and analytic number theory, and they harmoniously combine arithmetic, geometric, and analytic methods.

In mathematics there are individuals who appear suddenly, their obsession with their science is amazing, their discoveries excite the scientific community, attracting students and followers, they burn brightly and, burning out quickly, they define new directions of scientific research for many years to come. Georgy Feodosevich Voronoy (1868–1908) was just such a phenomenon in Russian mathematics. He was born on April 16, 1868, in the village of Zhuravka, Poltava governance, to the family of teachers Feodosy Yakovlevich and Cleopatra Mikhailovna Voronoy.

Georgy Voronoy published his first article while still at school. In 1884, Professor Vasily Ermakov of the Kiev University began publishing the Journal of Elementary Mathematics, which among other things offered topics for student papers in mathematics. Voronoy turned out to be the only one who submitted a paper on the topic *Factorization of polynomials based on the properties of the roots of quadratic equations*. Ermakov liked the work and published it in his journal in 1885.

The same year Voronoy graduated from grammar school and entered St. Petersburg University. At the university, Georgy diligently attended lecture courses in pure mathematics, which increasingly fascinated him. In his diary we read that "Professor Sochocki's lectures in the special course of higher algebra I prefer to all others." After graduation, Voronoy was retained to prepare for his master's examination (1889) on the recommendation of professors Andrei Markov, Alexander Korkin, and Julian Sochocki.

[1] Spelling variant: Voronoi.

Voronoy wrote in his diary in 1887

> What concerns me most is whether I have enough talent [...] at moments when the mind embraces an idea that used to slip away like a ball, I forget that I exist [...] my recent successes I owe to the habit of thinking without pen and paper. All the propositions I have proved had arisen quite independently [...] I hope that this habit of thinking in this way will serve me well.

After successfully defending his thesis, Voronoy was appointed professor of mathematics at the Imperial University of Warsaw in the Department of Pure Mathematics, where he worked for the rest of his life, with a short interruption. There he met and befriended professor of mathematics and mechanics Nikolai Borisovich Delone and his family. Boris Delone (the son of Nikolai Delone) liked to tell how Voronoy used to visit them and stay up late talking with his father. Voronoy's influence on Delone's work later turned out to be very significant.

Ukraine's coin *2 hryvnias* with Voronoy.

According to Delone, Voronoy thought geometrically but had to translate his reasoning into arithmetical language, because the leaders of the St. Petersburg school and especially Andrei Markov, the main opponent of the thesis, did not welcome the geometrical character of the presentation and a thesis written in geometrical language might not have been allowed. His doctoral dissertation was brilliantly defended in 1897 at St. Petersburg University. Both his master's theses, *On algebraic integers depending on the roots of an equation of third degree* (1894) and *On a generalization of the algorithm of continued fractions* (1896) were devoted to solving the most important problems in this field and were awarded the V.Ya. Bunyakovsky Prize in 1896.

Some works by Boris Delone and part of the famous monograph by Boris Delone and Dmitry Faddeev *Theory of irrationalities of the third degree* were devoted to the geometrization of Voronoy's algorithm. His dissertation was only printed in Russian, which was partly why its results remained little known abroad for a long time, and some of them were rediscovered in the following decades.

The article by Georgy Voronoy in analytic number theory *Sur un problème du calcul des fonctions asympototiques* (1903) stimulated the development of this branch of research in modern mathematics. Another Voronoy's discovery was the Voronoy method of generalized summation of series (1902). Unfortunately, it did not become widely known and was rediscovered in 1919 by the Swedish mathematician Niels Erik Norlund (1885–1981).

Georgy Voronoy was married to Olga Mitrofanovna Kritskaya, a girl from a noble family whose estate Bogdany was near his Zhuravka. Olga Kritskaya was his great love from his youth. They had six children. In addition to his own large family, Voronoy also took care of his sister's family (she widowed early) with seven children. All his children, except one of his daughters who died in childhood, were well educated and became specialists: physicians and teachers. The two elder daughters Alexandra and Maria and the eldest son Alexander and their families became victims of Stalinist repression. His younger son Yury Voronoy (1896–1961) was a well-known surgeon and a doctor of medical sciences; he became famous for performing the world's first human-to-human kidney transplant in 1933. Voronoy's scientific legacy was collected in a three-volume edition and commented on by famous mathematicians Boris Venkov, Boris Delone, Yury Linnik, Nikolai Chudakov, and Igor Shafarevich [3].

Professor Voronoy taught at the University of Warsaw (1894–1906, 1908) and the Polytechnic Institute (1898–1906, Warsaw; 1907–1908, Novocherkassk); his students included Waclaw Sierpinski, Tadeusz Banachiewicz, and others.

Unfortunately, Georgy Voronoy passed away untimely, at the age of 40 in Warsaw, on November 7 (20), 1908, from cholelithiasis and was buried in Zhuravka (Ukraine).

Natalia Lokot

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G.F. Voronoy: from Numbers to Parallelohedra

Georgy Feodosevich Voronoy (1868–1908) was an outstanding mathematician, a student of Andrei Markov, one of the most prominent specialists in the St. Petersburg number theory school.

During his study at St. Petersburg University, Voronoy wrote a remarkable paper on Bernoulli numbers (1890). This work was admired by professor Andrei Markov, who offered Georgy Voronoy the chance to continue his study at the university "to prepare for acquiring a professorship."[1] Under Markov's guidance, Voronoy wrote his candidate[2] thesis "On algebraic numbers depending on the root of a 3rd degree equation," devoted to the problem of finding the basis in the ring of algebraic integers of a cubic field (1894).

In his doctoral thesis *On a Generalization of the Algorithm of Continuous Fractions* (1897), Voronoy constructed an exceptionally efficient algorithm for finding fundamental units of cubic fields. In this work, Voronoy succeeded in gaining much deeper insight into the essence of the question than his famous predecessors such as Egor Zolotarev, Hermann Minkowski, and others. The Voronoy construction essentially generalized the existing algorithm for dimension two to three-dimensional lattices.

Voronoy's research in algebraic number theory was awarded the Bunyakovsky Prize of the St. Petersburg Academy of Sciences.

In 1903, Voronoy published a memoir on the analytic number theory. Let

$$S(n) = \tau(1) + \tau(2) + \ldots + \tau(n),$$

where $\tau(k)$ denotes the number of divisors of k. In the work "On a problem in the theory of asymptotic functions" Voronoy made significant progress in the Dirichlet divisor problem, namely, for the function $S(n)$ he found an asymptotically more accurate value than Dirichlet's estimation. This important work was the starting point for the research of such outstanding mathematicians as Wacław Sierpiński (who was a student of Voronoy) and Ivan Vinogradov.

[1] This is more or less similar to modern postgraduate study.

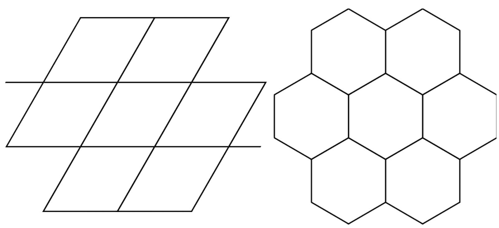
[2] Master degree.

In the late 1890s, influenced by the work of Minkowski on the geometry of numbers, Voronoy begins his study in the theory of quadratic forms

$$f(x_1, \ldots, x_n) = \sum_{i,j}^{n} a_{ij} x_i x_j.$$

Minkowski, the creator of the geometry of numbers, showed deep interest in these studies of Voronoy during their personal meeting[3] at the Third ICM (Heidelberg, 1904). In his extensive memoir "Properties of positive-definite quadratic forms" (1908), Voronoy gave an algorithm for finding all locally extremal positive-definite quadratic forms in n variables. The geometric meaning of this problem is to find all locally densest lattice sphere packings of the space $\mathbb{R}^n$.

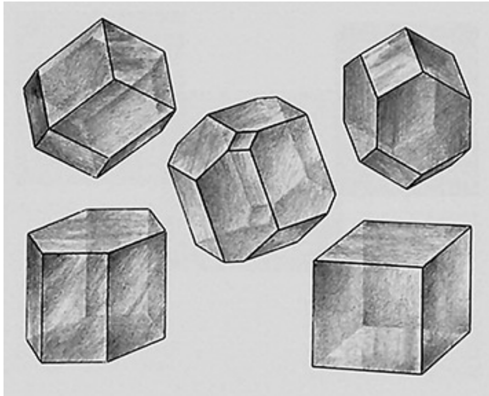
The last, most profound Voronoy's memoir "Studies on the primitive parallelohedra" (published posthumously in 1909) was devoted to the study of polyhedra[4] of a special kind. In 1885, the remarkable crystallographer Evgraf Fedorov introduced the concept of a parallelohedron as a convex polyhedron, parallel copies of which, attached to each other along whole faces, can fill the entire space without overlapping. It is easy to see that the two-dimensional analog of the parallelohedron is either a parallelogram or a centrally symmetrical hexagon (see the illustration below). Parallelepiped and regular hexagonal prisms are examples of three-dimensional parallelohedra. Fedorov found all five combinatorial types of three-dimensional parallelohedra (see the illustration on page 176).



Planar parallelohedrons.

In the second half of the 1890s, H. Minkowski studied parallelohedra and found a number of their fundamental properties. In the course of this research, Minkowski discovered one of the most remarkable theorems about convex polyhedra. Namely, he has shown that there exists one, and only one convex polyhedron with assigned face directions and areas. Because a parallelohedron tessellates the space with shifted self-copies, each of its faces (hyperface in the high-dimensional case) has an opposite, equal and parallel to it. Using this fact and the mentioned above general theorem, Minkowski deduced the

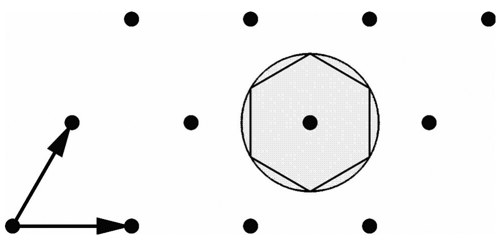
[3] Their both were among the invited congress speakers.
[4] We use the words *polyhedron* and *polytope* as synonyms.



Five types of three-dimensional parallelohedra.

following corollary. Any parallelohedron is centrally symmetric and all its faces (hyperfaces) are also centrally symmetrical. Later, Boris Delone added another property: if we project a parallelohedron along its edge (a face of codimension two) onto the plane, we always get either a parallelogram or a centrally symmetrical hexagon. In 1954, Boris Venkov established that these three parallelohedron properties are not only necessary, but also sufficient.

There is an easy way to build parallelohedra. To do this, we need to take the lattice Λ of integer points with respect to a chosen arbitrary basis in $\mathbb{R}^d$ (see the illustration below). For a point $\lambda \in \Lambda$ we construct its Voronoy cell $V_\lambda := \{x \in \mathbb{R}^d \colon ||x-\lambda|| \leq ||x-\lambda'||, \forall \lambda' \in \Lambda\}$. It is easy to see that the cell V_λ is a convex polyhedron, which is also a special case of parallelohedra. Such polyhedrons are now called Voronoy parallelohedra.



Lattice. Voronoy's cell.

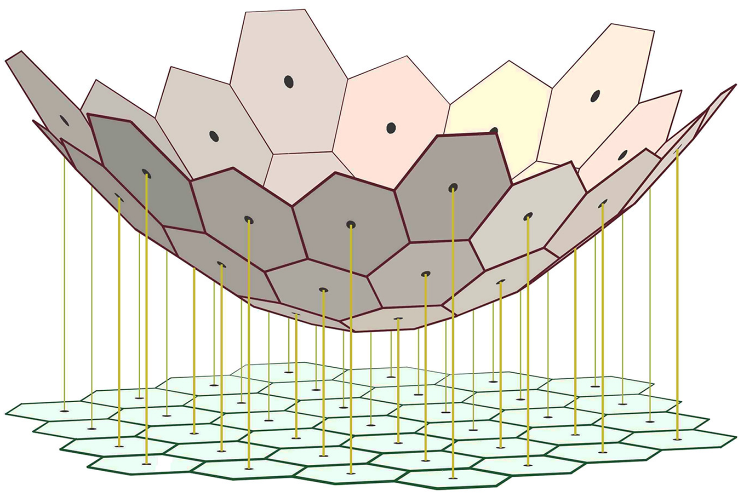
The geometry of the Voronoy parallelohedron is uniquely defined by the lattice Λ and, ultimately, by its basis. In turn, the basis in the n-dimensional Euclidean space is determined by $n(n+1)/2$ parameters: the lengths of the vectors and the angles between them.

However, not every parallelohedron is a Voronoy parallelohedron. For example, affine transformations preserve the class of parallelohedra, but, generally, not the class of Voronoy parallelohedra. In the case of dimension two, the class of parallelohedra consists of parallelograms and centrally symmetrical hexagons. But Voronoy parallelohedra are those that can be inscribed in a circle.

Voronoy, having developed the geometry of positive definite quadratic forms, constructed the theory of n-dimensional Voronoy parallelohedra and, in particular, proposed an algorithm that calculates, in a given dimension, the total number of combinatorial types of Voronoy parallelohedra.

What about finding other types of parallelohedra? In his last memoir Voronoy considered *primitive* parallelohedra defined by the following property. At every vertex of n-dimensional space partition by polyhedra, exactly $n+1$(the minimal possible number) polyhedra meet. It is known that for $n=2,3$ there exists only one primitive parallelohedron, and for $n=4$ there are three primitive parallelohedra out of 52 polyhedra types. As n grows, the number of primitive parallelohedra grows rapidly, but their share in the total number of all types of parallelohedra decreases. Voronoy proved that (in arbitrary dimension) any primitive parallelohedron is affine equivalent to some Voronoy parallelohedron.

Two key ideas used in Voronoy's proof are both related to the possibility of lifting a space partition to a paraboloid. The first idea: the Voronoy diagram of any discrete set $X \subset \mathbb{R}^n$ (and not just a lattice) can be "lifted" to a polyhedron circumscribed about a circular paraboloid $y = x_1^2 + \cdots + x_n^2$ in the space $\mathbb{R}^n \oplus \mathbb{R}^1$. Voronoy called such a polyhedron a generatrix. The generatrix is projected into a Voronoy tiling of the space $\mathbb{R}^n$ (see the illustration below). In this case, the set of the generatrix face tangent points is projected into the set of



The generatrix.

points, for which the Voronoy cells are projections of the corresponding faces. This fact is actively used in the computational geometry to study Voronoy diagrams.

The second Voronoy's idea happened to be difficult to implement. The statement was the following. Any partition T of the space $\mathbb{R}^n$ in primitive parallelohedra can be lifted to a polyhedron circumscribed about some elliptic paraboloid Π. Let the partition T be lifted to such a polyhedron Π. Consider the affine transformation φ taking the elliptic paraboloid Π to the circular one Π_V. Then, the map φ takes the partition T to the Voronoy tiling T_V, and primitive parallelohedra to Voronoy parallelohedra, respectively.

This fragment of the last Voronoy's essay, in which he proves the affine equivalence of primitive parallelohedra to Voronoy parallelohedra is, according to Delone, the deepest part of the whole memoir.

The problem of a complete parallelohedra classification is still far from being solved. The keystone to this problem is the following Voronoy's question (1908): Whether any (not necessarily primitive) n-dimensional parallelohedron is affine equivalent to a certain Voronoy parallelohedron. Now, more than a century later, this question is affirmatively answered (besides the cases $n=2, 3$) for $n=4$ (Delone, 1929) and $n=5$ (A. Garber and A. Magazinov, 2019).

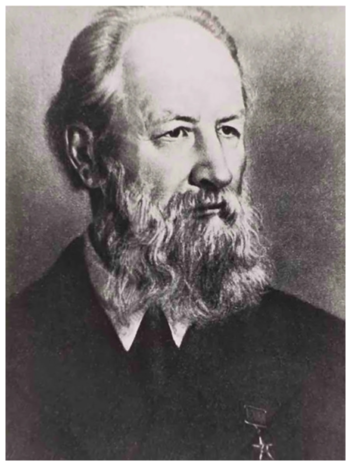
Nikolai Dolbilin

First years of Soviet period

Aleksey Nikolaevich Krylov (1863–1945)

The prominent Russian engineer and mathematician, academician Aleksey Nikolaevich Krylov, made major contributions to the mathematics of shipbuilding and naval engineering, and to the design of the first dreadnought vessels. After the revolution, he was one of the organizers of the Physico-Mathematical Institute of the Academy and helped to retain the Russian Academy of Sciences under new political power.

A.N. Krylov was born in 1863 into a family of Russian provincial gentry, his father being a former artillery officer. Among his close relatives were the famous physiologist Sechenov, the opthalmologist Filatov and the future mathematician Lyapunov (who was a few years his senior). Over the past century the highly talented Lyapunov family counts several first-rank scientists and intellectuals. When Aleksey was ten years old, the family moved to Marseille, where he obtained a solid knowledge of French, and then to Riga, where he studied German and Latin. This had been the intention of Aleksey's father who always stressed the importance of juvenile immersion into the linguistic environment for the study of foreign languages.



At the age of 15 Aleksey chose the Naval School in St. Petersburg to continue his education. One of the reasons for this choice, apart from the long-standing military tradition of the family, was his aversion to ancient languages which were not taught to future Naval officers (in contrast to university students). However, his solid background in Latin acquired during his studies in Riga would later prove valuable, as it enabled him to read the memoirs of Gauss and Euler; much later his Russian translation of Newton's *Principia* became a classic.

Very soon, the young student showed a profound interest in mathematics and naval engineering. After three years in Naval School he was appointed ensign. His first scientific work, published in 1887, treated the theory of compass deflection and was based on Gauss' treatise on geomagnetism and on the works of Fourier. From 1888 to 1890, he continued his studies at the Naval

Academy. Upon graduating he was appointed lecturer at the Naval School while continuing his mathematical education at St. Petersburg University where he attended the courses of Professors A.A. Markov, A.N.Korkin, D.A. Gravé, along with many others. His important early contribution to the teaching of Mathematics at the Naval School rested on the correct use of approximate calculations. At that time students and engineers often carried out extensive calculations with dozens of decimal places without a clear understanding of the fact that most of these decimal places were in fact superfluous and often totally erroneous. Krylov's simple yet effective remarks on the matter led to a drastic simplification of these calculations, and they became the basis of his innovative treatise on approximate calculations (*Lectures on Approximate Calculations* (1911)) which has often been republished, each time with additions.

In 1895 Krylov started his fundamental study of oscillatory ship motion in waves, and in 1898 his work in this area was awarded a Gold Medal by the Royal Institution of Naval Architects. In this work Krylov deduced simple differential equations for the 6 parameters (3 coordinates of the center of mass and 3 Euler angles) determining the position of a ship following a course of constant angle to the direction of undulation. These differential equations are then integrated by the method of successive approximation; this in turn allows explicit computation of the forces acting on the ship and determination of the resulting tensions in the ship's body.

In the following years Krylov actively participated in the naval construction program of the Russian Navy with a special emphasis on the concepts of stability, buoyancy, and survivability of military vessels. In simple terms, he insisted that a damaged ship may lose floatability but even in this extreme situation it should preserve stability and not capsize. The importance of this concept (at that time completely new) was dramatically confirmed by the tragic and heavy losses suffered by the Russian navy during the Russo-Japanese War of 1904–1905. Three years later Krylov was appointed Inspector General of Naval Construction in Russia. This appointment came at a time of great technological changes in international naval engineering following the invention of the heavily armed dreadnought vessels. Krylov actively participated in the rearmament of the Russian navy in the years preceding the First World War; he made a decisive contribution to the construction of the first Russian naval vessels of the dreadnought class.

In 1916 Krylov was elected a full member of the Russian Academy of Sciences. After the revolution, which took place the following year, he contributed much to the survival of the Academy and carried out numerous administrative duties. Together with academician Steklov, he played a decisive role in the organization of the new Physico-Mathematical Institute of the Academy which he headed for several years after the premature death of Steklov. He also played an essential role in the revival of the Russian trade fleet, which was badly damaged by the Revolution and the Civil War. He was sent to Western Europe to acquire railway equipment, particularly locomotive engines. In 1918 he wrote a famous 70-page memoir on numerical methods in ballistics, drawing and improving on recent methods of the Norwegian Carl Stormer (Adams–Stormer-method). This was published in France in 1927 on the initiative of Jacques Hadamard. Krylov remained totally loyal to the new Soviet authorities, although both his two sons perished during the Civil War fighting on the opposite side. While many of his former colleagues were compelled to emigrate (such as the eminent chemist and military engineer, V. Ipatiev, also a member of the Academy) — or perished during the purges, Krylov escaped prosecutions and remained an eminent expert in Naval Architecture. In 1944, as a tribute to his lifelong service in Naval Engineering, he was elected an Honorary Member of the British Royal Institution of Naval Architects. He died the following year at the age of 82.

Mikhail Semenov-Tian-Chanski

Vladimir Andreevich Steklov (1864–1926)

Vladimir Steklov was a leading mathematical physicist who obtained many significant results in mathematical analysis, mechanics, fluid dynamics, theory of elasticity, etc. In particular, he developed a technique for solving boundary value problems by an expansion of solutions in terms of eigenfunctions; also, he extended numerous results in potential theory and in heat conduction theory to a large class of nonconvex domains. He reorganized the Russian Academy of Sciences (RAS) into the Academy of Sciences of the Soviet Union and was the founder of its school of mathematical physics.

The Steklov Mathematical Institute of the RAS (or "steklovka" for short) is widely known as the leading mathematical center of the Soviet Union and later Russia. Steklov, its eponymous founder, combined the talents of a pragmatic administrator and a first-class mathematician. During the day, he was involved in administrative affairs, then slept for a few hours in the evening, and then worked on scientific problems until four or five in the morning.

Steklov was born in 1864 into the family of the priest Andrey Ivanovich Steklov, a teacher of Russian history and Hebrew at the Nizhny Novgorod Theological Seminary. His mother was Ekaterina Alexandrovna, née Dobrolyubova.[1] From his childhood, he was interested in natural sciences. Together with his uncle Ivan Steklov, also a priest, he read Mendeleev's *Chemistry* and conducted many experiments in their own physics room. Steklov, like his relatives, had a fine baritone voice. He sang at parties and celebrations in his youth, and even contemplated an opera career, but his passion for mathematics took over.

Relying on his excellent memory, Steklov did not study particularly hard until the sixth form of the gymnasium.[2] His father, wanting to taunt him, once said: "I thought you were only lazy, but apparently you are just incapable."

[1] Her brother, Nikolai Dobrolyubov (1836–1861), was a well-known literary critic. Family conversations about his late uncle and his beliefs influenced Steklov's worldview.

[2] *Gymnasium* is a grammar school.

After these words, Steklov spent the holidays diligently studying the course (Greek, Latin, Russian, history, etc.) of the previous five years. The same thing happened at university: He was expelled from the first-year course of the Faculty of Physics and Mathematics of Moscow University (he failed to answer the trivial question "what is the longest day in Moscow?"). He tried to enter the Medical Faculty but ended up in Kharkov at the Faculty of Physics and Mathematics. There, he took up his studies seriously.[3] Steklov organized clubs in mathematics and physics, and after lectures read extracurricular books of his choice for up to 8 hours a day.

He kept a detailed diary but it has not yet been published, so his most complete biography is his own *Memoirs* [1], written around 1923. In them, he explains how in 1886, at the age of 22, after having fallen in unrequited love with his second cousin, he decided to live solely for science. At that time, Lyapunov moved to Kharkov, and Steklov became his pupil and later his best friend. In 1890, Steklov married Olga Drakina; they had a daughter, Olga, who died of meningitis in 1901.

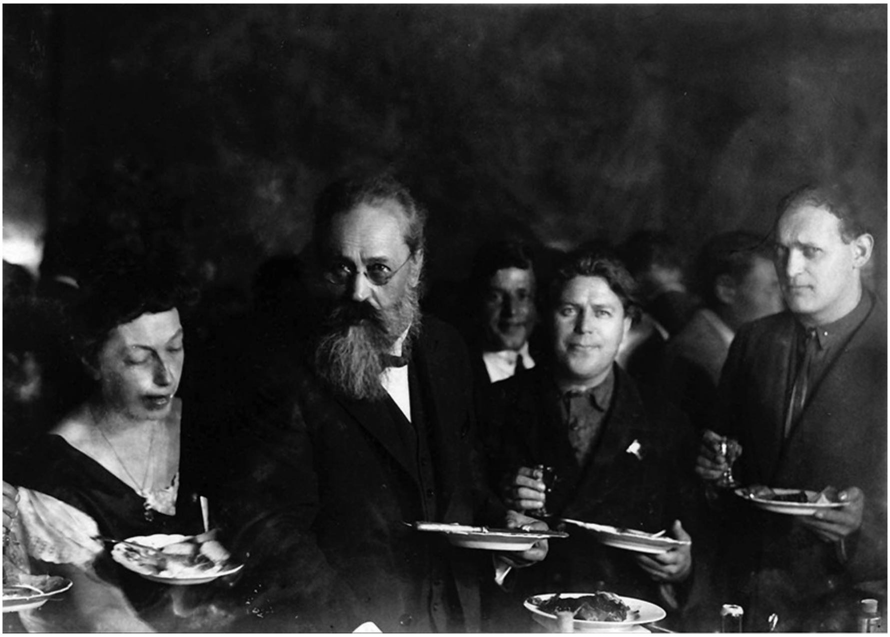
In Kharkov, Steklov was active in public life and fought against the university charter of 1884, which considerably curtailed the autonomy of the universities. He was once even challenged to a duel by a medical professor, which he only laughed at wryly. In 1904, Steklov was elected Rector of the University, but he refused to accept the post, taking only the Dean of the Faculty of Physics and Mathematics duties. His last year in Kharkov, 1905, was a violent one, with student uprisings and barricades. As Dean, Steklov mediated between the insurgents and the troops who arrived to quell the unrest, preventing the situation from escalating.

In 1906, Steklov moved to St. Petersburg, taking up a position in the Mathematics Department vacated by the retirement of Andrei Markov. Professors' salaries were not very high, so many taught at several institutions, "wasting their remarkable strength and destroying their talents," as Steklov put it, resolutely refusing any part-time positions in favor of science. At the 1912 International Congress of Mathematicians in Cambridge, he was one of the vice presidents, together with Vito Volterra and Jacques Hadamard among others.

Even then, Steklov saw only two outcomes for Russia: either the destruction of the monarchy and the ensuing civil war or the imposition of brutal despotism. Steklov was a man of science first and foremost and cared about its needs, so after the revolution of 1917, he, along with Krylov and others, cooperated with the new government,[4] and being concerned about the Academy of Sciences

[3] He did, however, occasionally allow himself to go to the opera.

[4] It can be argued that Steklov (together with Aleksey Krylov, Sergey Oldenburg, and others) saved the Academy of Sciences, as many Bolsheviks who came to power believed that a new academy should be established on Marxist ideological grounds; it cost Steklov great effort to defend both the existence of the Academy and its funding. For example, it was not



Academician Steklov among guests on the 200th Anniversary celebration of the Academy of Sciences. Leningrad, 1925, [6].

took on the role as its vice president from 1919. He succeeded in having seismic stations built all over Russia and establishing the Institute of Physics and Mathematics in 1921, which after his death was named after him.[5]

In the years of hunger during the Civil War, Steklov, together with Gorky, lobbied Lenin and Lunacharsky for a special 'scientific ration' for members of the Russian Academy of Sciences. In 1920, his wife Olga died of malnutrition and scurvy. Left all alone, Steklov was absorbed in mathematics and the needs of the Academy.[6]

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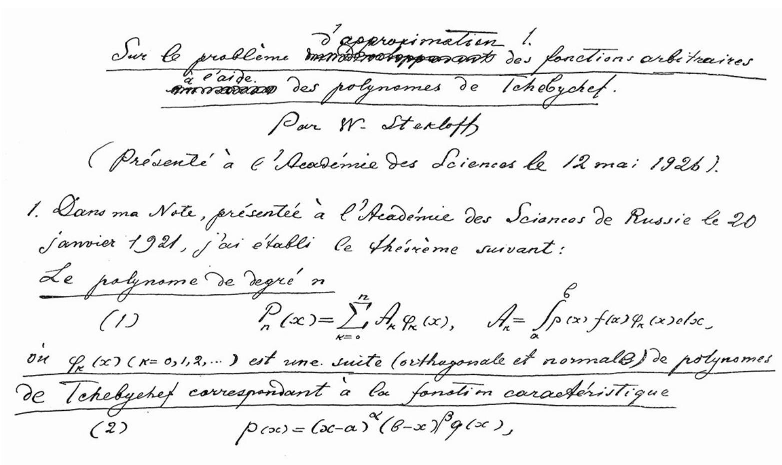
at all clear to the new government whether it was worth spending money on firewood to heat the zoological and anatomical museums in wartime and famine conditions.

[5] Initially, the Institute was made up of mathematics, physics, magnetology, and seismology departments. In 1926, the Institute of Physics and Mathematics was named after Steklov. In 1934, the institute was divided into two independent institutes: Physics and Mathematics, the latter being named the Steklov Mathematical Institute of the Academy of Sciences.

[6] In 1923, Steklov managed to achieve the adoption of a regulation according to which the Russian Academy of Sciences became the main administrator of its budget. It was in force until 2013 [4]. As Vladimir Smirnov wrote, "V.A. [Vladimir Andreevich] took over the work on both the administrative and scientific parts at a time when it seemed that nothing could be done, everything was falling apart. But it was not in V.A.'s temperament to fold his hands at a dire moment. The more difficult the situation, the more energetically he took up his work. He organized the printing of scientific works and purchasing of books and appliances from abroad, he worked hard on the restoration of the ruined seismic network, and

Steklov took part in the study of the Kursk Magnetic Anomaly.[7] In 1925, he caught a severe cold but endured the illness on his feet without reducing his activity in the slightest. Due to complications, Steklov died in 1926 in Gaspra, where he had gone for treatment. He is buried next to his uncle Nikolay Dobrolyubov.

As for Steklov's scientific achievements, we shall only briefly describe the beginning of his career. Alexander Lyapunov asked Steklov to investigate the question of the motion of a billiard ball on a rough surface, building on the work of Coriolis. The task was to test not only Steklov's knowledge but also his ability to work independently. Steklov coped brilliantly with the task, presented his work in February 1888, and was retained by the university on Lyapunov's recommendation. In the summer of 1888, Steklov studied, this time quite independently, Gustav Kirchhoff's *Lectures on Mathematical Physics.* Having studied all the material available on the motion of a solid body in a liquid, he began to think about the problem by himself and found a case not mentioned by Kirchhoff. Steklov wrote: "Such a thrill from my own scientific work, I must say, I never experienced after." Lyapunov was delighted with his student's success but added that, unfortunately, this case had already been discovered by Clebsch.


Sur le problème d'approximation des fonctions arbitraires à l'aide des polynomes de Tchebychef.

Par W. Stekloff

(Présenté à l'Académie des Sciences le 12 mai 1926).

1. Dans ma Note, présentée à l'Académie des Sciences de Russie le 20 janvier 1921, j'ai établi le théorème suivant:

Le polynome de degré n

(1) $P_n(x)=\sum_{k=0}^{n} A_k \varphi_k(x), \quad A_k=\int_a^b p(x) f(x) \varphi_k(x)\,dx,$

où $\varphi_k(x)$ $(k=0,1,2,\dots)$ est une suite (orthogonale et normale) de polynomes de Tchebychef correspondant à la fonction caractéristique

(2) $p(x)=(x-a)^{\alpha}(b-x)^{\beta} q(x),$

An example of Steklov's handwriting.

In 1893, Steklov defended his master's thesis *On the Motion of a Solid Body in a Fluid*, providing a missing conservation law, which is a quadratic form

he organized the Mathematical Study and then the Institute of Physics and Mathematics" [2].

[7] It is the largest magnetic anomaly in the world. In the course of its research, Steklov wrote several scientific papers on the theory of Eötvös torsion balance and on determining the size and depth of the magnetic layer from several observations.

of linear and angular velocity. This problem has four cases of integrability: two of them were found by Clebsch in 1871, the third one by Steklov in his thesis, and the fourth by Steklov's supervisor Lyapunov in 1893. In 1902, he defended his doctoral thesis *General Methods of Solving Fundamental Problems in Mathematical Physics.* In 1908–1909, Steklov published in France a major work *Problème du mouvement d'une masse fluide incompressible de la forme ellipsoidale dont parties s'attirent suivant la loi de Newton* [Problem of the movement of an incompressible fluid mass of ellipsoidal shape whose parts attract each other according to Newton's law]. This is a problem relating to the shape of celestial bodies, and it was one on which Lyapunov was working.

Steklov also worked on the classical problem of the motion of a solid body around a fixed point. The solution of the problem for the Euler and Lagrange cases made it possible to construct the theory of gyroscopes, which found wide application in technology, including space navigation. Another case was found by Sofia Kovalevskaya. Later, researchers considered other special cases corresponding to a particular choice of initial conditions. Two such cases were found by Steklov.

A full and rather impressive list of Steklov's scientific achievements may be found in [3, 4, 5].

Nikita Kalinin

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Steklov top

Most people are familiar with the most significant motions of the Earth: the Earth rotates on its axis every 24 hours, and it orbits the sun every 365 days. But the direction of the Earth's rotational axis itself is not fixed: analogous to the behavior of a spinning top or a gyroscope, the axis traces out a circular path, known as a precession, with a period of about 26,000 years, and wobbles slightly about that circle, or undergoes a nutation, with a period of 18.6 years. This precession and nutation (see the illustration below) are driven by the interaction of the Earth with the gravitational forces of the Sun and Moon.

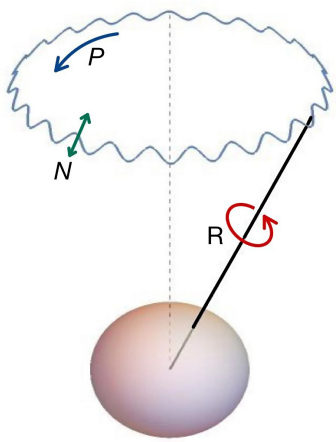

Illustration of precession and nutation for Earth.

Another form of nutation was predicted in 1765 by Leonhard Euler, who suggested that an axially symmetric rigid body, with a difference between the equatorial and polar moments of inertia, will freely wobble as it rotates. Using equations describing rotations of a free rigid body around its center of mass

$$\dot{M} = M \times \omega\,, \quad M = \mathbf{I}\,\omega,$$

where $\mathbf{I}$ is the Earth's tensor of inertia and M, ω are the angular momentum and angular velocity vectors, Euler predicted that free nutation of Earth would have a period of about 10 months, but Chandler's observations, first published in 1891, established that the actual period is about 14 months. In 1892, Newcomb explained the discrepancy between theoretical and experimental periods, considering Earth to be a body with cavities filled with a fluid.

In 1908 Steklov [1] obtained equations of motion for an ellipsoidal rigid body with a cavity filled with the ideal incompressible fluid being in a state of homogeneous vortex motion. In his approach, Steklov used general methods developed by him for solving boundary value problems of mathematical physics and reduced the problem of fluid motion to the solution of Neuman's problem which he called the main problem of hydrodynamics.

In [2] Steklov studied a partial case of equations of motion from [1] when ellipsoid has a fixed center and fluid being in the Dirichlet motion. Steklov proved that this system of equations has three-dimensional invariant manifolds fulfilled by periodic orbits except a zero Lebesgue measure set by using the last Jacobi multiplier and the Poincaré small parameters methods. Then he discussed new and known cases integrable by quadratures and estimated the period of the Chandler precession, the thickness of the Earth's crust, etc.

A more rigorous and perfect theoretical description of precession and nutation was given by Poincaré in [4] which continues a set of Poincaré works devoted to the construction of the Euler type equations on finite and infinite Lie algebras. According to Poincaré and V.I. Arnold, an extension of Euler's equations for arbitrary Lie algebra is equal to

$$\dot{M} = \mathrm{ad}^*_{\omega(M)} M \,,$$

where the vector M belongs to a space $\mathfrak{g}^*$ conjugate to the Lie algebra $\mathfrak{g}$,

$$\omega(M) : \mathfrak{g}^* \to \mathfrak{g}$$

is a linear self-conjugate operator defining the Hamiltonian $H = (M, \omega)/2$, and the map $\mathrm{ad}_\xi : \mathfrak{g} \to \mathfrak{g}$ is the linear map $\eta \to [\xi, \eta]$, where $[\xi, \eta]$ denotes the Lie bracket.

In a suitable basis Euler's equations on $\mathfrak{so}^*(4) \cong \mathfrak{so}^*(3) \oplus \mathfrak{so}^*(3)$ have the form

$$\dot{s} = s \times \frac{\partial H}{\partial s}\,, \quad \dot{t} = t \times \frac{\partial H}{\partial t}\,, \quad s, t \in \mathbb{R}^3 \cong \mathfrak{so}^*(3)\,.$$

In the Steklov case the Hamiltonian $H = b_1 H_1 + b_2 H_2$ is a linear combination of the functions

$$H_1 = (\mathbf{A}s, \mathbf{A}s) - 2(s, \mathbf{A}^\vee t)\,,$$
$$H_2 = (\mathbf{A}^\vee t, \mathbf{A}^\vee t) - 2\det \mathbf{A}(s, \mathbf{A}t)\,,$$

defined by the symmetric matrix $\mathbf{A} = \mathbf{I}^{-1}$, which is inverse to the Earth's tensor of inertia $\mathbf{I}$, and by its co-factor matrix $\mathbf{A}^\vee = \det \mathbf{A} \cdot \mathbf{A}^{-1}$. The matrix $\mathbf{A}$ has the diagonal form $\mathbf{A} = diag(a_1, a_2, a_3)$ in the so-called reference frame, which is firmly attached to the ellipsoid, and its axes coincide with the principal inertia axes. The physical meaning of the parameters $b_{1,2}$ may be also found in [2].

The basic Steklov's Hamiltonians $H_{1,2}$ define bi-Hamiltonian vector fields

$$\dot{x} = \{x, H_1\} = \{x, H_2\}',$$
$$\{H_1, H_2\} = \{H_1, H_2\}' = 0,$$

where $[.\,,.]$ is the Lie–Poisson bracket on $\mathfrak{so}^*(4)$ associated with Lie bracket $[.\,,.]$

$$\{s_i, s_j\} = \varepsilon_{ijk} s_k\,, \quad \{s_i, t_j\} = 0\,, \quad \{t_i, t_j\} = \varepsilon_{ijk} t_k$$

and $[.\,,.]'$ is the linear Poisson bracket compatible with $[.\,,.]$

$$\{s_i, s_j\}' = 0, \quad \{s_i, t_j\}' = \frac{\varepsilon_{ijk}}{2a_i a_k} s_k, \quad \{t_i, t_j\}' = \frac{\varepsilon_{ijk}}{2}\Big(\frac{s_k}{a_i a_j} - a_k^{-2}\, t_k\Big)\,.$$

The linear map

$$f\,:\,\mathfrak{so}^*(4) \cong \mathfrak{so}^*(3) \oplus \mathfrak{so}^*(3) \to e^*(3) \cong \mathfrak{so}^*(3) \ltimes \mathbb{R}^3$$

defined by

$$f\colon p = s, \quad M = (\mathbf{A}^2 - \mathrm{tr}\mathbf{A}^2)s - 2\mathbf{A}^{\vee} t\,,$$

relates compatible linear Poisson brackets on $\mathfrak{so}^*(4)$ and $e^*(3)$

$$f\colon \quad \begin{array}{lcc} \text{Lie–Poisson brackets} & \{\cdot\,,\cdot\}_{\mathfrak{so}^*(4)} & \{\cdot\,,\cdot\}_{e^*(3)} \\ & \nwarrow\!\!\!\nearrow & \\ & \swarrow\!\!\!\searrow & \\ \text{Second linear brackets} & \{\cdot\,,\cdot\}'_{\mathfrak{so}^*(4)} & \{\cdot\,,\cdot\}'_{e^*(3)} \end{array} \,.$$

In the last formula, the Lie–Poisson bracket on $e^*(3)$ is equal to

$$\{M_i, M_j\} = \varepsilon_{ijk} M_k\,, \quad \{M_i, p_j\} = \varepsilon_{ijk} p_k\,, \quad \{p_i, p_j\} = 0\,.$$

This Poisson map f identifies Steklov top on $\mathfrak{so}^*(4)$ with the Steklov–Lyapunov top on $e^*(3)$ describing the motion of a rigid body in a fluid [3]. It allows us to use separated variables and Abel's quadratures proposed by Kötter [5] in both cases. The corresponding 2×2 Lax matrices are suitable to standard finite-gap linearization of dynamics on the Jacobian of genus two hyperelliptic curve.

There are also non-linear Poisson maps relating the Steklov and Steklov–Lyapunov tops with a potential motion on a 2D sphere

$$p = \alpha J - \beta(x \times J)\,, \quad M = J - \mathbf{C}z\,, \quad z_i = \sum_{j,k=1}^{3} |\varepsilon_{ijk}| x_j (x \times p)_k\,,$$

where α, β and $\mathbf{C}$ satisfy some algebraic equations and vectors x and J are coordinates on the cotangent bundle T^*S^2 to unit sphere S^2. The corresponding Hamilton–Jacobi equation is separable in elliptic or spheroconical coordinates, which after inverse Poisson map coincide with Kötter's separated variables [5].

For Steklov top, it is also known that Hamiltonians $H_{1,2}$ commute with respect to a few non-linear Poisson brackets on $\mathfrak{so}^*(4)$ having a common symplectic foliation with the Lie–Poisson brackets $[.\,,.]$. Eigenvalues of the corresponding recursion operator on the generic symplectic leaves of $\mathfrak{so}^*(4)$ are so-called Darboux–Nijenhuis variables which are separated variables for the Steklov top different from the Kötter variables. Divisors of these separated variables are obtained from divisors of the Kötter's separated variables by scalar divisor multiplication on genus two hyperelliptic curve.

Summing up, Steklov studied a specific polynomial system of differential equations on $e^*(3)$ and $\mathfrak{so}^*(4)$ associated with the concrete Earth model. He did not use underlying Lie symmetries and Poisson brackets, Lagrangian or Hamiltonian formalism, he only explicitly constructed integrals of motion and solutions of differential equations similar to permanent rotations, analyzed the effect of the viscosity on dynamics, found the period of the Chandler precession, thickness of the Earth's crust, etc. Now we have another model of Earth, nevertheless, we can continue Steklov's work to solve various pure mathematical problems. For instance, we can classify compatible polynomial and rational Poisson brackets on $\mathfrak{so}^*(4)$ using arithmetic of divisors and isogenies of algebraic curves, consider Dirac brackets associated with other types of fluids with viscosity in electric and magnetic fields, study various deformations of the Steklov top on the Lie algebras $\mathfrak{so}^*(4)$ and $\mathfrak{so}^*(3) \oplus \mathfrak{so}^*(3) \oplus \cdots \oplus \mathfrak{so}^*(3)$, etc.

Andrey Tsiganov

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Boris Grigoryevich Galerkin (1871–1945)

Boris Galerkin was an expert engineer in structural mechanics, a specialist in the theory of elasticity, a full member of the USSR Academy of Sciences, and an engineer-lieutenant general. He developed several parts in structural mechanics and the theory of elasticity. Despite the complexity of the mathematical apparatus, Boris Galerkin's results were presented down to very concrete results that were ready to be applied directly. He was one of the founders of the finite element method.

Berka Girshevich Galerkin, as was his original name, was born on 20 February 1871 to a poor Jewish artisan family in the village of Prudok in the Lepel district of Vitebsk governance of the Russian Empire. His parents were Girsh–Shleima and Pearl–Basya Galerkin. The elementary education of the future academician lasted for only two years, as his parents, who were engaged in handicrafts, could not afford more. From the age of 12, Boris had to take side work as a scribe in the Orphans' Court. In Polotsk, the oldest city in Belarus, he graduated from a Realschule (practical school).

At the age of 22, Boris Grigoryevich took an examination in Minsk, which, if passed, would give him the right to enter a higher school. Anatoly Filin, author of the monograph *Essays on Mechanical Scientists,* learned of the following episode in 1971 in a trolleybus from a relative of Boris Grigoryevich, who was taking part in the conference devoted to the 100th anniversary of Boris Galerkin. According to Filin,

> B.G. Galerkin's father was of extreme Jewish nationalistic views, he forbade his son to study Russian language, and his son did it secretly from his father, and then passed the gymnasium course as an external student.

Boris Galerkin arrived in St. Petersburg in 1893 and entered the mechanical department of the St. Petersburg Institute of Technology in the same year.

He graduated from the St. Petersburg Institute of Technology in 1899. His tutor was Professor Viktor Kirpichev, a renowned specialist in the mechanics of materials and structural strength.

Boris Grigoryevich started working at the Kharkov plant of the Russian Locomotive and Mechanical Society. In 1903 he became an engineer at a line of the Chinese Eastern Railway which was under construction at the time, and six months later he became head of the technical department of the Northern Mechanical and Boiler Plant in St. Petersburg.

During his student years, Boris Galerkin became involved in politics and joined a social democratic group. In 1899, the year of graduation, he became a member of the RSDLP,[1] and in 1906 a member of the St. Petersburg Committee of the RSDLP (he was not a member of the Bolshevik faction, however). During the 1905 revolution, he was arrested as a member of the Union of Engineers bureau and imprisoned for 35 days. In early 1906, Galerkin was actively involved in the organization of the Russian Union of Metalworkers. On 5 August 1906, police surrounded house number 13 in Alekseevskaya Street and detained almost all the members of the Saint Petersburg RSDLP Committee. On March 26, 1907, the Petersburg Court Chamber sentenced Boris Galerkin to one and a half years of imprisonment. He served this term in the "Kresty"[2] prison.

"Kresty" prison.

In the prison, he wrote his first scientific paper *The theory of longitudinal bending and its application to the calculation of structures* of 130 pages (published in 1909). The work became one of the classics in structural mechanics.

[1] The Russian Social Democratic Labor Party was founded in 1898 with the idea of uniting the various revolutionary organizations in the Russian Empire. In 1903 the party split into Bolsheviks and Mensheviks factions. The former eventually became the Communist Party of the Soviet Union.

[2] "Kresty" means "crosses," this political (before and after 1917) prison was named so because it consists of two cross-shaped buildings.

In 1913–15, Galerkin designed a steel frame structure — a metal boiler house in St. Petersburg, the first large metal building in Russia to withstand heavy loads. It was one of Europe's most outstanding buildings in terms of its boldness and originality. In 2001, the house was included by the Committee for State Control, Use and Protection of Monuments of History and Culture of St. Petersburg in the "List of Newly Identified Objects of Historical, Scientific, Artistic or Other Cultural Value." Later, Boris was a consultant in the design and construction of all major hydroelectric power stations.

During the same years he published several important scientific papers: *Rods and plates*, *Rectangular plates*, *Bending and compression*, *To the calculation of thin loosely supported plates*, etc. The calculations for building structures required a clear task description, the development of new mathematical methods for determining the stress state of a structure and its strength and stability.

In 1915, Boris Galerkin published a paper in which he proposed an approximate method for solving differential equations, which had a significant influence on the development of the theory of partial differential equations. Nowadays, the Galerkin method serves as the basis of algorithms for solving various problems in mathematical physics, mechanics, thermodynamics, classical electromagnetism, etc.

Since 1909 he also taught a course in structural mechanics at the mechanical department of the Imperial Saint Petersburg Polytechnic Institute, founded on February 19, 1899. In the same year Boris traveled abroad to examine structures he was interested in. He used the next four summers before the war to travel to Germany, Austria, Switzerland, Belgium, and Sweden for scientific purposes. It should be noted that he spoke three foreign languages: English, French, and German.

Between 1924 and 1929, Galerkin taught at Leningrad State University and the Institute of Transport Engineers. In January 1928, he was elected a corresponding member of the USSR Academy of Sciences, and in 1935, he became a full member.

From 1931 to 1941, Galerkin was a member of the Research Institute of Hydraulic Engineering (NIIG).

In 1934, B.G. Galerkin was awarded two academic degrees: Doctor of Technical Sciences and Doctor of Mathematics, as well as the title of Honored Worker of Science and Technology of the RSFSR. He was a recognized authority among design engineers. He was invited as a consultant for the design and construction of several major buildings. In April 1936, by decree of the Council of People's Commissars, Boris was appointed chairman of the commission of the Construction Council for the expertise of preliminary design of steel frame with wall and ceiling structures of the Palace of Soviets in Moscow, which would have become, if built, the most pompous building on the planet, 495 meters high with a spire.

The project of the Palace of Soviets in Moscow.

The foundation of the building was ready by 1939, but in 1941 the construction of the Palace of Soviets was postponed.

Boris Grigoryevich was one of the founders (in 1939) and the first director of the Institute of Mechanics of the USSR Academy of Sciences, as well as the editor-in-chief of the journal *Applied Mathematics and Mechanics.* In 1939, he chaired the department of structural mechanics at the Military Engineering and Technical University (VITU) in Leningrad and was promoted to the rank of Lieutenant General Engineer.

Boris Galerkin died in Moscow in 1945 and was buried in the academic section of the Literatorskie Mostki in St. Petersburg's Volkov cemetery.

Irina Demidova

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The Galerkin method

The theory of partial differential equations (PDEs) has its origins in the 19th century, when mathematicians began to study the problem: find a function $u(x, y)$ such that

$$\Delta u + f = 0 \quad \text{in } \Omega, \qquad u = 0 \qquad \text{on } \partial\Omega, \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth (or piecewise smooth) boundary $\partial\Omega$. It was unknown how to solve this equation and how the solution could depend on the problem data. Moreover, the existence of a solution to this problem was also the subject of active discussion.

Similar questions arose in connection with other problems $\mathcal{L}u = f$ associated with a partial differential operator $\mathcal{L}$ mapping a Banach space X to a Banach space Y. At that time, the equations were studied in the framework of classical analysis, and, therefore, the space X was considered as the space of continuous functions having sufficiently many classical derivatives. It was discovered that some problems have equivalent variational formulations, e.g., (1) is equivalent to minimization of the energy functional

$$\int_\Omega \left(\frac{1}{2}|\nabla u|^2 - fu\right) dx.$$

In 1909, W. Ritz suggested finding approximate solutions of variational problems in the form

$$u_N(x) = \sum_{i=1}^{N} \alpha_i w_i(x), \tag{2}$$

where N is a natural number and the coordinate functions w_i belong to X and form a linearly independent system. The weights α_i should be chosen to minimize the functional.

However, many problems are not generated by a certain (energy) functional. It was necessary to create a unified and mathematically justified approach valid for differential equations of all types. Intuitively, it was clear that a suitable approximation v should somehow minimize the residual $\mathcal{L}v - f$, but which form of the residual should be used, and how to select the set of coordinate functions? Without the right answers to these questions, it is impossible to prove that u_N converges to the exact solution u.

The idea of Galerkin's method [3] is to find α_i from the condition: *the residual must be orthogonal to a finite-dimensional subspace of test functions* *with integral type orthogonality conditions.* For the problem (1), this principle yields (after integration by parts)

$$\int_{\Omega} \nabla u_N \cdot \nabla w_i dx = \int_{\Omega} f w_i dx \quad \forall w_i \in X_N, \tag{3}$$

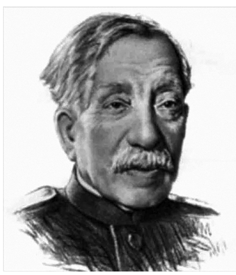
where X_N contains functions of the form (2) vanishing on the boundary $\partial\Omega$. In more general cases, the sets of coordinate and test functions may not coincide and the orthogonality relation has the form

$$\langle \mathcal{L} u_N - f, \eta \rangle = 0, \quad \forall \eta \in Y_M, \tag{4}$$

where u_N is defined by (2) and Y_M ($\dim Y_M = M$) is a set of test functions, $Y_M \subset Y'$, where Y' is the space conjugate to Y and $\langle \cdot, \cdot \rangle$ denotes the duality pairing of Y and Y'. Certainly, Y_M and X_N must be selected in such a way that the system (4) is solvable.

This approach is very flexible. If X and Y are Hilbert spaces and the spaces X_N and Y_M coincide, then the method is called Bubnov–Galerkin. This name was used by S. Mikhlin [6], who was the first to prove its convergence. If the integral orthogonality relation originates from the Euler equation generated by a quadratic functional, then we have the Ritz–Galerkin method. More general schemes (such as (4)) were studied by G. Petrov, who also extended the method to eigenvalue problems.

Portraits of W. Ritz, B. Galerkin, I. Bubnov, and G. Petrov (from left to right) are presented below.

"How to guarantee the existence of solutions to boundary value problems for PDEs?"

At the end of the 19th century this question was open. In particular, solvability of the simplest problem (1) was intensively discussed in the literature (typically in the classical sense, i.e., u was sought in $C^2(\Omega) \cap C(\bar{\Omega})$). This question was answered after many years of studies that have completely reconstructed the theory of PDEs.

A new concept of *generalized* or *weak* solutions was created by D. Hilbert, H. Poincaré, S. Sobolev, R. Courant, O.A. Ladyzhenskaya, and many other outstanding mathematicians. In fact, the Galerkin method served as a turning point in the study of this problem. This is easy to observe with this paradigm the integral relation of (3).

Indeed, let tend N to $+\infty$. If we expect that u_N will tend (in some sense) to the exact solution, then it is natural to consider the limiting form of (3), where X_N is replaced by an infinite dimensional functional space $\widetilde{X}$ (which should be a proper closure of $\{X_N\}$). This way leads to the *generalised statement* of (1): find $u \in \widetilde{X}$ satisfying the boundary conditions such that

$$\int_\Omega \nabla u \cdot \nabla w \, dx = \int_\Omega f w \, dx \quad \forall w \in \widetilde{X}. \tag{5}$$

Now we know that in the case of Lipschitz Ω the closure generates the Sobolev space $\overset{\circ}{H}{}^1(\Omega)$ (of functions vanishing on the boundary and having square summable generalized derivatives of the first order).

Integral type definition of solutions to PDEs is nowadays commonly accepted (a systematic exposition can be found in O.A. Ladyzhenskaya and N.N. Uraltseva, *Linear and Quasilinear Elliptic equations*).

If $\widetilde{X}$ is a reflexive Banach space, then from (3) it follows that

$$\|\nabla u_N\|_\Omega \le C\|f\|_\Omega$$

with a constant independent of N. Hence, there exists a subsequence of u_N weakly converging to a function $u \in X$. Using this fact, it is not difficult to show that u_N converges to a function u satisfying (5). Similar arguments can be used in other boundary value problems. Therefore, Galerkin approximations suggest a method for proving the existence of weak solutions.

E. Hopf used this idea in order to prove the existence of weak solutions to nonstationary Navier–Stokes equations. A similar approach was often used by O.A. Ladyzhenskya and N.N. Uraltseva [4, 5], for various nonlinear problems in the theory of viscous fluids and other PDEs.

It is worth noting that the concept of a generalized (weak) solution and the Galerkin method are closely related to the *Virtual Work Principle* in mechanics, which dates back to J. D'Alembert (who used it for a mechanical system of rigid bodies) and J.-L. Lagrange (who suggested a generalization for continuum media problems). This principle was known already at the beginning of the 19th century. The development of mathematics at that time was insufficient to correctly determine what should be considered as "the set of virtual displacements" and properly state the corresponding boundary value problems.

In 1943, R. Courant [2], suggested a version of the method with locally supported test functions, which generated geometrically flexible numerical schemes with dispersed resolving matrixes. Later it was named the *Finite Element Method* (FEM). Mathematical analysis of FEM is based upon two fundamental relations: the *Galerkin orthogonality* and the *projection estimate*. The latter estimate forms the basis of error analysis. It states that the distance between u and u_N is controlled by the distance between u and the respective finite dimensional space X_N.

In this or other form, the majority of modern computational technologies use approximations of the Galerkin type. For example, the Discontinuous Galerkin method uses (3) (or (4)) with discontinuous test functions. Many other methods (spectral, finite volume, weak Galerkin, isogeometric, meshless) can be viewed as advanced versions of the Galerkin concept. The need to calculate Galerkin approximations for real-life scientific and engineering problems stimulated studies in numerical linear algebra and generated the creation of multigrid iteration methods and domain decomposition method (DDM).

Multigrid methods allow very large systems of linear equations to be solved by using several different scales of discretizations in order to optimize the process of computations. The domain decomposition method originates from the method of Schwarz, who suggested to decompose domains with complicated geometry into a collection of simple subdomains (e.g., rectangles, convex polygons), for which the corresponding sets of test functions can be constructed by simple methods.

In [7], the reader can find more about further development of the method and useful references.

A systematic consideration of the Galerkin method in application to elliptic, parabolic, and hyperbolic equations is presented in [8].

For the Discontinuous Galerkin method (DG) see [1] and references therein.

Sergey Repin

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Nikolai Maksimovich Günther (1871–1941)

Nikolai Günther,[1] professor and corresponding member of the USSR Academy of Sciences (1925), one of the founders of the Leningrad school of mathematical physics, President of the Leningrad Physical and Mathematical Society, devoted his whole life to science and teaching in St. Petersburg-Petrograd-Leningrad higher education institutions.

Working on partial differential equations, Günther was one of the first to realize the importance of considering solutions whose smoothness is less than required by the equation in the literal sense. This idea, a key one for the mathematics of the XXth century, led him to develop the theory of functions of domains. Although forgotten, his ideas served as the basis for the theory of distributions constructed by his student Sergei Sobolev and by the French mathematician Laurent Schwartz.

His mother was Maria Petrovna Andreeva, a peasant woman of the Tver; Nikolai was born in St. Petersburg, baptized in the Kazan Cathedral there, and surnamed Ivanov. Only in 1893, when he was a student, Nikolai Günther was adopted by his father, the merchant Max Efimovich Günther, who had just been christened into the Orthodox faith.

Pafnuty Chebyshev, Alexander Korkin, and Andrei Markov were his teachers at St. Petersburg University. Markov recommended Günther for preparation for the professorship. In 1915, Günther defended his doctoral thesis and, in 1916, he became an ordinary professor at the department of pure mathematics. In addition to the university, Günther taught at many other higher educational institutions in the city, and the numerous lecture courses he published were almost never stereotypical but took into account the specifics of each institution.

The second decade of the XXth century in the mathematical life of St. Petersburg was characterized by the considerable interest of physicists in new

[1] Spelling variants: Gyunther, Gunter, Gjunter.

ideas of mathematics, which formed a toolkit for developing new fields in physics. Communication with Paul Ehrenfest, Yury Krutkov, and later with Vladimir Fock, Alexander Friedmann, Jacob Tamarkin, the brothers Abram and Jacob Besicovitch, Vladimir Smirnov, Sergei Bernstein, and Stephen Timoshenko, in whose scientific circle the demand for the development of mathematical physics was clear, prompted Günther to address mathematical problems connected to physics.

In the 1920s, he initiated a special seminar at the university on the applications of function theory to problems of fluid dynamics, potential theory, and mathematical physics. At that time, the Leningrad Physics and Mathematics Society was founded under the leadership of Vladimir Steklov and Nikolai Günther, which included Alexander Friedmann, Abram Besicovitch, Boris Delone, Jacob (James) Uspensky, Jacob Tamarkin, Vladimir Smirnov, Grigory Fichtenholz, Vladimir Fock, and Yury Krutkov. Meetings were held twice a month, both physicists and mathematicians addressed issues in astronomy, mechanics, potential theory, fluid dynamics, theory of relativity, meteorology, and theory of elasticity.

Mathematical societies existed earlier in St. Petersburg. The first one was founded in St. Petersburg in 1890, chaired first by Vasily Imshenetsky and later by Julian Sochocki. It had 89 members, and its meetings were held monthly at the Academy of Sciences and then, since 1895, at the University. In the 1920s, the Physical and Mathematical Society was based on the mathematical circle inspired by Alexander Vasiljev. In 1925, Günther became chairman of the Society, and the Society was registered at his address.

The year 1929 was "the year of the great turnaround (*perelom*) on all fronts of socialist construction." Stalin proclaimed a route of mobilization and called for a change in the ideological struggle on all fronts, including the scientific one. Loyalty was placed above competence. A group of "left-wing" Marxist mathematicians began the struggle against "Güntherism." In 1929, elections were to be held for the Academy of Sciences, and in terms of the significance of his work, Günther was a candidate for full membership. Ivan Vinogradov, who worked in number theory, had also applied for the same position.

This situation was skillfully exploited as a part of the ongoing policy to depose the old intelligentsia. In December 1928, a mathematical section of the Scientific Society of Marxists, consisting of five people, was formed. In 1931, they called themselves the Society of Materialist Mathematicians at the Leningrad Branch of the Communist Academy.[2] From 1931 onwards, the "Society of Marxist-Materialists under the auspices of the Institute of Philosophy" was first registered in the book "The Whole of Leningrad." Its functions were: "Consolidation of mathematicians-materialists of the Leningrad region

[2] The Communist Academy was a higher education and research institution meant to allow research independent of the old Academy of Sciences. It was subsumed within the Academy of Sciences in 1936.

in scientific work on mathematical research, the teaching of mathematics and its application to technology, on the basis of dialectal materialism and the revolutionary practice of economic socialist construction."

The Physical and Mathematical Society was accused of idealism, a "closed club of professors," detachment from the tasks of socialist construction, and allegedly pursuing "science for the sake of science," i.e., "güntherism." At the national level, such attacks aimed to subordinate the Academy of Sciences, the universities, and the scientific community to the Party's leading role, and at the level of Leningrad, to diminish the role of the old professors, to crush the Physical and Mathematical Society, all the while advancing the career ambitions of the "leftist" figures.

The notable number theorist Ivan Vinogradov became academician,[3] bypassing the status of corresponding member. By that time, colleagues close to Günther, like Abram Besicovitch, Jacob Tamarkin, Yakov Uspensky, and James Shohat, had left Russia. Vladimir Steklov and Alexander Friedmann were no longer alive. Grigory Fichtenholz and Boris Delone moved to the Society of Marxist Mathematicians. On March 10, 1931, the "Declaration of the Initiative Group for the Reorganization of the Leningrad Physics and Mathematics Society" condemning Günther was signed by Academician Ivan Vinogradov, Professors and researchers Boris Delone, Leonid Kantorovich, Grigory Fichtenholz, and others.

On the same day, Günther was forced to write a letter to the editorial office of the newspaper *Leningrad University* saying,

> Having read the declaration of the Society of Materialist Mathematicians, I find it necessary to make the following statement. The life of our country is advancing so rapidly that many people have to think over their past activities and have to reevaluate them seriously. For my part, I saw a year ago that I had made fundamental mistakes. I consider my main mistake to be that when I was the president of the Physical and Mathematical Society, I could not connect its activity with the needs of socialist construction so that the Society actually remained on the ground of the old slogan "science for the sake of science." [...] Recognizing a year ago my unsuitability to occupy leading posts, I spoke in December 1929 to the rector of the university about my desire to give up the chair in the university, which I have since done; then, in January, I asked the board of the Society to release me from the presidency, and I agreed to remain only temporarily to bring in order financial affairs of the Society; Using my pedagogical experience and scientific competence, however, I expect to be useful further on. N. Günther.

In 1930, Günther resigned as chairman of the Society, and in 1931 the latter was dissolved on the advice of Vladimir Smirnov. The Society of Materialist

[3] In these elections Vinogradov, Bernstein, Krylov became academicians, Egorov and Grave became honorary members of the Academy of Sciences USSR.

Mathematicians, despite the support of the Communist Academy, ceased to exist two years later. Nikolai Günther was forced to give up his position as chair of the department but remained a professor. He continued to lecture until the last year of his life.

Students with excellent marks talk with Prof. Günther, 4 March 1941, [6].

The years from 1932 to 1934 were his most productive scientifically. He published two voluminous monographs, on the Riemann–Stieltjes integral and on potential theory. In 1939, he resumed his chairmanship at the department, and in 1941 he was awarded the title of Honored Worker of Science. Günther's many years of teaching were embodied in 47 published books, as well as in the *Higher Mathematics Problem Book*, the thirteenth edition of which was published in 2003. A large number of mathematicians and engineers regard him as their teacher.

Galina Sinkevich

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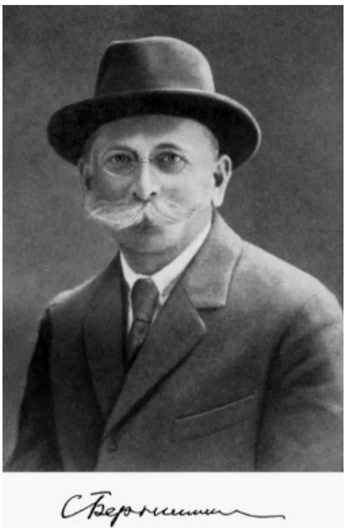
[4] LOKA is the Leningrad Branch of the Communist Academy.

Sergei Nathanovich Bernstein (1880–1968)

In his master's thesis, Sergei Bernstein gave a partial solution to 19th Hilbert's problem. In his doctoral dissertation *On the Best Approximation of Continuous Functions by Polynomials of Given Degree*, he laid the foundation for a new field of mathematics: constructive function theory. Bernstein wrote more than 200 scientific papers, most of which were included in his lifetime collection of works, prepared under his own editorship.



Sergei N. Bernstein was born in Odessa into the family of Nathan Bernstein, a Doctor of Medicine and privatdozent of the University of Novorossiysk. There were two other siblings: a brother, later a famous psychiatrist, and a sister, who studied in Paris, became a prominent microbiologist, and worked at the Pasteur Institute until her old age. Sergei became interested in mathematics while in high school at Richelieu gymnasium and independently studied analytic geometry, the basics of abstract algebra, and mathematical analysis.

He completed his education in Paris: it took him three years to complete a four-year course at the Sorbonne and two years for a full course at the École Supérieure Electrotechnique de Paris, from which he graduated as an electrical engineer. In search of problems for independent research, Sergei Bernstein moved to Göttingen to participate in a seminar of David Hilbert, who invited the young colleague to take up the 19th problem concerning analytic solutions of elliptic differential equations. Hilbert asked for a proof that all solutions of regular analytic variational problems were analytic.

Bernstein defended the problems' partial solution as a doctoral dissertation before a committee consisting of Jacques Hadamard, Émile Picard, and Henri Poincaré (Sorbonne, 1904). The thesis began with the following words: "All the mathematicians and physicists of our day seem to agree that the field of application of mathematics has no boundaries other than those of knowledge itself." The full solution to Problem 19 was obtained by Ennio de Giorgi in 1956.

After returning to Russia, Bernstein lived in St. Petersburg for three years, teaching mathematics in private high schools and at St. Petersburg Higher Women's Polytechnic Courses, then went to Kharkov, where he taught at the Higher Women's Courses and the university, and organized and headed a mathematical research institute in 1929. He became a corresponding member of the USSR Academy of Sciences in 1924 and a full member in 1929.

In 1930, at the congress of Soviet mathematicians in Kharkov, a confrontation between the old professors and the new "Soviet" scientists arose. To persuade foreign scientists (Jacques Hadamard, Arnaud Denjoy, and others) to agree to take part in the congress, Bernstein obtained guarantees from the Commissar of Education of Ukraine, Nikolai (or Mykola) Skrypnik, that no political statements would be made at the congress. Nevertheless, during the congress, the head of the Department of Algebra at Moscow State University, Otto Schmidt, proposed to send greetings to the XVI Congress of the Communist Party. Bernstein strongly objected, Nikolai Gunter, Dmitri Egorov, and others supported him. As a compromise, greetings from "party members" of the congress were sent. After that, rallies were held in Kharkov at which Bernstein was denounced as an "idealist," a "fellow traveler,"[1] and even a "monarchist" (in the words of the future Rector of Kharkov University, Yakov Bludov). Everyone was forced to speak out, and only a few found the courage to refuse. As a result of this persecution, Bernstein left for Leningrad in 1933.

In 1941, as Nazi Germany invaded Russia and Hitler's troops besieged Leningrad, Sergei Bernstein, together with other academicians, was evacuated; he did not return to Leningrad after the war. He settled in Moscow and worked at the Steklov Mathematical Institute of the Academy of Sciences until his death. Until the spring of 1947, he gave lectures and led a scientific seminar on constructive function theory at Moscow State University; later, from 1947 to 1957, he was the head of the Department of Constructive Function Theory.

His lecture notes on *Probability Theory* (1911) were reprinted several times and his book *The Present State of Probability Theory* (1933) became widely known. Both books were written for higher education institutions.

In 1942, under the direction of Sergei Bernstein, a manual for fixing the vessel location by wireless direction finder was developed, which made navigational calculations about ten times faster. The long-range aviation headquarters, praising the work of the mathematicians, noted that no other country in the world had tables equal to these in their simplicity and originality.

Sergei Bernstein gained recognition not only as a mathematician and teacher but also as a historian and popularizer of science, reviewer, and translator. His book *The Scientific Legacy of P.L. Chebyshev* is especially noteworthy.

[1] A term in Soviet political jargon meaning "a temporary, unreliable ally of the Communist regime; should be treated with suspicion."

The contribution of Sergei Bernstein was highly appreciated by the world scientific community: he was a member of the Paris Academy of Sciences (only the third Russian to become one after Peter the Great and Pafnuty Chebyshev) and of many foreign mathematical societies. He received numerous scientific awards in the USSR.

His contemporaries described him as a very caring, gentle, and benevolent person, fully immersed in science, and at the same time deeply decent, honest, and principled. In Moscow, being a member of the commission of the Presidium of the Academy of Sciences of the USSR on "the Luzin case"[2] (1936), Sergei Bernstein was one of the very few who openly defended Luzin. His intervention in the fate of Nikolai Koshlyakov, a corresponding member of the Academy of Sciences of the USSR arrested in the fabricated "case of the Union of old Russian intelligentsia"[3] and sentenced to 10 years' imprisonment in forced labor camps, resulted in an improvement of Koshlyakov's incarceration conditions: he was provided with an enhanced diet and given a paper for work. In addition, one of his scientific papers, which was written in the camp and miraculously found its way to the Steklov Mathematical Institute in Moscow, was published in 1949 thanks to the efforts of Ivan Vinogradov and Sergei Bernstein, under the editorship of Yuri Linnik and under the pseudonym "N.S. Sergeev."

Incidentally, it was Academician Bernstein's integrity that prevented the 5th edition of his university textbook on probability theory from being published. In the 4th edition, among many examples, one of Mendel's fundamental laws of genetics was given and substantiated. However, after genetics was declared a pseudoscience at the infamous session of the All-Union Academy of Agricultural Sciences in 1948 ("Lysenko Case"), Bernstein was asked to exclude any examples related to Mendel's laws and biological traits' inheritance in general from the 5th edition of his textbook. He categorically refused to do so.

Sergei Bernstein died on 26 October 1968 in Moscow and is buried at the Novodevichy Cemetery.

Natalia Lokot

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[2] See more details in a footnote in the article about Sobolev.

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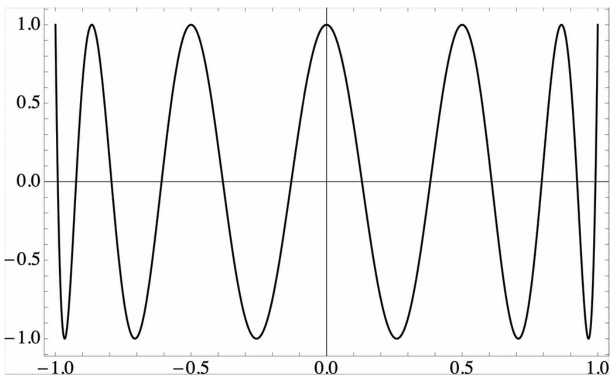
The Bernstein Inequality in Approximation Theory

The inequality

$$\|P^{(m)}\|_{[-1,1]} \le T_n^{(m)}(1) \cdot \|P\|_{[-1,1]}$$

for every algebraic polynomial P of degree at most n with complex coefficients was first proved by Vladimir Markov in 1892. Here T_n denotes the Chebyshev polynomial of degree n defined by $T_n(\cos\theta) := \cos(n\theta)$, $\theta \in \mathbb{R}$, and $\|f\|_A$ denotes the supremum norm of a complex-valued function f defined on A. V.A. Markov was the brother of the more famous Andrei Markov who proved the above inequality for $m=1$ in 1889 by answering a question raised by the prominent Russian chemist, Dmitri I. Mendeleev. Sergei Bernstein presented a shorter variational proof of V.A. Markov's inequality in 1938.

Note that $T_n'(1) = n^2$. The picture below shows the graph of T_{12} on $[-1, 1]$ suggesting why $T_n'(1)$ is large. The simplest known proof of Markov's inequality



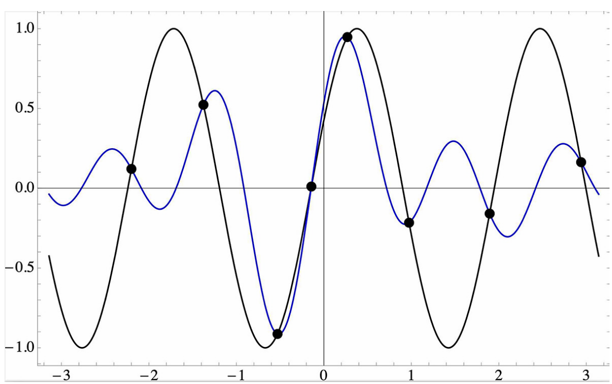
for higher derivatives is due to Richard J. Duffin and Albert C. Schaeffer [6], who gave various extensions as well. The inequality

$$|P'(y)| \le \frac{n}{\sqrt{1-y^2}} \|P\|_{[-1,1]}, \quad y \in (-1, 1),$$

holds for every algebraic polynomial P of degree at most n with complex coefficients, and is known as The Bernstein Inequality. Various analogs of the above two inequalities are known in which the underlying intervals, the supremum norms, and the family of functions are replaced by more general sets, norms (or other metrics) and families of functions, respectively. These inequalities are called Markov- and Bernstein-type inequalities. If the norms are the same in both sides, the inequality is called Markov-type, otherwise it is called Bernstein-type (this distinction is not completely standard). Markov- and Bernstein-type inequalities are known on various regions of the complex plane and the n-dimensional Euclidean space, for various norms such as weighted L_p norms, and for many classes of functions such as polynomials with various constraints, rational functions, exponential sums of n terms, just to mention a few. Markov- and Bernstein-type inequalities have their own intrinsic interest. In addition, they play a fundamental role in approximation theory. The inequality

$$\|T'\|_{\mathbb{R}} \leq n\|T\|_{\mathbb{R}}$$

for all complex trigonometric polynomials T of degree at most n is also called The Bernstein Inequality. It was proved by Sergey N. Bernstein in 1912 with $2n$ in place of n, but Charles Jean de la Vallée Poussin found an error in it. Paul Nevai has done an extensive research on the early history around The Bernstein Inequality. In his book [14] in preparation he writes "So there is published evidence that in 1914, just before World War I broke out, the Hungarian juggernaut produced at least four independent proofs of The Bernstein Inequality with n. The count is as follows: Fejér and F. Riesz have one proof each, and M. Riesz has three proofs, maybe more." The sharp inequality might have appeared first in a paper by Marcel Riesz [16]. Its clever proof based on zero-counting may be found in many books dealing with approximation theory. For real trigonometric polynomials this proof goes as follows. Suppose that T is a real trigonometric polynomial of degree at most n with $\|T\|_{\mathbb{R}} < 1$ and $T'(t_0) > n$. Let $Q_\alpha(t) := \cos(nt - \alpha)$, and choose an $\alpha \in \mathbb{R}$ such that $Q_\alpha(t_0) = T(t_0)$ and $Q'_\alpha(t_0) > 0$. Observe that $T'(t_0) > Q'_\alpha(t_0)$, and hence $T - Q_\alpha$ is a real trigonometric polynomial of degree at most n having at least $2n+2$ distinct real zeros in a period. Hence $T - Q_\alpha$ must be identically 0, which contradicts $T'(t_0) > n$. This proof is illustrated by the picture below.



In books, Markov's inequality for the first derivative is then deduced by a combination of The Bernstein Inequality and the Riesz–Schur inequality

$$\|P\|_{[-1,1]} \le (n+1)\|P(x)(1-x^2)^{1/2}\|_{[-1,1]}$$

for every algebraic polynomial P of degree at most n with real coefficients. It was observed by Nevai [13] that The Bernstein Inequality for real trigonometric polynomials and the above Riesz–Schur inequality are equivalent in the sense that each can be obtained from the other one with the aid of brief elementary arguments. Bernstein used his inequality to prove inverse theorems of approximation, and several other inverse theorems of approximation can be proved by straightforward modifications of Bernstein's method. That is why Bernstein- and Markov-type inequalities play a quintessential role in approximation theory. Direct and inverse theorems of approximation and related matters may be found in many books on approximation theory, including [4, 9, 10]. Let $\mathcal{T}_n$ be the collection of all real trigonometric polynomials of degree at most n. Let $\mathbb{T} := \mathbb{R} \pmod{2\pi}$. For $f \in C(\mathbb{T})$, let

$$E_n(f) := \inf\{\|T-f\|_{\mathbb{T}} : T \in \mathcal{T}_n\}.$$

An example of a so-called direct theorem of approximation is the following. If f is an m times differentiable function on $\mathbb{T}$ and $f^{(m)} \in \mathrm{Lip}_\alpha$ for some $\alpha \in (0,1]$, then there is a constant C independent of n so that $E_n(f) \le Cn^{-(m+\alpha)}$ for every n. A proof may be found in [9], for example. The inverse theorem of the above result can be formulated as follows. If $m \ge 1$ is an integer, $\alpha \in (0,1)$, $f \in C(\mathbb{T})$, and there is a constant $C > 0$ independent of n such that $E_n(f) \le Cn^{-(m+\alpha)}$ for every n, then f is m times continuously differentiable on $\mathbb{T}$ and $f^{(m)} \in \mathrm{Lip}_\alpha$.

The identity

$$\begin{gathered} T'(\theta) = \sum_{\nu=1}^{2n} (-1)^{\nu+1} \lambda_\nu T(\theta+\theta_\nu)\,, \\ \lambda_\nu := \frac{1}{4n\sin^2(\theta_\nu/2)}\,, \qquad \theta_\nu := \frac{2\nu-1}{2n}\pi\,, \end{gathered} \tag{1}$$

for all complex trigonometric polynomials T of degree at most n has been established by M. Riesz [16] and it is called the Riesz Interpolation Formula. Here, choosing $T(\theta) := \sin(n\theta)$ and the point $\theta = 0$, we obtain that $\sum_{\nu=1}^{2n} \lambda_\nu = n$. Identity (1) can be used to prove not only Bernstein's inequality, but an L_p version of it for all $p \ge 1$. Namely, combining the triangle inequality and Hölder's inequality in the Riesz Interpolation Formula, and then integrating both sides, we obtain

$$\int_0^{2\pi} |T'(\theta)|^p \, d\theta \le n^p \int_0^{2\pi} |T(\theta)|^p \, d\theta$$

for all complex trigonometric polynomials T of degree at most n. This Bernstein Inequality was extended to all $p > 0$ by Vitaly V. Arestov [1]. It followed the paper [11] by Attila Máté and Nevai, where the Bernstein factor $11^{1/p}n$ was proved for every $0<p<1$. A short and elegant proof of Arestov's result due to Manfred Golitschek and George G. Lorenz is presented in [4]. See more on Arestov's Theorem in [7]. For *real* trigonometric polynomials T the inequality

$$T'(\theta)^2 + n^2 T(\theta)^2 \le n^2 \|T\|_{\mathbb{R}}^2\,, \quad \theta \in \mathbb{R}\,,$$

is known as the Bernstein–Szegö inequality. Various extensions and generalizations of this inequality have also been established. There is a Bernstein inequality on the open (or closed) unit disk D, or equivalently, on the unit circle C of the complex plane. It states that

$$\|P'\|_D \le n\,\|P\|_D$$

for all algebraic polynomials P of degree at most n with complex coefficients. It was conjectured by Paul Erdős and proved by Peter Lax in 1944 that

$$\|P'\|_D \le \frac{n}{2}\,\|P\|_D$$

for every algebraic polynomial P of degree at most n with complex coefficients having no zeros in the open unit disk D. For Erdős, Markov- and Bernstein-type inequalities had their own intrinsic interest and he explored what happens when the polynomials are restricted in certain ways. It had been observed by Bernstein that Markov's inequality for monotone algebraic polynomials is not essentially better than that for arbitrary algebraic polynomials. Bernstein proved that if n is odd, then

$$\sup_P \frac{\|P'\|_{[-1,1]}}{\|P\|_{[-1,1]}} = \left(\frac{n+1}{2}\right)^2,$$

where the supremum is taken over all algebraic polynomials P of degree at most n with real coefficients which are monotone on $[-1,1]$. This is surprising since one would expect that if a polynomial is this far away from satisfying the equioscillating property of the Chebyshev polynomial T_n, then there should be a more significant improvement in the Markov inequality. In his short paper [8], Erdős gave a class of restricted algebraic polynomials for which the Markov factor n^2 improves to cn. He proved that there is an absolute constant c such that

$$|P'(y)| \le \min\left\{\frac{c\sqrt{n}}{(1-y^2)^2}, \frac{en}{2}\right\}\|P\|_{[-1,1]}$$

holds for every algebraic polynomial P of degree at most n that has only real zeros outside $(-1,1)$, and for every $y \in (-1,1)$. This result motivated several people to study Markov- and Bernstein-type inequalities for polynomials with restricted zeros and under some other constraints. Generalizations of the above Markov- and Bernstein-type inequality of Erdős have been extended in many directions by several authors including G.G. Lorentz, John T. Scheick, József Szabados, Arun Kumar Varma, Attila Máté, Quazi Ibadur Rahman, Narendra

K. Govil, and Ram Mohapatra. Many of these results are contained in the following essentially sharp result, due to Peter Borwein and Tamás Erdélyi [3]: there is an absolute constant c such that

$$|P'(y)| \le c \min\left\{\sqrt{\frac{nk}{1-y^2}},\, nk\right\} \|P\|_{[-1,1]}$$

for every algebraic polynomial P of degree at most n with real coefficients having at most $k-1$ zeros in the open unit disk, $0 \le k-1 \le n$, and $y \in (-1,1)$.

An entire function f said to be of exponential type $\tau > 0$ if for every $\varepsilon > 0$ there exists a real-valued constant A_ε such that

$$|f(z)| \le A_\varepsilon e^{(\tau+\varepsilon)|z|}, \quad z \in \mathbb{C}.$$

In 1926 Bernstein proved that

$$\|f'\|_{\mathbb{R}} \le \tau \|f\|_{\mathbb{R}}$$

for all entire functions f of exponential type $\tau > 0$. Assuming that $\|f\|_{\mathbb{R}} < \infty$, equality holds if and only if $f(z) := ae^{i\tau z} + be^{-i\tau z}$, where $|a| + |b| = \|f\|_{\mathbb{R}}$. A nice proof of this may be found in the book [15] by Rahman and Schmeisser by using the interpolation formula

$$f'(x) = \frac{4\tau}{\pi^2} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(2k+1)^2} f\left(x + \frac{2k+1}{2\tau}\,\pi\right)$$

proved by Ralph Boas in 1937.

The books listed below are some of good sources to find inequalities of Markov-, Bernstein-, and Nikolskii-types and related results. The results mentioned without a reference in this short note may be found with complete proofs in some of them.

We thank Michael Mossinghoff for making the illustrating pictures. We dedicate this survey to the birthplace of Sergei Natanovich Bernstein.

Tamás Erdélyi and Paul Nevai

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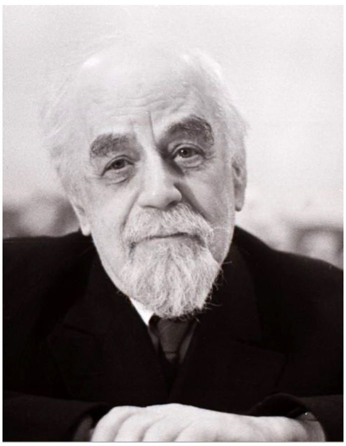
Vladimir Ivanovich Smirnov (1887–1974)

Vladimir Smirnov was born on May 29, 1887, in St. Petersburg into the family of an archpriest, a teacher of religion at the Imperial Alexander Lyceum. After graduating from the famous Second Gymnasium (the oldest gymnasium in the Russian Empire) with a gold medal, he entered the Physics and Mathematics Faculty of St. Petersburg University, from which he graduated with a First class Diploma[1] in 1910. In 1918, he defended his Master's thesis[2] (under Vladimir Steklov's supervision). During the Civil War, from 1918 to 1921, Smirnov taught at Tavria University in Simferopol, Crimea.



After the Bolsheviks took Crimea, his wife was shot, while Smirnov escaped death by a whisker. In 1921 he returned to Petrograd.[3] In 1934, he married again, and in 1935 his son Nikita was born. Vladimir Smirnov taught at many educational institutions throughout the city, and from 1929 to 1935, he worked at the Seismological and Mathematical Institutes of the USSR Academy of Sciences. However, St. Petersburg (Leningrad) University, where he worked from 1915 until his death, was his number one affiliation. During the Great Patriotic War,[4] Smirnov was evacuated to Elabuga, a town 200 kilometers east of Kazan in the Republic of Tatarstan, together with a part of the university, where he organized a group that carried out important defense work. He became a Corresponding Member of the USSR Academy of Sciences in 1932 and a Full Member in 1943.

Vladimir Ivanovich was an outstanding organizer of science. On his initiative, the Research Institute of Mathematics and Mechanics, which now bears his name, was created at Leningrad State University. A whole array of

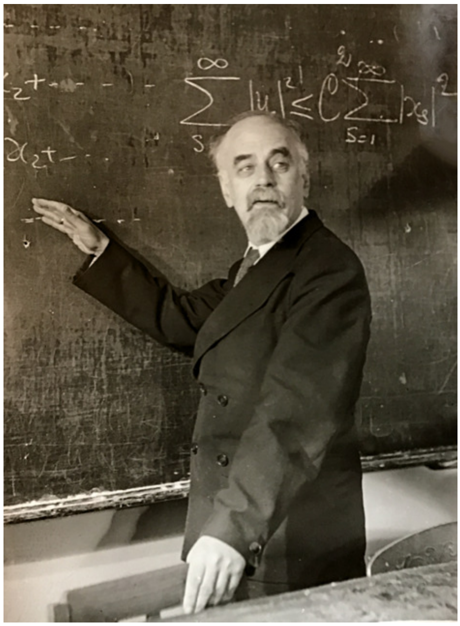
[1] It corresponds to today's Master's degree with honors.

[2] It corresponds to today's PhD thesis.

[3] Petrograd, Leningrad and St. Petersburg all refer to the same city. The city was referred to as Petrograd from 1914 to 1924, Leningrad from 1924 to 1991, and St. Petersburg before and after the aforementioned years.

[4] The Great Patriotic War (1941–1945) is a term used in Russia and former Soviet Republics to describe the war between the Soviet Union and Nazi Germany during World War II.

research fields and even schools appeared in Leningrad thanks to the efforts of Vladimir Smirnov, in particular, due to the seminars he started. Among them is the scientific seminar on Mathematical Physics, now named after V.I. Smirnov. This seminar celebrated its 75th anniversary in 2022. Smirnov was able to promote fields of mathematics that he had never even worked in, such as functional analysis and spectral theory of operators, because of his phenomenally broad education and deep understanding of mathematics as a whole.



During the post-war years, because of the overwhelming lack of professors, Smirnov was in charge (successively, and sometimes simultaneously) of several departments at LSU: Elasticity Theory, Hydroaerodynamics, Complex Analysis, and Real Analysis. But as soon as suitable candidates showed up, Smirnov passed the chair to younger colleagues. He only held the chairs of Higher Mathematics (Physics Department, since 1933) and Mathematical Physics (Department of Mathematics and Mechanics, since 1956) permanently. They were both departments that he had created.

Smirnov was also a consummate teacher. According to Frantisek Janouch, "at Professor Smirnov's lectures, mathematical functions came to life, they had their fates, experienced their misfortunes and accidents, which made it possible to calculate an integral or solve a differential equation."

As Victor Zalgaller recalled,

> Vladimir Ivanovich would begin with benchmark examples, with the exposition of classical theorems, and then he would change the timbre

> of his voice and in the last 10 minutes of each lecture would expound approximately three times as much material as he had presented before, with all sorts of generalizations, both purely mathematical and with the physical sense retained... Each of us could choose up to what point we were able to understand, without the slightest affront to the listener's self-esteem... This amazing talent for teaching people with different backgrounds at the same time distinguished the manner of V.I. Smirnov's teaching.

His life's work was the creation of the extraordinary *Course in Higher Mathematics*, which he began together with Jacob Tamarkin in the 1920s. The work on the course lasted for more than 50 years. With time, it grew into a real mathematical encyclopedia in five volumes which has been reprinted many times and translated into eight languages. It would not be a stretch to say that the creation of such a course was quite a feat.

The range of Smirnov's scientific interests was very broad. His main works are in complex analysis. His works on partial differential equations are also widely known. However, according to Sergei Vallander,

> he was very modest about his role in science, considering that [...] he was mainly a catalyst of scientific life in the mathematical world, who accelerated the processes already taking place within it.

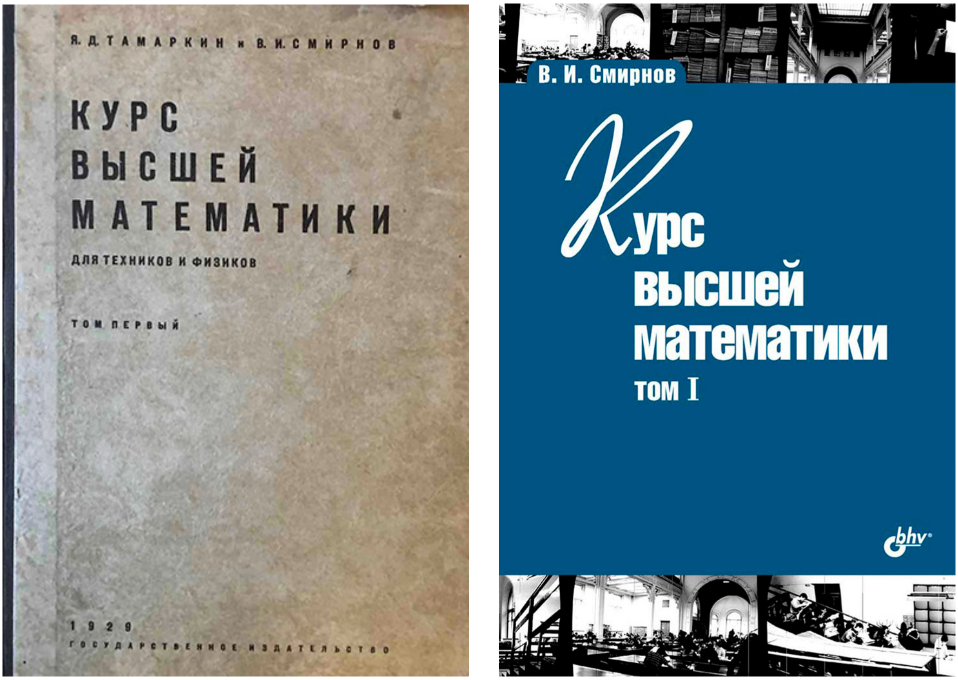
Vladimir Smirnov made several fundamental contributions to the history of mathematics. Thanks to him, we have a detailed history of the development of mathematics in Russia from the 1840s to 1970. He was the editor and author of texts on the life and work of Daniel Bernoulli, Pafnuty Chebyshev, Alexander Lyapunov, and other outstanding mathematicians.

After the premature death of his talented pupil Ivan Lappo-Danilevsky, Smirnov, together with Nikolai Kochin, studied his manuscripts and drafts, filled in all the lacunae, and published 12 works under Lappo-Danilevsky's name. The work, which took several years, was much like the work of a fine arts restorer and is perhaps unique in the history of science.

In the late 1950s, a systematic study of Euler's archives — manuscripts, letters, and notebooks — was initiated under Smirnov's supervision. The result was the publication of several volumes of Euler's writings, including a 438-page annotated index to his letters.

Vladimir Smirnov always strived to unite mathematicians in Leningrad. In 1921, he was one of the founders of the Leningrad Physical and Mathematical Society (it was disbanded in 1930 for political reasons). In 1953, Smirnov organized a general mathematical seminar, which in 1959 became the foundation for the revived Mathematical Society. He refused to become its President and was unanimously elected Honorary President.

One of Smirnov's main hobbies throughout his life was music. Although he had no musical education, he played the piano beautifully, having learned to play as a child from his elder brothers, who were professional musicians.



Cover pages of the 3rd (1929) and 24th (2008) editions of “A course of higher mathematics.”

Smirnov regularly gave home concerts, performing symphonies and quartets arranged for piano four hands. His musical partners over the years included Jacob Tamarkin, Dmitry Faddeev, and Dmitri Shostakovich.

He loved to take long walks and usually walked so fast that his companions could hardly keep up with him. In his younger years, another of Smirnov’s hobbies was the card game *vint.*[5] He was said to have played the game with ingenuity and skill.

Vladimir Smirnov was a deeply religious man. Although during Soviet times practicing religion was, to put it mildly, discouraged and sometimes even life-threatening, he was a member of the Parish Council at the Prince St. Vladimir’s Cathedral in Leningrad for many years. It is important to note that there was never anything ostentatious about his religiosity.

During the Stalinist terror, Vladimir Ivanovich wrote letters on numerous occasions to the authorities and spoke in defense of arrested acquaintances. Once, having been summoned to the NKVD,[6] when he refused to confirm slander that was being mounted against his colleague, he heard, after an eerie silence: “And you are a very brave man!”

[5] “Russian whist,” a complicated card game.

[6] NKVD (The People’s Commissariat for Internal Affairs) was the leading Soviet secret police organization from 1934 to 1946.

A commission of the USSR Academy of Sciences on the history of mathematics and physics. Sitting: S.I. Vavilov, A.N. Krylov, V.I. Smirnov. Standing: M.I. Radovsky, T.P. Kravets.

The number of people who had to thank Smirnov for the opportunity to stay in mathematics is huge. Vladimir Ivanovich fought like a lion for talented students.

Let us end our essay with the words of Vladimir Koshlyakov:

> During his long life, this extraordinary man did a lot of good, wise, and useful things. They were as inseparable from him as he was from them.

Darya Apushkinskaya and Alexander Nazarov

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V.I. Smirnov in complex analysis: Canonical Factorization and other fundamental results

V.I. Smirnov (1887–1974) obtained a few most useful and basic results of the Hardy space theory. The majority of these results were published in three papers quoted below as [1], [2], and [3].

Some sorrowful and dramatic circumstances (mostly related to Stalin's "Iron Curtain", which isolated the Russian scientists from any links with abroad) entailed that the results were totally overlooked by the international community for many decades (the first two papers even have been reviewed neither in *Math Rev* nor in *ZbMath*). Nevertheless, they anticipated some most important achievements in the Hardy space theory. These results, much later rediscovered in the literature, were employed in an uncountable quantity of research, everywhere where the Hardy space techniques are used.

Below, we comment mostly on the following themes:

(1) the proof of the "canonical factorization theorem" (1929),

(2) the cyclicity of "Beurling's outer functions" (1928),

(3) an integral ("universal") maximum principle and Smirnov class D (1932);

(4) Smirnov domains Ω in $\mathbb{C}$ and the Cauchy formula in $E^1(\Omega)$ (1932).

Canonical Factorization Theorem

V. Smirnov's Canonical Factorization Theorem CFT [2], states that any non-zero Hardy class function $f \in H^p(\mathbb{D})$, $p > 0$, in the unit disc

$$\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$$

can be uniquely represented as the product

$$f = \lambda B[f]S$$

of a complex constant $\lambda \in \mathbb{C}$, $|\lambda| = 1$, a product B of fractional linear functions (Blaschke factor), a "G. Szegö maximal function" $[f]$, and a "singular zero free function" S.

The Blaschke factor B fo f was previously separated by F. Riesz (1923), $f=BF$, $F\in H^p(\mathbb{D})$, where F has no zeros in $\mathbb{D}$, and

$$B(z)=\prod_n b_{\lambda_n}(z), \quad b_\lambda=\frac{\lambda-z}{1-\bar{\lambda}z}\cdot\frac{|\lambda|}{\lambda},$$

where $(\lambda_n)_n$ stands for the sequence of all zeros of f in $\mathbb{D}$ (counted with their multiplicities), known to satisfy the "Blaschke condition" $\sum_n(1-|\lambda_n|)<\infty$; $b_0(z):=z$. The "maximal function" $[f]$ of f was defined by G. Szegö (1921) in his study of the Toeplitz forms on $H^2(\mathbb{D})$ as

$$[f](z)=\exp\left\{\int_{\mathbb{T}}\frac{\zeta+z}{\zeta-z}\log|f(\zeta)|\,dm(\zeta)\right\}, \quad z\in\mathbb{D},$$

m being the normalized Lebesgue measure on the torus $\mathbb{T}$, and the singular factor S of f appeared in Smirnov's paper (without any specific name),

$$S(z)=\exp\left\{\int_{\mathbb{T}}\frac{\zeta+z}{\zeta-z}\,d\mu_f(\zeta)\right\}, \quad z\in\mathbb{D},$$

where μ_f is a singular (with respect to m) nonpositive measure on $\mathbb{T}$.

Notice that the Hardy spaces $H^p(\mathbb{D})$ were defined by G.H. Hardy in 1915 and rapidly became a strategic crossroad of holomorphic function theory and the real analysis. A fast development of the H^p-theory during the XXth century played the central role for both disciplines.

Smirnov's CFT is a cornerstone of the theory and a source of its most important applications (some of them are given by Smirnov himself). For the Hardy space theory, the CFT plays a role similar to the "fundamental theorem" of algebra (factoring a polynomial in linear factors), or the Weierstrass factorization theorem for entire functions.

However, by a joy of circumstances, V. Smirnov was not recognized during several decades as the author of the CFT, creating a longlasting historical injustice. In particular, in many publications, as a source for the CFT, a reference book by R. Nevanlinna (1936) was quoted, whereas it even does not contain such a statement. Some other researchers were disoriented with historical remarks in influential monographs by K. Hoffman (1962) (where the theorem was attributed to F. Riesz and G. Herglotz), and then by J. Garnett (1980), P. Koosis (1980), and W. Rudin (1987) (in three latter books, no attributions are given). To the contrary, V. Smirnov's authorship was restored in many other sources, see I. Privalov (1950), P. Duren (1970), M. Pavlovic (2014), N. Nikolski (2020).

In fact, V. Smirnov proved the CFT for a larger class (A) (now, often denoted N, for R. Nevanlinna) of functions f holomorphic in $\mathbb{D}$ and such that

$$\sup_{0<r<1}\int_{\mathbb{T}}\log^+|f(rt)|\,dm(t)<\infty,$$

exactly in the same form (here μ_f is simply singular, not necessarily non-positive). He called his theorem the "*parametric representation theorem*", and insisted on the importance of *free (integrability type) character of the parameters* defining a holomorphic function f: 1) a sequence $(\lambda_n) \subset \mathbb{D}$ with $\sum_n (1-|\lambda_n|) < \infty$, 2) a function $w = |f| \geq 0$ with $\log(w) \in L^1(\mathbb{T})$, and 3) a singular measure μ_f on $\mathbb{T}$. Conceptually, this point of view strongly contributed to the successful applications of the CFT to the spectral theory of Hilbert space contractions on the Sz.–Nagy–Foias functional model.

The CFT is the principal ingredient in the proofs in A. Beurling's Acta Mathematica paper (1949) on invariant subspaces of the shift operator, [5]. Beurling grouped unimodular functions in the canonical factorization as $BS := f_{in}$ and called f_{in} the inner factor, speaking on $\lambda[f] := f_{out}$ as the outer factor, $f = f_{in} f_{out}$.

Beurling's closure problem

Beurling's closure problem (1949)... had been solved in 1928. A. Beurling's just mentioned paper (1949) appeared to give a very influencial impact to the H^p-theory (and beyond). The main analytical problem considered was the question for which $f \in H^2$ the "weighted polynomials"

$$fP = \{pf : p \text{ is a polynomial}\}$$

are dense in H^2 (called the "closure problem", and also the "cyclic function problem"). A. Beurling's *answer* ("f is cyclic if and only if it is outer"), in its essential "if" part (the "only if" part is trivial by the CFT), *in fact, was proved in Smirnov's paper* [1]: if f is outer, fP is dense in H^2, this proof is contained in Theorem 2 of the 1928 paper, in a slightly different language; the needed part of the proof is "(26) $\Rightarrow$ (20)" in the notation of Smirnov's paper (and it is based on G. Szegö's theory of weighted orthogonal polynomials, 1920/1921).

35 years later, V. Smirnov returned to the theme and used exactly the same idea (in his 1964 book with N. Lebedev, [4, Ch. III § 2]), contributing to the study of cyclic functions in the Bergman space $L^2_a(\mathbb{D}, dxdy)$ with some useful sufficient conditions for cyclicity. These facts were also overlooked by the holomorphic space community.

An integral maximum principle

An integral maximum principle (1929), Smirnov class D, and a generic maximum principle (1932). Smirnov's paper [3] contains the definition, properties, and important applications of a remarkable "*class D*", now often called N_+: a function f in the unit disc $\mathbb{D}$ belongs to D if and only if it is in N and $\mu_f \leq 0$. Smirnov showed that $\bigcup_{p>0} H^p(\mathbb{D}) \subset D$ and proved a "*universal maximum principle*" (UMP): if $f \in D$ and $(f|\mathbb{T}) \in L^r(\mathbb{T})$ $(1 \leq r \leq \infty)$ then $f \in H^r$ (a preliminary form of UMP is contained already in his 1929 paper).

Then, he found a conformally invariant description of the class D and extended the UMP (at least for $r=\infty$) to any simply connected domain with sufficiently regular boundary.

Much later, H. Helson in his *Lectures on invariant subspaces* (1964) developed a similar approach to the classical Phragmen–Lindelöf principles (overlooking, as the whole community, Smirnov's results...). We refer to [6] for a careful analysis of these approaches and a derivation of a "most general" Phragmen–Lindelöf principle using the Smirnov class D.

An important series of applications of the class D, initiated by Smirnov himself [3], lies in the polynomial approximation theory: roughly speaking, if the polynomials are not dense in a certain (weighted) holomorphic space, the obstructions to an approximation (and so, the closure of the polynomials) can be described in terms of the class D; we refer to [6] for details.

Smirnov domains and the Cauchy formula in E^1 (1932)

In his paper [3], V. Smirnov introduced and studied an important analog of Hardy spaces for domains $\Omega\subset\mathbb{C}$ conformally equivalent to the unit disc $\mathbb{D}$ (and having a rectifiable boundary $\partial\Omega$): given $p>0$, the class $E^p(\Omega)$ consists of holomorphic functions f such that

$$\sup_{0<r<1}\int_{\partial\Omega(r)}|f(z)|^p ds<\infty,$$

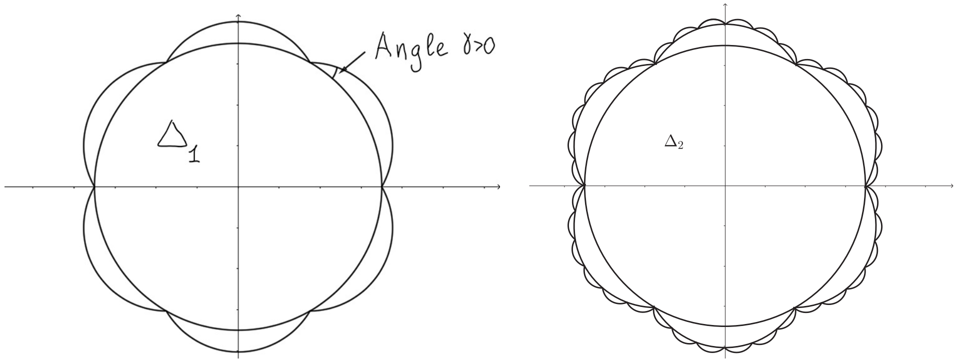
where $\Omega(r)=\varphi(r\mathbb{D})$ and φ stands for a conformal mapping $\varphi=\varphi_\Omega:\mathbb{D}\to\Omega$. Smirnov's fundamental theorem claims that a holomorphic in Ω function f is in $E^1(\Omega)$ if and only if it has almost everywhere boundary limits and the classical Cauchy formula holds. Smirnov's spaces $E^p(\Omega)$ are largely employed in the duality theory for extremal problems developed much later by G. Tumarkin and S. Havinson (the 1960s).

The same 1932 paper contains the definition and some applications of what is now called *"Smirnov's domains"* (or, in written, $\Omega\in(S)$): these are $\Omega's$ for which the derivative φ'_Ω is an outer (maximal) function in $\mathbb{D}$. V. Smirnov showed that the polynomials are dense in the space $E^2(\Omega)$ if and only if $\Omega\in(S)$, and similarly for the classical maximum principle:

$$|f(z)|\leq\sup_{\zeta\in\partial\Omega}|f(\zeta)|,\quad\forall z\in\Omega,$$

holds for every $f\in N_+(\Omega)$ if and only if $\Omega\in(S)$. Later on, many other interesting and important properties of Smirnov's domains were discovered, see [6] for an (incomplete) list.

V. Smirnov remarked that if the argument $\zeta\longmapsto\arg(\varphi'_\Omega(\zeta))$ is in $L^1(\partial\Omega, ds)$ then $\Omega\in(S)$, and he was inclined to the opinion that all $\Omega's$ are Smirnov... The first example of a non-Smirnov domain was discovered by M. Keldysh and M. Lavrentiev (1937): every nontrivial (different from the circle $\mathbb{T}$) pseudocircle bounds a non-Smirnov domain; an example of such a curve was presented.



An illustration to the Keldysh–Lavrentiev example of a non-Smirnov domain. The construction is similar to the 1904 von Koch snowflake, however, the boundary is isometric to $|z| = 1$, as in the famous J. Nash's embedding theorem (1954). The domains $\Delta_1 \subset \Delta_2 \subset \ldots$ provide approximations.

Later on, P. Duren, H. Shapiro and A. Shields (1966) showed that a conformal mapping φ whose derivative φ' is a singular inner function defines a pseudocircle $\varphi(\mathbb{T})$ if the corresponding singular measure is small and satisfies a "Zygmund smoothness condition."

Many other V. Smirnov's achievements in complex analysis and their numerous applications are overviewed in [6].

Nikolai Nikolski

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Alexander Alexandrovich Friedmann (1888–1925)

Friedmann is best known for his two papers in which theoretical studies of the expanding Universe were initiated. Since he used the general theory of relativity, Einstein's reaction to his first paper was quick and negative. In a note published in *Zeitschrift für Physik*, Einstein wrote: "The results concerning the non-stationary world, contained in the work, appear suspicious to me. In reality, it turns out that the solution given in it does not satisfy the field equations." Several months later, Einstein admitted his error and wrote back to the journal:

> In my previous note I criticized [Friedmann's work *On the curvature of Space*]. However, my criticism, as I became convinced by Friedmann's letter, communicated to me by Mr. Krutkov, was based on an error in my calculations. I consider that Mr. Friedmann's results are correct and shed new light on the subject.

Two other fields in which Friedmann's contributions were substantial are hydrodynamics and meteorology. His work in these areas was already acknowledged during his short life; indeed, in a letter sent from Delft in 1924, he wrote the following about the Congress for Applied Mathematics:

> Everything went well at the congress, the attitude towards the Russians was wonderful; in particular, I was included among the members of the committee for convening the next conference. [. . .] Blumenthal, Kármán and Levi-Civita got interested in my and my colleagues' work.

Friedmann was awarded the Lenin Prize posthumously in 1931 for his "outstanding work in science."

Alexander Friedmann was born in St. Petersburg into an artistic family; his father was a ballet dancer and his mother a pianist. At the 2nd St. Petersburg Gymnasium, Alexander and Yakov Tamarkin were the top two pupils in their class and graduated in 1906 with gold medals. Being close friends, they were almost always together: during their university studies at St. Petersburg University (in 1906–1910, both were students in the Mathematical Division of the Faculty of Physics and Mathematics), and in their political activity. Both were leaders of student strikes during the 1905 Revolution. They are always mentioned together in V.A. Steklov's diaries for 1908, when they attended his lectures and visited him at his home many times, bringing their lecture notes with them, to be looked through and then printed lithographically. The friends also had mathematical questions to discuss with their professor; a characteristic item in Steklov's diaries is:

> November 22. Tamarkin and Friedmann came to see me this evening [...] Kept asking me about their findings from delving deep into the theory of orthogonal functions. They are having an article published in Crelle's journal. Sharp fellows! They left at half past twelve, after supper.

The aforementioned 11-page-long paper by Friedmann and Tamarkin, published in the *Journal für die reine und angewandte Mathematik* in 1909, concerns the function $[x]$ and Bernoulli numbers. Their interest in these numbers began when they were still gymnasia pupils, and they sent the paper *Sur les congruences du second degré et les nombres de Bernoulli* to Hilbert in 1905; it was published in *Mathematische Annalen* the following year.

Steklov recommended Friedmann and Tamarkin to continue their studies for Master's Degrees, and they completed the necessary examinations by 1913. Then Friedmann was appointed to the Aerological Observatory located near St. Petersburg, where he began studying meteorology, and even spent some time in Leipzig with Bjerknes, the leading expert in theoretical meteorology, just before World War I broke out in 1914.

Friedmann decided to join the volunteer aviation detachment and soon began flying aircraft. He was involved in bombing raids; in a letter to Steklov written on February 5th, 1915, he writes:

> My life is progressing pretty smoothly, if you don't count such accidents as a shrapnel explosion twenty feet away, the explosion of an Austrian bomb within half a foot, which turned out almost happily, and falling down on my face and head, which resulted in a ruptured upper lip and headaches. But one gets used to all this, of course, especially when you see things around you that are a thousand times more awful.

Friedmann was decorated with the Cross of St. George for his courage, but he remained a mathematician at heart even at the front lines, and developed an approach to computing the trajectory of bombs. Since this approach was based on Steklov's suggestion, Friedmann reported to him: "The bombs turned out to be falling almost the way the theory predicts." In the summer of 1915, Friedmann was sent to Kiev to lecture on aeronautics for pilots at the Central

Aeronautical Station; several months later, he was appointed its Head and moved with the station to Moscow in April of 1917. The Bolsheviks came to power six months later and began peace negotiations with Germany, however. The Central Aeronautical Station ceased its operations and Friedmann wrote to Steklov about how depressed he was with the uncertainty surrounding his future.

Fortunately for Friedmann, Perm University was established in 1916 and Steklov, who participated in its organization, sent them a letter of recommendation in February of 1918 supporting the recruitment of Friedmann, emphasizing that "Recruiting him as an instructor in mechanics at Perm University is, in my opinion, highly desirable. The university will find in him a worthy lecturer and researcher." Friedmann was elected an extraordinary professor in the Department of Mathematics and Physics and arrived in Perm in April 1918, where he found A.S. Besicovitch, I.M. Vinogradov, A.F. Gavrilov and R.O. Kuzmin, who had arrived earlier from Petrograd (the name of St. Petersburg from 1914–24).

In the autumn of 1918, the Physico-Mathematical Society was founded, and Friedmann assumed the duties of its Secretary and member of the editorial board of the *Journal de la Société des Math. et de Phys. a l'Université de Perm.* (In the 2nd volume of this journal, Besicovitch, another member of the editorial board, published his solution to a problem equivalent to the famous Kakeya's problem.) Friedmann carried out a great many tasks and duties (this was the case with him earlier in his life as well as later on) that acted as his proud civic contribution.

However, Perm became a frontline city soon and was occupied by the White Army on the 24th of December, 1918; the period of 'Kolchakovia' (Friedmann used this term because admiral Kolchak was the 'Supreme Leader' of the anti-Bolshevik movement in Siberia) lasted until August 1919. In January, Friedmann was accused of being sympathetic towards the Soviet powers during a session of the University Council; fortunately, without consequences. As the Red Army approached Perm in the summer, the university staff, except Besicovitch, left the city. Unlike his colleagues, who positioned themselves against the Soviets, Friedmann returned soon, and his last letter to Steklov from Perm, sent in November 1919, was a cry for help: "The University needs staff!"

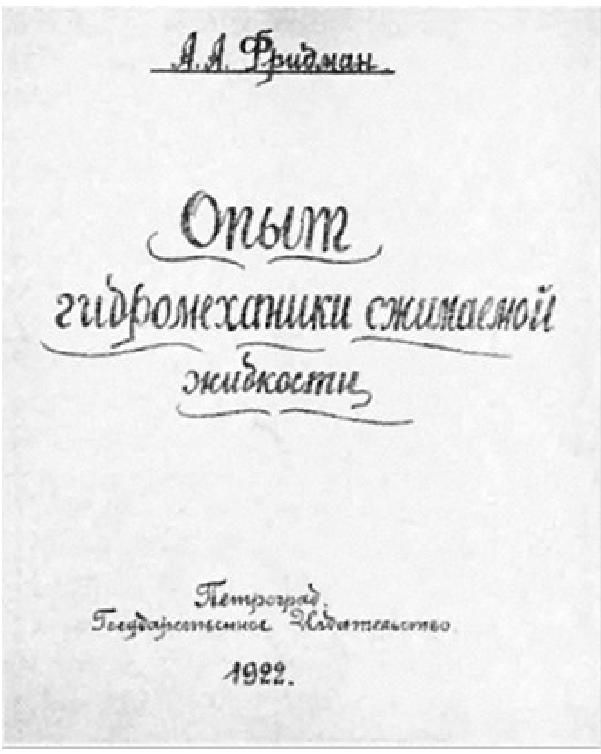
Friedmann's portrait, M.M. Devyatov, [5].

In the spring of 1920, Friedmann returned to Petrograd to take up an impressive number of appointments. He headed the newly-established Mathematical Bureau at the Central Geophysical Observatory, kept teaching positions at Petrograd University, the Polytechnic Institute, the Institute for Railway Engineering and the Naval Academy; finally, he undertook research at the Atomic Commission[1] of the Optical Institute. Of course, the Mathematical

[1] Atomic Commission dealt with atoms and X-rays, no idea of an atomic bomb existed at that time.

Bureau was his main occupation; even at the very beginning of his military service, Friedmann had already been thinking of setting up a theoretical department at the Observatory. To realize this project he employed a number of brilliant young researchers, including I.A. Kibel, B.I. Izvekov, V.A. Fock, N.E. Kochin and P.Y. Polubarinova–Kochina (the last three being future Academicians); each researcher had his/her own topic for investigation and the results were to be presented at the scientific seminar.

In his own studies, Friedmann considered general equations describing the motion of a compressible fluid, important for the field of dynamic meteorology. The work in this area had been initiated in 1916, when the brief note *Sur les tourbillons dans un liquid a température variable* appeared in Comptes Rendus. At the beginning of 1921, he completed his Master's dissertation on this topic; it was published lithographically in 1922 under the unassuming title of *An Essay on the Hydrodynamics of a Compressible Fluid* (see its front page below). It was in fact a comprehensive treatise reprinted as a monograph, with Kochin's comments, in 1934. Along with some important results, the dissertation included many interesting open problems that were subsequently solved by Friedmann himself and by his disciples.



Soon after returning to Petrograd, Friedmann began his studies of the general theory of relativity, which he mentioned in a letter to P. Ehrenfest

(Leiden) written circa the end of 1920: “I have been working on the axiomatics of the relativity principle.” Along with his two research papers mentioned at the beginning of this article, he published the booklet The World as Space and Time, intended for a thoughtful but amateur readership. Another of his projects, the five-volume monograph Fundamentals of the Theory of Relativity, to be written in collaboration with V.K. Frederiks, was not realized due to Friedmann's untimely death from typhoid fever in August of 1925. A few weeks before, he participated in a record-breaking, 7400-meter-high ascent in a hot air balloon during which he recorded interesting meteorological and medical observations.

Nikolay Kuznetsov

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The impact of Friedmann's work on cosmology

Historical introduction. The impact of Friedmann's work on cosmology can hardly be overestimated. By training, Friedmann was a mathematician, but one of exceptional versatility, who made important contributions also in other fields, such as meteorology. In the summer of 1917 and in the middle of tumultuous events in Russia, he founded and was the first director of the "Aeropribor" factory in Moscow, which produced tools for airplanes, and which still exists to this day.

Nevertheless, his greatest contribution to science is undoubtedly contained in the two pioneering papers in 1922 and 1924 which appeared in the German journal *Zeitschrift für Physik* [11, 12].[1] In these papers, he demonstrated that Einstein's field equations with a cosmological constant (called by him *Weltgleichungen*, i.e. world equations) do not only allow Einstein's 1917 static solution with matter and de Sitter's 1917 apparent static vacuum solution, but also dynamical solutions describing an expanding or collapsing Universe. The corresponding equations, today called Friedmann or Friedmann–Lemaître equations, form the basis of modern cosmology. In 1923, Friedmann published a book on cosmology in which he also presents insights into his general philosophical ideas [13].[2]

In the 1920s, Friedmann's work had little impact [8]. The main question in those years was trying to find out whether there is an observational difference between Einstein's and de Sitter's solution. Friedmann's papers were apparently also unknown to Georges Lemaître, who in 1927 wrote another groundbreaking paper that was little appreciated at the time: he related the formal solutions for an expanding or contracting Universe to redshifts and thus to observations. Einstein, after having read Friedmann's first paper, first thought that the solutions were wrong. Later he admitted that the solutions are mathematically correct, but (in his opinion) physically irrelevant. This demonstrates how deeply the idea of a static Universe was rooted in people's imagination at the time.

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[1] In [11], the German transcription of the Russian name was chosen "Friedman", but we stick to the common practice of writing "Friedmann."

[2] The editor of the German translation [13] speculates that the title *Мир как пространство и время* (Die Welt als Raum und Zeit) alludes to Schopenhauer's opus magnum *Мир как воля и представление* (Die Welt als Wille und Vorstellung).

It is often stated that Friedmann was only interested in the mathematics of the equations, not in their physical content. In our opinion, this is only partially true. He was certainly strongly influenced by the mathematicians Weyl and Hilbert, especially the latter's idea of axiomatization.[3] But in his work he strongly emphasized that the geometry of the world should be determined by theoretical physics *and* observational astronomy.[4] At the end of his 1922 paper, he gives an estimate of 10^{10} years for the duration of a recollapsing Universe, which is close to the current estimate for the age of our Universe.

Friedmann was, in particular, interested in the question of whether the world (three-dimensional space) is finite or infinite. This motivated him to study the case of negative curvature in 1924 [12]. He found that, in contrast to the spatially closed case discussed in 1922, the case of negative curvature leaves this question open. He concludes the 1924 paper with the words: "This is the reason why, according to our opinion, Einstein's world equations without additional assumptions are not yet sufficient to draw a conclusion about the finiteness of our world."[5] The question whether it makes sense to talk about actual infinities in physics (in contrast to mathematics) is an intriguing one and continues to be discussed up to the present day [10], as Friedmann's insights continue to inspire modern research.

Friedmann's equations[6]**.** The starting point is Einstein's field equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^3} T_{\mu\nu}. \tag{1}$$

Observations indicate that the Universe is approximately isotropic around our position. These come mainly from the anisotropy spectrum of the Cosmic Microwave Background (CMB). Adopting the Cosmological Principle ("all places in the Universe are alike"), one is led to assume (approximate) isotropy around *every* position. One can then mathematically prove that our Universe must also be (approximately) homogeneous. The geometry of a homogeneous and isotropic spacetime is characterized by the line element

$$\mathrm{d}s^2 = -c^2 \mathrm{d}t^2 + a^2(t) \left(\frac{\mathrm{d}r^2}{1 - kr^2} + r^2 \mathrm{d}\Omega^2 \right), \tag{2}$$

where $a(t)$ is the scale factor. For the parameter k, we have the possible choices $k = 0$ (flat spatial geometry), $k = 1$ (positive curvature), $k = -1$

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[3] Friedmann had paid a visit to Göttingen in 1923.

[4] In 1924, he even gave a thesis topic to his student A.B. Schechter dealing with the question whether trigonometric measurements at astronomical dimensions can lead to a decision between different world geometries. A paper on this was published three years after Friedmann's death by Frédericksz and Schechter [8].

[5] The German original reads: "Dies ist der Grund dafür, daß, unserer Meinung nach, Einsteins Weltgleichungen ohne ergänzende Annahmen noch nicht hinreichen, um einen Schluß über die Endlichkeit unserer Welt zu ziehen."

[5] A comprehensive discussion of the material in this and the following section can be found in many textbooks, see, e.g., [17], [16], [9] and [4].

(negative curvature); only the latter two cases were treated by Friedmann. Current observations favor a spatially flat Universe, although there is still a controversy [7]. A given value for k does not fix the topology of our (spatial) Universe, and it is a most intriguing question to determine the cosmic topology from observations [15].

Inserting the ansatz (2) into (1), one is led to Friedmann's equations.[7] The first equation is the restriction of the general Raychaudhuri equation to a homogeneous and isotropic Universe,

$$\ddot{a} = -\frac{4\pi G}{3}(\rho + 3p)a, \tag{3}$$

where ρ and p denote energy density and pressure of matter, respectively. If matter obeys the strong energy condition $\rho + 3p \geq 0$, (3) leads to concave solutions for $a(t)$, that is, to a world model with a singular origin. The second Friedmann equation reads

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - k. \tag{4}$$

In contrast to (3), this equation only contains temporal derivatives up to *first* order, so it has the interpretation of a *constraint*. In fact, it is the Friedmann version of the Hamiltonian constraint in general relativity [14].

From (3) and (4), one can derive a third equation,

$$\dot{\rho} + 3H(\rho + p) = 0, \tag{5}$$

where $H := \dot{a}/a$ is the Hubble parameter (its evaluation at the present day is called Hubble constant, denoted by H_0). The combination $\rho + p$ occurring in (5) is called inertial mass density. In these Friedmann equations, we have followed the modern practice of including the cosmological constant Λ into the density ρ (although this was already suggested by Schrödinger in 1919), because it contributes an "energy density of the vacuum" $\rho_\Lambda := \Lambda/8\pi G$. Its equation of state reads $p_\Lambda = -\rho_\Lambda$, so from (5) we see that ρ_Λ is constant. For barotropic equations of state $p = w\rho$, $w \neq -1$, we find from (5) the solution

$$\rho a^{3(1+w)} = \text{constant}, \tag{6}$$

which includes as particular cases:

- dust $(p = 0) \longrightarrow \rho \propto a^{-3}$,
- radiation $(p = \rho/3) \longrightarrow \rho \propto a^{-4}$,
- stiff matter $(p = \rho) \longrightarrow \rho \propto a^{-6}$.

By the kinematic relation $a_0/a = 1 + z$, with a_0 as the present scale factor, we can relate ρ to the observable redshift z of objects. The case of radiation is relevant for the early Universe, while stiff matter so far seems unrealistic.

[7] From here on, we set $c = 1$.

Today, the Universe is dominated by dust (about one-third) and vacuum energy (about two-thirds), leading to the temporal evolution

$$a(t) = a_0 \Big(\frac{3\Omega_0 H_0}{\Lambda}\Big)^{1/3} \sinh^{2/3}\Big(\frac{3}{2}\sqrt{\frac{\Lambda}{3}}t\Big), \qquad (7)$$

where Ω_0 is today's matter density in terms of the critical density, observationally determined to be about 1/3. Observations also indicate that the age of our Universe is about 13.8 billion years. For large times, the evolution law (7) asymptotes to de Sitter space.[8] From observations of the CMB, there are strong indications that our Universe underwent a quasi-exponential expansion (with very large Λ) already very early in its history, a phase called inflation. Inflation offers the means to explain the origin of structure in the Universe.

Instead of barotropic equations of state, one often employs dynamical matter models, typically with a scalar field ϕ. In the Friedmann limit, this field depends, of course, only on time. In the case of a massless field, it corresponds in (4) to the choice of a density $\rho_\phi = \dot{\phi}^2/2$.

Beyond the Friedmann approximation. Beyond the immediate and obvious utility of the Friedmann equations for cosmological applications there are several important and promising directions for future development that build on Friedmann's achievements. For lack of space we here mention only two of these, namely (i) their use for taking the first steps toward a theory of quantum gravity, and (ii) the generalization of the isotropic ansatz (2) in order to search for a fundamental symmetry of Nature.

When adapting the ansatz (2) to a quantum mechanical context one speaks of the so-called *minisuperspace approximation*, in which the full superspace of geometrodynamics, being the moduli space of all three-metrics modulo spatial diffeomorphisms, is restricted to few homogeneous degrees of freedom such as the scale factor a. This limit was first discussed by DeWitt in his pioneering paper on canonical quantum gravity [6]. This is a huge simplification because key technical issues such as the non-renormalizability of perturbative quantum gravity can be ignored in this approximation. Furthermore, various conceptual issues of quantum gravity and quantum cosmology can be studied. Namely, the direct canonical quantization of the second Friedmann equation (4) leads to a special case of the Wheeler–DeWitt equation [14, 4], here given for the case of a massless homogeneous scalar field,

$$\left[\frac{4G\hbar^2}{3\pi a^2}\frac{\partial}{\partial a}\left(a\frac{\partial}{\partial a}\right) - \frac{\hbar^2}{a^3}\frac{\partial^2}{\partial\phi^2} - \frac{3\pi}{4G}ka\right]\Psi(a,\phi) = 0,$$

where a particular factor ordering has been adopted. The wave function $\Psi(a,\phi)$ is a simple example of the "wave function of the universe." It is, in

[8] For late time expansion with constant positive ρ_Λ one speaks of *dark energy*, but there is also the possibility that the effective vacuum energy density varies with time, in which case one speaks of *quintessence*. The latter is thought to originate from matter sources and is often modeled by means of a time-dependent scalar field ϕ.

particular, possible to analyse the behaviour of Ψ near the singularity where $a \to 0$. The Wheeler–DeWitt equation has no external time parameter, but one can employ the scale factor a so as to track the evolution of the matter degrees of freedom with respect to this "intrinsic time." (Note that the minisuperspace Wheeler–DeWitt equation is hyperbolic with respect to a.) Key open issues concern the physical interpretation of Ψ, the construction of a suitable Hilbert space, and the meaning of observables; for a survey and further discussion, see [14].

The cover page of Friedmann's book *The world as space and time.*

The other extension concerns the inclusion of *non*-homogeneous degrees of freedom. On the phenomenological side, the evolution of our Universe, if approximated by a homogeneous and isotropic spatial part, is successfully described by Friedmann's equations, but small inhomogeneities *must* be taken into account in order to understand the properties of the CMB in the framework of cosmological perturbation theory. Furthermore, a precise understanding of the formation of galaxies and clusters of galaxies requires the numerical treatment of the Einstein equations (1) and their Newtonian limit. Incorporating inhomogeneities is likewise crucial for a better understanding of the origin of the Universe, because inhomogeneities scale like a^{-6}, like stiff matter, and thus dominate very close to the Big Bang singularity. This is a crucial issue for inflationary cosmology, which hinges on the ansatz (2). Finally, there remain difficult issues related to defining a generally covariant averaging procedure in Einstein's theory that would provide a rigorous basis for the Friedmann approximation [3].

On the more mathematical side, a key insight came from the Belinski–Khalatnikov–Lifshitz (BKL) analysis [1, 2] of the generic behavior of solutions of Einstein's equations near a spacelike singularity. There, one generalizes the ansatz (2) to

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}x^2 + b^2(t)\mathrm{d}y^2 + c^2(t)\mathrm{d}z^2,$$

thus giving up isotropy, but retaining spatial homogeneity. A surprising result to come out of this analysis is the appearance of chaotic oscillations of the metric coefficients a, b, c as one approaches the singularity. This result indicates that the "near singularity limit" of the metric exhibits a far more complicated behavior than inspection of, say, the Schwarzschild metric would suggest, thus also showing the limitations of the assumption of isotropy.

The BKL analysis has been generalized in many directions. In particular, a closer study of the BKL limit has revealed evidence for a huge infinite-dimensional symmetry of indefinite Kac–Moody type, vastly generalizing the known duality symmetries of supergravity and string theory. This novel symmetry can possibly serve as a guiding principle towards unifying the fundamental interactions [5].

Claus Kiefer and Hermann Nicolai

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Jacob David (Yakov Davidovich) Tamarkin (1888–1945)

Jacob Tamarkin's research interests were very wide, and this was reflected in the topics of his research: number theory and function theory, general theory of summability and summability of Fourier series, moments problems and differential and integral equations, boundary value problems and Green's formula, problems of mathematical physics and approximation theory. All these works dealt with the most important problems of modern mathematics, and the results they yielded immediately resonated with the research of contemporaries and followers, which undoubtedly contributed to the further development of mathematics. Twenty-two of his students defended their theses; many of them became well-known professors of mathematics (e.g., Elmer Tolsted). His special lectures for postgraduate students were unique and updated annually, and he used the latest articles from Polish, French, German, English, and Russian scientific journals to prepare them.

Jacob Tamarkin was born in Chernigov into a family of a physician but grew up in St. Petersburg, where he studied at one of the best high schools, The Second St. Petersburg Gymnasium, from which he graduated with honors in 1906. He was taught mathematics by Jakov Jodynsky, a remarkable teacher, who influenced his students to begin studying the basics of higher mathematics and to attend a city seminar for gymnasium students, with classes given by university professors, including Andrei Markov. In 1906, together with his classmate and friend Alexander Friedmann, the 18-year-old Tamarkin wrote his first scientific paper on Bernoulli numbers, which was published in the German journal *Mathematische Annalen* (see [1]).

The desire to study mathematics led the friends to the Mathematics Department of the Physics and Mathematics Faculty of St. Petersburg University, where they studied number theory intensively in the scientific circle headed by Yakov Uspensky. In their second year, in 1909, the paper *Quelques formules* *concernant la théorie de la fonction* $[x]$ *et des nombres de Bernoulli*, written by Tamarkin and Friedman, was awarded a gold medal and was published in another serious periodical *Journal für die reine und angewandte Mathematik* on the recommendation of David Hilbert (see [2]).

In addition, Jacob Tamarkin and Alexander Friedmann were actively involved in the work of the publishing committee at the Faculty, taking notes of Vladimir Steklov's lectures and lithographing them. They also set up a mathematical reading room and a mathematical circle where they studied the results of the latest research by Western scientists, in particular, in the theory of orthogonal functions. From 1907, Tamarkin and Friedmann (they were called "the Second Gymnasium boys") attended Paul Ehrenfest's seminar on new physics.

During his final year at University, Tamarkin studied the problem of vibration of elastic solid rods and thin plates under Vladimir Steklov's guidance, who recommended leaving him to prepare for a professorship (1910), because "Mr. Tamarkin passed perfectly all the tests, has sufficient knowledge of French and German, reads mathematical books in English and Italian... In twenty years of my teaching activity, I have not met such gifted young people [Tamarkin and Friedmann]... They must, by all means, be retained for science and given the opportunity to devote themselves to scientific work..."

From 1913 to 1919, Tamarkin taught at higher educational institutions in Petrograd: the Institute of Transport Engineers, the Electrotechnical Institute, and the Polytechnical Institute. Having been invited in 1919 to take up a professor's position at Perm University, he left for Perm, in the hope of a more peaceful and secure existence in the aftermath of the war, together with his wife, who was sent there by Petrograd University as a secretary. In Perm, he lectured, took part in setting up the office for approximate computations, and briefly served as dean of the Faculty of Physics and Mathematics.

But soon the civil war reached the Urals, and Perm was passing from the Whites to the Reds[1] and back... When Tamarkin was sent to Petrograd and Moscow to purchase equipment for the office for approximate computations in March 1920, he did not return to Perm. On May 20, 1920, Steklov made a record: "At 11.00, Tamarkin and Friedmann appeared unexpectedly, traveling for 12 days in a special teplushka."[2] They were carrying books which the library of Perm University was returning to Petrograd and also some food, which had almost been requisitioned: the authorities fought "speculators" and "bag people"[3] at the time. Everything, however, was sorted out.

[1] The Whites was a loose confederation of anti-communist forces that fought the communist Bolsheviks, i.e., the Reds, in the Russian Civil War of 1917–1922/1923.

[2] A heated freight car used for transportation of people.

[3] A "speculator" was a person, who, basically, traded with the intent of gaining profit, which was considered illegal since prices were fixed. "Bag people" referred to persons doing small trade for personal profit, recognizable by their large sack.

In Petrograd, Tamarkin resumed teaching at his previous institutions, to which he added the Naval Academy, the Central Weather Bureau, the Physics and Technology Institute, and the Atomic Commission[4] at the State Optical Institute. The reason for such an extensive list of jobs was simple: he had a family to support (his son Pavel was born in 1922), and his salary was accompanied by a set of ration cards needed to survive.

Jacob David was not only successful in science, publishing a large number of papers on various questions of applied mathematics together with Vladimir Steklov, Alexander Friedmann, Jacob Besicovitch, Grigory Fichtenholz, and Aleksei Krylov, but also wrote textbooks for students.

Tamarkin's Menshevik[5] past, the GPU's[6] interest in him, well-founded fears for his family's life, and fear of famine led him to consider emigrating. His final decision to leave the country was probably also influenced by his meeting with James (Yakov) Shohat, at the International Congress of Mathematicians in Toronto in 1924. Shohat had already left Russia by that time and was teaching at the University of Michigan.

In 1924, Jacob Tamarkin crossed the Latvian border with professional smugglers and came before the American consul, who doubted at first that a strange visitor was a professor of mathematics from Petrograd and tried to examine him. Afterward, Tamarkin often told his American students about this "exam" (the consul asked the equation of the ellipse and something else from analytical geometry), whose positive outcome played such an important role in his later fate.

In March 1925, Tamarkin reached the shores of America by ship, and the first day of acquaintance with the USA remained in his memory: he often shared these unforgettable impressions with his friends. Straight "from the ship to the ball",[7] he, a passionate music lover with a hunger for music, went to a concert at the New York Philharmonic to hear a Brahms symphony, and after the performance allowed himself the "Royal Banana" — an enormous ice cream with all sorts of sauces and gravies.

In exile, Jacob Tamarkin lived and worked in the USA, first at Dartmouth College (1925–1927), then at Brown University (1927–1945), where he lectured on integral equations and topological groups, theories of series and of polynomial approximation, partial differential equations and subharmonic functions.

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[4] It was created in 1920 by Dmitri Rozhdéstvenski and comprised many leading scientists such as Ioffe, Khvolson, Krutkov, Krylov, Friedmann etc.

[5] The Mensheviks had used to be one of the three dominant factions in the Russian socialist movement, the others being the Bolsheviks and Socialist Revolutionaries. Tamarkin was a member of the Mensheviks faction in 1905.

[6] The GPU, short for Glavnoe Politicheskoe Upravlenie (State Political Directorate), was the intelligence service and secret police.

[7] It is a quotation from Alexander Pushkin's *Eugene Onegin*, meaning "to get unexpectedly from mundane circumstances to more solemn or strictly official."

The American scientific community highly valued Jacob Tamarkin's contribution. His obituary said, "He appeared as a stranger on our shores two decades ago, and soon began to play an active role in American mathematical life..." (cited by [6]). Tamarkin was a board member (from 1931) and vice-president (from 1942–1943) of the American Mathematical Society, a member of the American Mathematical Association, and the American Academy of Arts and Sciences, one of the initiators and editor of the abstracts journal *Mathematical Reviews*, and on the board of the German abstracts journal, *Zentralblatt für Mathematik*, where he published many reviews.

His colleagues valued him for his erudition, critical mind, and readiness to help and support the authors of new ideas. It was Tamarkin who persuaded Norbert Wiener, the future "father of cybernetics," to present his results concerning Tauberian theorems systematically. Friends and acquaintances loved the cheerful character of Tamarkin, loved his hospitable home with its constant musical concerts and unprecedented Russian hospitality, calling him behind his back simply "J.D." (for Jacob Davidovych).

Natalia Lokot

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On the works of Ya.D. Tamarkin, their influence and development in asymptotic theory and spectral theory of operators

Yakov Davidovich Tamarkin (Jacob David Tamarkin) entered the Mathematics Department of the Physics and Mathematics Faculty of St. Petersburg University in 1906. Already in his first years of study at the university, he showed outstanding ability for research work in mathematics and, together with his friend A. Friedmann, published two articles in German journals *Mathematische Annalen* and *Journal für die reine und ungewandte Mathematik.* It should be noted that these first works were related to number theory and Diophantine equations. They were highly appreciated by specialists, but did not significantly affect his further interests in mathematics. V.A. Steklov became his scientific advisor, and Yakov was attracted by the problems of expanding functions in series of eigenfunctions of the Sturm–Liouville equation and the fourth-order equation describing the vibrations of a rod. These questions were of the greatest interest to V.A. Steklov at that time. Note that Steklov [1, 2] was the first to prove the possibility of expanding an arbitrary continuously differentiable function vanishing at the endpoints of a segment in a series of the eigenfunctions of the Sturm–Liouville problem with Dirichlet conditions. This theorem of Steklov anticipated the famous Hilbert–Schmidt theorem on the representation of an arbitrary function from a Hilbert space by a series in the eigenfunctions of a self-adjoint operator. Note that Steklov's and Hilbert–Schmidt's theorems are different. In the first theorem, the uniform convergence, in the norm of the space of continuous functions, is asserted; in the second theorem, the convergence is in the norm of the space of square-summable functions L_2.

V.A. Steklov highly appreciated Tamarkin's bright talent and recommended him to stay and prepare for a professorship. Since 1910, Tamarkin had been teaching at St. Petersburg University and other universities of St. Petersburg, he had been closely cooperating with V.A. Steklov, A.A. Friedmann, G.M. Fichtenholz, A.N. Krylov. He had written several joint papers with these famous mathematicians and he had independently published the book *The Course of Analysis.* But the main work was his master's thesis, published as the book [3] later in 1917. Tamarkin subsequently published some of its fragments together with A. Besicovitch [4] in German, and later he presented the main results in a more concentrated form in [5] in English. An important addition to these works was the article [6], where it was essentially proved (in the particular case of boundary value problems for ordinary differential equations) the now well-known theorem on a holomorphic operator function.

After emigrating to the United States in 1924, Tamarkin quickly became one of the recognized international authorities in mathematics. He wrote joint works with M. Stone and A. Zygmund. His main co-author was the young, brilliant mathematician E. Hille, who began to work with Tamarkin essentially as his student. Later, Hille gained international recognition as one of the founders of such an important section of modern functional analysis as the theory of operator semigroups.

The main results of Hille–Tamarkin's works are devoted to the summation theory of the Fourier series by various methods. The results of these works at one time were highly appreciated. But after the publication of the encyclopedic monographs of A. Zygmund and N.K. Bari the previous articles and books were cited less frequently. Over time, the relevance of this topic has decreased. However, the book [3] has a lucky destiny. This book is the main one in Tamarkin's scientific heritage and is still widely cited in works on the theory of ordinary differential operators. There are several reasons for this, below we give more details about these reasons as well as about the results obtained in [3].

The key problem is to obtain asymptotic expansions for solutions of ordinary differential equations with a large (spectral) parameter. Asymptotic theory is the main tool for investigating the spectral characteristics of ordinary differential operators and studying the convergence of expansions in eigenfunctions of such operators. To the best knowledge of the author, the first formulas for large eigenvalues of the Sturm–Liouville problem were obtained by Horn [7] (see also Horn's earlier works referenced in [7]). Horn considers the following self-adjoint Sturm–Liouville problem:

$$\frac{d}{dx}\Big(p(x)\frac{dy}{dx}\Big)+q(x)y=\lambda^2\rho(x)y, \tag{1}$$

$$y'(0)=hy(0),\quad y'(1)=Hy(1),\qquad h,H\in\mathbb{R}, \tag{2}$$

where p, q, ρ are smooth functions, p and ρ are positive on the interval $[0,1]$. Horn reduces (1) to an integral equation and uses the method of successive approximations to show that there exists a pair of solutions to this equation with the following asymptotic representation on the positive semiaxis:

$$y_\pm(x)=e^{\pm\lambda\beta(x)}\Big(\varphi^0_\pm(x)+\frac{\varphi^1_\pm(x)}{\lambda}+\dots\Big),\qquad \lambda\in\mathbb{R}^+,\ \lambda\to\infty, \tag{3}$$

where $\beta(x)=\int_0^x\sqrt{\rho(t)}\,dt$. Substituting these solutions into the boundary conditions, Horn finds the now well-known asymptotic formulas for the large eigenvalues and the corresponding eigenfunctions. Definitely, this was a significant achievement of its time. Moreover, in contrast to numerous previous works on the asymptotic theory (for asymptotics not in a large parameter, but in an independent variable), Horn understood and substantiated that solutions (3) not only formally satisfy equation (1), but they are real solutions to this equation.

Horn's work served as a source for the general theory developed by J. Birkhoff [8], [9]. He considered the following general equations with a large parameter:

$$\frac{d^n}{dx^n} + \lambda p_{n-1}(x, \lambda)\frac{d^{n-1}}{dx^{n-1}} + \ldots + \lambda^n p_0(x, \lambda)y = 0, \tag{4}$$

where the functions

$$p_j(x, \lambda) = \sum_{k=0}^{\infty} p_{jk}(x)\lambda^{-k}$$

are analytic for large $|\lambda|$ and smooth in x on a finite interval $[a, b]$. If $p_{jk}(x) \equiv 0$ for $k \geqslant j$, then we obtain the following equation with polynomial dependence on the large parameter λ:

$$l_n(y) + \lambda l_{n-1}(y) + \ldots + \lambda^{n-1} l_1(y) + \lambda^n l_0(y) = 0, \tag{5}$$

where

$$l_j(y) = p_{j0}(x)y^{(j)} + p_{j1}y^{(j-1)} + \ldots p_{jj}(x)y, \quad p_{n0}(x) \equiv 1, \quad j = 0, 1, \ldots, n-1, \tag{6}$$

are differential expressions of order j. It is clear that the leading coefficients p_{j0} play a central role in the studies of the asymptotic behavior for (5). Equation (5) is associated with the following characteristic equation:

$$\omega^n + p_{n-1,0}(x)\omega^{n-1} + \ldots + p_{1,0}(x)\omega + p_0(x) = 0.$$

If the roots $\omega_j(x)$ of the above equation depend on x, then no significant results are obtained. If ω_j are constants, then the equations

$$Re\lambda\omega_k = Re\lambda\omega_j, \quad k \neq j,$$

define on the complex plane $\mathbb{C}$ straight lines passing through the origin and splitting the plane $\mathbb{C}$ into sectors Γ_s (their number is at most $n^2 - n$). Birkhoff proves that in each sector Γ_s, there exists a fundamental system of solutions to equation (4) with the following asymptotic representations:

$$y_j(x, \lambda) = e^{\omega_j \lambda x}\Big(\tau_0(x) + \frac{\tau_1(x)}{\lambda} + \ldots\Big), \quad j = 1, \ldots, n, \tag{7}$$

as $\lambda \to \infty, \lambda \in \Gamma_s$. One can define the functions τ_k in the above expansions sequentially via the functions p_{jk} in (4) and their derivatives, although the information on how to define them is not specified by Birkhoff.

Next, Birkhoff considers an important special case where all differential expressions $l_k(y)$ in (5) are equal to zero for $k = 1, \ldots, n-1$ and

$l_0(y) = q_0(x)y$, $q_0(x) \equiv 1$. This case corresponds to the eigenvalue problem for the following general ordinary differential operator of order n:

$$y^{(n)} + q_{n-1}(x)y^{(n-1)} + \cdots + q_1(x)y' + q_0(x)y = -\lambda^n y,$$

to which Birkhoff adds boundary conditions of the form

$$U_j(y) = \sum_{k=0}^{n-1} a_{kj}y^{(k)}(0) + b_{kj}y^{(k)}(1) = 0, \quad j = 1, 2, \ldots, n, \tag{8}$$

where a_{kj} and b_{kj} are arbitrary numerical coefficients. For such operators (boundary value problems) Birkhoff defines the notion of regularity, the definition involves the coefficients a_{kj}, b_{kj} and the function q_{n-1}. For the regular operators, he obtains asymptotic formulas for the eigenvalues and proves that the inverse to a regular operator is an integral one, whose kernel $G(x, \xi, \lambda)$ (called the Green's function) admits the following estimate:

$$|G(x, \xi, \lambda)| \leqslant \text{const}\, |\lambda|^{1-n}$$

outside disks of fixed radius centered at the eigenvalues. He proves that any n times differentiable function f subject to boundary conditions is representable by a uniformly converging series in the eigenfunctions and associated functions of the operator.

These works are definitely a huge step forward in comparison with the results of Horn on the operator $l(y) = -(p(x)y')' + q(x)y$ with the Sturm boundary conditions $y'(0) + hy(0) = 0$, $y'(1) + Hy(1) = 0$, under the additional assumption that the functions p and q are real-valued, h and H are real and, hence, the operator is self-adjoint.

Based on the outlined prehistory, we are now in a position to describe the contribution of Ya.D. Tamarkin to the development of the asymptotic theory. Tamarkin [3] considered the following systems of ordinary differential equations of the first order:[1]

$$\mathbf{y}' + \mathbf{Q}(x, \lambda)\mathbf{y} = \lambda\mathbf{A}(x)\mathbf{y}, \quad x \in [a, b], \tag{9}$$

where $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, $\mathbf{Q}$ and $\mathbf{A}$ are $n \times n$ matrices with elements sufficiently smooth in x, the matrix $\mathbf{Q}$, as a function of λ, is analytic at infinity with a regular point, the matrix $\mathbf{A}$ is reducible to the following diagonal form:

$$\mathbf{A}(x) = \operatorname{diag}(a_1(x), a_2(x), \ldots, a_n(x)),\, a_k(x) \neq a_j(x) \text{ for } k \neq j. \tag{10}$$

Under the above condition on $\mathbf{A}$, it is not possible to obtain significant results. However, for the important case where the diagonal of the matrix $\mathbf{A}$ contains collinear functions $a_j(x) = a_j\rho(x)$, $\rho(x) > 0$, and the (complex) numbers a_j are pairwise different, Tamarkin develops a sufficiently complete theory. In this case, the lines $\Re(a_j\lambda) = \Re(a_k\lambda)$, $k, j = 1, \ldots, n$, $k \neq j$, split the complex

[1] Schlesinger [10] earlier considered systems of the first order with a large parameter, but his results did not have a sufficiently complete form.

λ-plane into several (at most $n^2 - n$) sectors Γ_l; and he proves that in each sector, there exists a fundamental matrix of solutions of the following form:

$$\mathbf{Y}(x, \lambda) = \{y_{jk}(x, \lambda)\}_{j,k=1}^{n} = e^{a_k \lambda x} \tau_k(x) \left(\delta_{kj} + r_{jk}(x, \lambda)\right), \tag{11}$$

where δ_{kj} is the Kronecker symbol, $|r_{jk}(x, \lambda)| = O\left(\lambda^{-1}\right)$ uniformly for $x \in [a, b]$ and $\lambda \to \infty$ in the selected sector Γ_l. Tamarkin like Horn and Birkhoff shows that the solutions with the indicated asymptotics are genuine and not only formal. Next, in the case under consideration, he introduces the notion of regularity for the boundary value problem generated by equation (9) with the following boundary conditions:

$$\mathbf{U}_0 \mathbf{y}(0) + \mathbf{U}_1 \mathbf{y}(1) = 0, \tag{12}$$

where $\mathbf{U}_0$, $\mathbf{U}_1$ are numerical $n \times n$ matrices. For the regular problems, he proves that the spectrum is discrete and obtains the following estimate for the Green's function of the inverse operator:

$$|G(x, \xi, \lambda)| \leqslant \text{const}$$

outside disks of fixed radius centered at the eigenvalues; he proves a theorem on expansion of a smooth vector function, subject to the boundary conditions, into a uniformly converging series in eigenfunctions and associated functions.

Next, Tamarkin starts the study of boundary value problems generated by equations (4), (5) and the boundary conditions (8). He defines the regularity of such boundary value problems (Birkhoff gave the definition of regularity only for a linear operator) and proves that an arbitrary smooth function, subject to boundary conditions, decomposes in a series in terms of the root functions of the problem. However, he does not observe that the decomposition is not unique in this case. Tamarkin obtains these results by reducing the problem (5; 8) to the system (9; 12). Thus, the results obtained for systems are more general than those for boundary value problems polynomially depending on the spectral parameter. However, the problems with a polynomially dependent spectral parameter were rapidly developed later and contributed to the frequent citation of Tamarkin's book [3]. More details are provided below.

Consider the following general equation:

$$A_0 \frac{d^n u}{dt^n} + A_1 \frac{d^{n-1}}{dt^{n-1}} + \dots A_{n-1} \frac{du}{dt} + A_n u = 0, \quad u = u(t), \tag{13}$$

where $t \geqslant 0$ and A_j are operators in a Hilbert space H. The equation is given in an abstract form, however, we mean that A_j are differential operators on $\Omega \subset \mathbb{R}^n$ defined in the space $H = L_2(\Omega)$. If (13) is a hyperbolic equation in the variables (t, x), $x = (x_1, \dots, x_n) \in \Omega$, then it is natural to formulate the Cauchy problem with the following n initial conditions:

$$u(0) = f_0, \ \dots, \ u^{(n-1)}(0) = f_{n-1}. \tag{14}$$

Solving equation (13) by the Fourier method (by substitution $u(x, t) = e^{\lambda t} v(x)$), we obtain the following spectral problem with nonlinear spectral parameter (polynomial operator pencil):

$$(\lambda^n A_0 + \lambda^{n-1} A_1 + \cdots + \lambda A_{n-1} + A_n) v(x) = 0. \tag{15}$$

The eigenfunctions v_k corresponding to the eigenvalues λ_k of this pencil generate the elementary solutions $u_k(x,t) = e^{\lambda_k t} v_k(x)$ of equation (13). If we represent a solution to problem (13; 14) as the series $u(x,t) = \sum_k c_k u_k(x,t)$, then the initial conditions are equivalent to the following equality:

$$\sum_k c_k \mathbf{v}_k(x) = (f_0, \ldots, f_{n-1}), \tag{16}$$

where c_k are numerical coefficients and $\mathbf{v}_k = (v_k, \lambda_k v_k, \ldots, \lambda_k^{n-1} v_k)$ are vector-functions in the space $H^n = H \oplus \ldots \oplus H$, they are called Keldysh's derived chains (hereinafter, for simplicity, we assume that the pencil (15) has no associated functions). It was M.V. Keldysh [11, 12] who introduced the concept of derived chains and posed the following problem of multiple completeness: *Under what conditions does the pencil (15) have in the space H^n, consisting of n copies of H, a complete system of derived chains $\{\mathbf{v}_k\}_k$?* Clearly, the completeness property guarantees that the initial conditions can be satisfied only with certain arbitrary accuracy by considering finite sums of elementary solutions. To construct exact solutions, it is necessary to prove expansion theorems or theorems on the basis property for the derived chains.

If (13) is a hyperbolic equation, then the spectrum of the pencil (15) lies in a strip containing the imaginary axis; all derived chains are allowable in expansion (16), since all elementary solutions $u_k(x,t) = e^{\lambda_k t} v_k(x)$ in this case grow not faster than a finite order exponential as $t \to \infty$. If (13) is an elliptic equation (we have $n = 2m$ in this case), then it is necessary to take elementary solutions with $Re\,\lambda_k < 0$ (damping solutions) and certain solutions with $Re\,\lambda_k = 0$ (the selection of such solutions is based on the radiation principles for specific physical problems; the concept of a sign characteristic for a real eigenvalue is introduced for abstract problems).

Therefore, we arrive at the so-called half range problem in the spectral theory of operator pencils. In particular, we have the following important problem: *Is the system of half-length derived chains $\mathbf{v}_k = (v_k, \ldots, \lambda_k^{m-1} v_k)$, constructed from only the selected half of the eigenfunctions, complete in the space H^m, $m = n/2$?*

This problem is closely related to the factorization of operator pencils, namely, to the problem of possible decomposition

$$A(\lambda) = A_-(\lambda) A_+(\lambda),$$

where the degree of the pencil $A_\pm(\lambda)$ is $m = n/2$, and the spectrum of $A_+(\lambda)$ contains all eigenvalues of the pencil $A(\lambda)$ in the right half-plane and certain eigenvalues on the imaginary axis. Thus, a series of sophisticated non-ordinary problems arise, these problems now belong to the general theory of non-selfadjoint operators. Even in the case of a finite-dimensional space H (i.e., for matrix coefficients of operator pencils), the solutions to these problems are complicated.[2]

The first significant paper, in which the foundations of the analytical direction in the theory of non-selfadjoint operators were built, was written by M.V. Keldysh [11] in 1951. Detailed proofs of the corresponding results were published only 20 years later in 1971; see [12]. In particular, Keldysh proves that, under a compact perturbation of a self-adjoint operator, the system of eigenvectors and associated vectors preserves the completeness property. More precisely, the system of eigenvectors and associated vectors of $A=(I+S)T$ is complete if S is a compact operator, I is the identity operator, $T=T^*$ is a finite order operator, that is, $\sum \lambda_k^{-p} < \infty$ for certain $p<\infty$, where λ_k are the eigenvalues of T, and T is assumed to be an unbounded self-adjoint operator. Observe that in the works of Keldysh and other mathematicians up to the 80s, the results were presented for more specific operators. In [11, 12], Keldysh also considers the following operator pencils:

$$A(\lambda) = A_0 + \lambda A_1 + \ldots, + \lambda^n A_n,$$

where

$A_0 = I + S_0$, $A_j = S_j T^j$, $A_n = (I+S_n)T^n$, S_j are compact operators, $j=1,\ldots,n$, T is a finite order operator; now such operator pencils are called Keldysh's pencils. Keldysh shows that the system of eigenvectors and associated vectors of such pencils is n-fold complete in the space H^n, that is, the derived chains mentioned above form a complete system in the space H^n. The proof is based on a very complex analytic machinery, the key result is the following statement: *The resolvent $A^{-1}(\lambda)$ of the Keldysh pencil under consideration is a meromorphic operator function of finite growth order p.* This is a deep result, requiring significant preliminary work. Namely, we have to define the notion of a regularized determinant for an operator of finite order p, develop the theory of singular numbers of compact operators (or unbounded operators with discrete spectrum), show that this determinant is an entire function of order $\leqslant p$, and obtain estimates for the resolvent of a finite-order operator for the case where this operator has no eigenvalues.

Clearly, Keldysh did not work from scratch. Papers on characteristic determinants and perturbation determinants for nuclear operators and finite order operators were published earlier. But there were no effective estimates. T. Carleman [13] used the estimates for the singular numbers of operators obtained by H. Weyl to study a Volterra Hilbert–Schmidt operator, i.e., an operator of order 2 with no eigenvalues. But there were no estimates for the resolvents as meromorphic functions, moreover, for the operator functions nonlinearly depending on the spectral parameter.

[2] The factorization problem for polynomial operator pencils with matrix coefficients is the subject of articles and books by many mathematicians, we mention works of I. Gohberg, R. Kaashoek, P. Lancaster, and L. Rodman.

Keldysh [11, 12] refers only to Birkhoff's articles [3, 4], Tamarkin's book [3] and Carleman's article [13]. Problems with a nonlinear spectral parameter were considered by Birkhoff and Tamarkin, but remained a "thing in itself" until the appearance of [11]; their connections with partial differential equations were not clear. Paper [11] had changed the situation dramatically. Topics related to the problems of completeness and basicity of eigenvectors of non-selfadjoint and normal operators attracted the attention of many famous mathematicians, in particular, I.M. Gel'fand, M.G. Krein, I.C. Gohberg, F. Bauder, S. Agmon, V.B. Lidsky, A.S. Markus, V.I. Matsaev, M.S. Agranovich and others. The history of the problem and the corresponding results are presented in the books by Naimark [14], Gohberg and Krein [15], Markus [16], in the reviews by Radzievsky [17] and Agranovich [18]. A detailed account of the history of the topic, including recent results, is available in author's article [19].

Although Birkhoff was the first to introduce the notion of a regular ordinary differential operator and to consider nonlinear (in particular, polynomial) spectral problems, the most important notion of regularity for the polynomial spectral problems was introduced by Tamarkin [3]. Only almost half a century later, the notion of regularity for partial differential operators appeared in the works of Ya.B. Lopatinskii [20] and other mathematicians. The importance of the concept of regularity is explained by the following phenomenon: the estimates obtainable for their resolvents are similar to those known for the self-adjoint operators. Estimates for the resolvent of a pencil of partial differential operators were obtained by S. Agmon [21] and independently, in a more general form, by M.S. Agranovich and M.I. Vishik [22]. A modern exposition of the corresponding theory, not only in L_2, but also in L_p spaces, can be found in Triebel's book [23].

As already mentioned, the concepts of multiple completeness, multiple expansions, and multiple basicity arise in the study of polynomial spectral problems. However, the study of multiple decompositions in the spaces H^n consisting of n copies of the original space H (as considered by Keldysh) is not natural. It was noted in [24] for pencils of ordinary differential operators and in [25] for general operator pencils in a Hilbert space H that, in general, there are no theorems on the basis property for the Keldysh derived chains in H^n. The point is that the linearization of pencils in H^n leads to operators that are not closable in this space. One should note that it is natural to require that the functions from the initial conditions (14) belong to spaces of different smoothness (the initial function f_j must belong to a space of greater smoothness than the function f_{j+1} defining the next derivative). In the abstract case of operator pencils given by (13), it is reasonable to impose natural restrictions on the coefficients, for example, to require that $A_j = B_j T^j$, where B_j are bounded operators and T is a self-adjoint positive operator used to construct the scale of Hilbert spaces $H_\theta = \mathcal{D}(T^\theta)$, $\theta > 0$, $H_0 = H$; here, $\mathcal{D}(T^\theta)$ is the domain of the operator T^θ. It is natural to assume that the smoothness of the initial functions in the collection $\mathbf{f} = (f_0, f_1, \ldots, f_{n-1})$ uniformly decreases as the index increases. Therefore, it is natural to embed the vector-function $\mathbf{f}$ not into H^n, but into the sum

$$H_{\theta+n-1} \oplus H_{\theta+n-2} \oplus \ldots \oplus H_\theta$$

of spaces of variable smoothness for certain $\theta > 0$. The choice of the parameter θ should take into account theorems on traces in the abstract Sobolev spaces $W_p^n(0, \infty; H)$; see [26]. This approach yields results on multiple completeness and basis property, which harmoniously combine the methods of interpolation theory, trace theory, perturbation theory, modern theory of equations in Hilbert and Banach spaces intensively developed after the publication of the fundamental work by Agmon and Nirenberg [27].

The problem of completeness, minimality, and basicity for the Keldysh derived half-length chains (the half range problem) for self-adjoint or dissipative operator pencils is also solved in the sum of spaces of variable smoothness (for $n > 2$, clearly). However, its solution is more difficult and requires the development of new ideas. The results available on this problem are presented in the author's work [24].

The most significant development of Tamarkin's theory of boundary value problems for ordinary differential equations with polynomial dependence on the spectral parameter was obtained in [24]. We have already noted that Tamarkin's theorem on the expansion of a smooth function in a series in terms of the eigenfunctions of a boundary value problem does not ensure the uniqueness of the expansions and does not harmonize with applications. Multiple expansions should be considered. In this case, it is important to select an appropriate space, in which we consider multiple expansions; this question has already been discussed in the abstract setting. Special difficulties arise if the boundary conditions also depend on the spectral parameter: the previously outlined abstract approach does not work in this case. In [24], a general approach to the boundary value problems with a polynomial presence of the spectral parameter was proposed; this method was later transferred in [28, 29] to similar problems with partial derivatives, without significant changes. A construction for linearizers of the boundary value problem (5), (8) was proposed in the general case, where the coefficients not only of the equation, but also of the boundary conditions a_{kj}, b_{kj} depend on λ polynomially.

The most significant result of [24] is as follows. *For any integer $r \geqslant 0$ and for sufficiently smooth coefficients of equation (5) (the smoothness is specified in terms of r), one can construct a linear operator $\mathcal{L}_r$ in a subspace $\mathcal{W}_{r,U}^r$ of $W_2^{r+n-1} \oplus \ldots \oplus W_2^r + \mathbb{C}^{N_r}$ (here, $W_2^k = W_2^k[a, b]$ is the Sobolev space and $N_r \geqslant 0$ is an integer), which linearizes the boundary value problem; in particular, the spectra of the problem and $\mathcal{L}_r$ coincide, and the eigenfunctions and associated functions of the problem coincide with the first coordinates of the eigenfunctions and associated functions of the operator $\mathcal{L}_r$. Constructions* *of the operator $\mathcal{L}_r$ and subspace $\mathcal{W}^r_{r,U}$ are presented explicitly. The number N_r monotonically decreases as r increases, $N_r = 0$ for sufficiently large $r \geqslant r_0$. If the boundary value problem is regular (the definition of Tamarkin's regularity [3] is refined and simplified), then the eigenfunctions of the linearizer $\mathcal{L}_r$ form an unconditional basis (possibly of finite-dimensional subspaces of bounded dimension) in the space $\mathcal{W}^r_{r,U}$. In particular, for large r, the Keldysh derived chains form an unconditional basis.*

Relatively recently, there has been a revival of interest in the study of operators generated by the first order systems studied by Tamarkin [3]. Djakov and Mityagin [30] prove that the eigenfunctions of the operator, generated by a 2×2 Dirac system with L_2-potential and regular boundary conditions, form an unconditional basis (possibly of subspaces of dimension at most 2) in the space $(L_2)^2$. This theorem for L_∞-potentials is deduced from general results of Markus and Matsaev published in the 1980s (see [19]). A weaker condition on the potential requires the development of a more subtle proof technique in [30]. Malamud and Oridoroga [31] establish a theorem on the completeness of the eigenfunction system of the operator generated by system (9) such that $A = \operatorname{diag}(a_1, \dots, a_n)$ with different numbers a_j and by boundary conditions that are called almost regular. Savchuk and Shkalikov [32] obtain a theorem on the unconditional basis property for the eigenfunctions of a regular Dirac operator with L_1-potential; also they obtain new theorems on the asymptotics of the eigenvalues of such operators with potentials from L_p.

Papers on the refinement of asymptotic formulas for the fundamental matrices of solutions of the systems considered by Tamarkin and on asymptotic formulas under less restrictive conditions than in [3] were published since the 50s of the last century. The history of these studies is described in sufficient detail by Savchuk and Shkalikov [33]. In this work, asymptotic formulas for matrices of fundamental solutions are obtained in the most general case, only under the assumption that the coefficients in the system are summable. Also, the remainder terms in the asymptotic expansions are investigated. In [34], the definition of Tamarkin and Birkhoff–Langer regularity from [35] is modified for systems; on the basis of results from [33] and methods from [19, 24], a sketch of the proof for the theorem on the unconditional basis property of the regular operators' eigenfunctions is presented. The class of hyperbolic systems is defined and the concept of regularity and semiregularity of operators generated by such systems is introduced in [36]. For the regular operators generated by the hyperbolic systems, a sketch of the proof for the theorem on the unconditional basis property of eigenfunctions is given; also, it is proved that the semiregular operators from this class generate strongly continuous semigroups in the spaces $(L_2)^n$.

The recent works mentioned above show that Tamarkin's ideas from [3] remain relevant and are further developed even after a century.

Andrey Shkalikov

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Boris Nikolaevich Delone (1890–1980)

Boris Nikolaevichich Delone[1], a student of Dmitry Grave, was a professor at Leningrad State University (1922–1935) and Moscow State University (1935–1958), and a corresponding member of the USSR Academy of Sciences (since 1929). He worked in algebra, number theory, computational geometry, mathematical crystallography, and the history of mathematics.

Boris Delone obtained an explicit estimate of the number of integer solutions to cubic Pell's equations and later of arbitrary third degree Diophantine equations with negative discriminant. Further development of the geometrical approach to solving equations in radicals led him to a geometrical exposition of Galois theory and then to the inverse Galois problem for solvable groups. He developed a complete classification of four-dimensional parallelohedra, investigated regular partitions of n-dimensional space with an arbitrary Fedorov group, proved that there exist 24 types of three-dimensional lattices based on the combinatorial structure of the Voronoy diagram and the arrangement of symmetry elements with respect to it. He was awarded the Fedorov Prize of the USSR Academy of Sciences (1959) for his work in crystallography.

Boris Delone was born in St. Petersburg into the family of Nikolai Delone, a professor of mechanics. Boris Delone's great-grandfather, Pierre Delaunay,[2] was a paramedic in Napoleon's army and was taken prisoner during the War of 1812. After being let out he stayed in Russia, worked as a doctor, was granted nobility, and married a Russian.

From an early age, Boris showed an interest in music and mathematics: he played all of Beethoven's sonatas and composed himself as well as knew the basics of analysis at the age of 12 and began his studies in algebra and number theory by himself. The family was often visited by Georgy Voronoy,

[1] Spelling variant: Delaunay.
[2] A nephew of Marquis de Launay, who was the governor of the Bastille for some time.

whose works later influenced Delone. In 1904, Boris traveled with his father to Heidelberg to attend the International Mathematical Congress and admired talks of the great mathematicians David Hilbert and Hermann Minkowski.

In 1906, the family settled in Kiev. Under the influence of Nikolai Zhukovsky, Boris' father organized Russia's first aeronautical club; among its members was Igor Sikorsky, the future famous airplane and helicopter designer. For two years Boris was building gliders, perfecting their design and flying them; he once fell from a height of 15 meters, fortunately on recently plowed land.

In 1908 Boris joined the Department of Physics and Mathematics at Kiev University, the same year as Otto Schmidt.[3] A year later, Nikolai Chebotarev[4] also enrolled there. They became actively involved in the work of the seminar of Professor Dmitry Grave. This seminar determined Delone's research interest: algebraic number theory.

Being interested in algebraic number theory, Delone wrote his essay *The interrelation between ideal theory and Galois theory* (1912), for which he was awarded the University Grand Gold Medal. Immediately after university he began a series of research studies on the theory of indefinite (Diophantine) equations, especially the cubic Pell equation.

After graduating from university (1913), Boris Delone taught there. From 1922 to 1935 he lived in Petrograd, taught at the university, became a professor in 1923, and was the head of the department of algebra and number theory from 1930 to 1934. He was a member of the Leningrad Physical and Mathematical Society. In 1928 he went to Germany and Italy to give lectures. In 1932 he joined the Institute of Mathematics, and in 1934 he became a member of its scientific council. From 1945 to 1960 he was the head of the department of algebra, and in 1960 he was appointed the head of the department of geometry. Since 1935 he was a lecturer at Moscow State University, where he was also the head of the department of higher geometry (1935–1943). From 1944 he was a member of the scientific council of the Institute of crystallography of the USSR Academy of Sciences.

Delone's research in the theory of cubic irrationalities also includes a brilliant geometrical exposition of the Voronoy algorithm. Delone also worked on the reduction theory of quadratic forms and the theory of sphere packings in space/lattice coverings of space by spheres. He wrote several papers on the history of algebra and geometry (on Euler, Gauss, Fedorov) and a book on Chebyshev and Zolotarev titled *Petersburg School of Number Theory* (1947). His results in the theory of Diophantine equations and the theory of cubic irrationalities as well as his ideas, which were developed in the works of his

[3] Otto Friedrich Julius Schmidt (1891–1956), a Soviet scientist (mathematician, astronomer, geophysicist) and statesman, a member of the USSR Academy of Sciences.

[4] Another student of Dmitry Grave, best known for the Chebotarev density theorem.

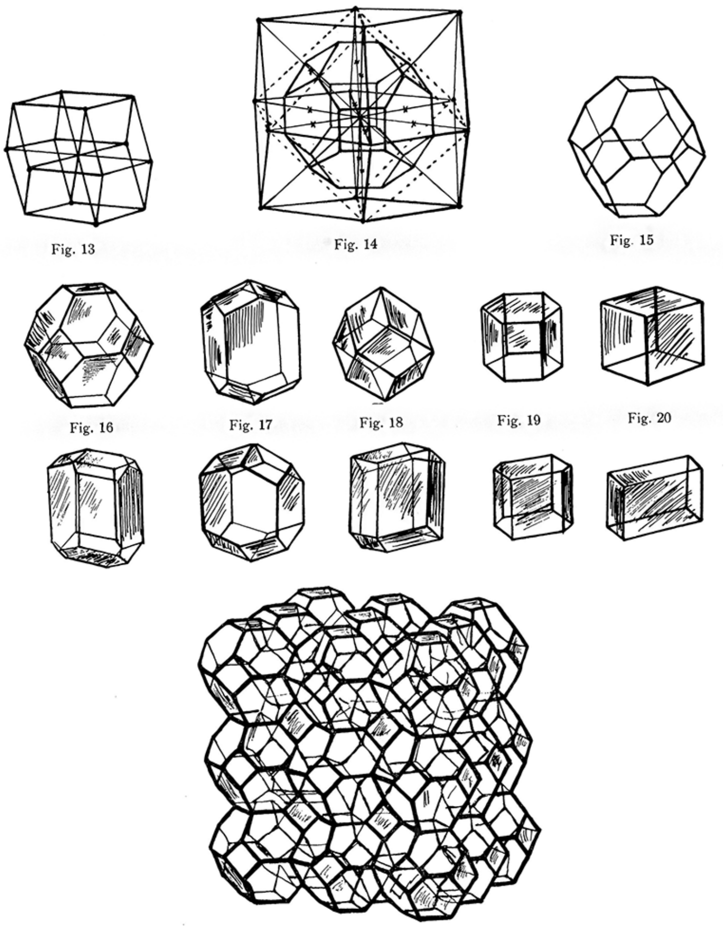


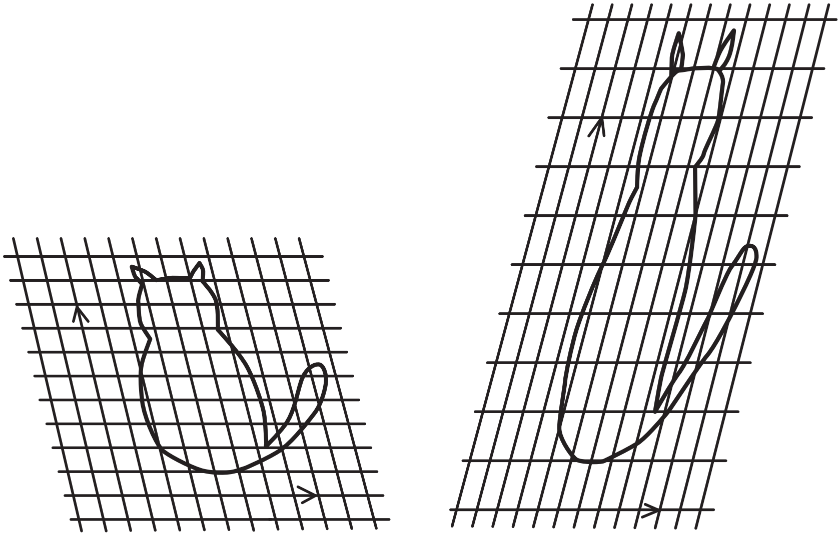
Illustration from Boris Delone's report "Sur la sphère vide" at the International Congress of Mathematicians, 1924.

students, are exposed in his monograph *Theory of Irrationalities of the Third Degree* (1940, together with Dmitry Faddeev).

Alexander Alexandrov, Igor Shafarevich, and Dmitry Faddeev considered themselves his pupils.

Boris Delone was an excellent teacher and worked hard to convey the geometrical essence of what was going on. In a lecture on affine transformations in analytic geometry, Delone drew a cat and showed how its image changed. Students from various courses attended this lecture each year.

In the spring of 1934 Boris organized the first mathematical Olympiad for school children in Leningrad.



The illustration that Delone used in his lecture on affine transformations.

Delone became one of the founders of Soviet mountaineering. His love for the mountains was instilled in Boris by his father, with whom he trekked in the Alps in 1903. He was the Master of Sports of the USSR in mountaineering; he wrote the book *The Peaks of the Western Caucasus* (1938). He has developed principles of classification of mountain ascents (five categories of complexity) and then classified more than 200 routes to the summits of the Caucasus, Central Asia, and the Altai. On the Katun Ridge of the Altai Mountains, the Delone Peak on the Akkem wall of the Belukha Mountain and the ice Delone Mountain Pass are named after him.

Even at a senior age, Delone did not give up his hobby. For example, in 1975, when he was 86, he made the following journey in one day: spent the night at an altitude of 4200 meters at the foot of Khan Tengri in Kyrgyzstan when it was $-20°$ C, flew by helicopter down to Frunze (now Bishkek) in the morning, where the temperature was $-40°$ C, and got to Moscow by plane the same day. Then, on the way to his dacha[5] in Abramtsevo near Moscow, he lost his way in the woods late at night. Nevertheless, by morning Boris reached home safe and sound, with only his rucksack having had to be left behind.

Galina Sinkevich

[5] A summer house.

The Delone peak (on the right).

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Boris Delone and Diophantine equations of degree three

In the early 1910s, Dmitry Alexandrovich Grave organized an algebraic seminar at Kyev University. Grave was a representative of the famous St. Petersburg number theory school, and it is not surprising that the seminar was attended by several exceptionally gifted students who later became outstanding mathematicians. Among them were B. Delone, O. Schmidt, N. Chebotaryov, A. Ostrowski. Here, we shall talk about the research of one of them — Boris Nikolaevich Delone.

For his excellent thesis work, Boris Delone was awarded the University's Grand Gold Medal and continued his stay at the university "to prepare for acquiring a professorship."[1] At that time, the First World War began, and Delone, without interrupting his scientific activities, worked in a military hospital as an X-ray machine operator. In the summer of 1915, the front approached Kyev, and the university was evacuated to Saratov. It resumed its work only in the fall of 1916.

In these extraordinary circumstances, Boris Delone chose one fundamental topic in number theory as the subject of his future research: indefinite equations of the third degree. Progress in number theory has always been determined by the achievements of outstanding mathematicians. In its ancient period, the theory of indefinite equations was associated with the names of Pythagoras and Diophantus. The solution of the simplest Diophantine equations — of the first degree with two unknowns — appeared in the works of the greatest Indian mathematician Aryabhata (6th century).

The solution of the simplest quadratic indefinite equation — Pell's equation — was fully investigated in the works of other Indian mathematicians Brahmagupta (7th century) and Bhāskara II (12th century). When asked why the equation is named after Pell, there is the following answer. In his work, Leonard Euler erroneously attributed the solution of this important equation to the British (17th century) mathematician John Pell. The last step in the theory of quadratic indefinite equations was made by the great Lagrange. Cubic Diophantine equations remained practically unexplored for about another hundred years, up to the works of A. Thue and Delone. Achel Thue showed that a Diophantine equation $f(x, y) = a$, where f is an irreducible

[1] This is similar to the modern postgraduate study.

form in two variables of degree ≥ 3, $a \in \mathbb{Q}$, $a \neq 0$, can have only a finite number of integer solutions.

Knowing about Lagrange's work on the theory of indefinite equations of degree two, Delone started with an equation like

$$x^3 a + y^3 = 1, \tag{1}$$

where a is an integer, but not the cube of an integer.

In the case a is the cube of an integer, the form $x^3 a + y^3$ decomposes into a product of linear and quadratic polynomials in two variables with integer coefficients, and the solution of equation (1) reduces to the solution of Diophantine equations of degree one and two, respectively.

Equation (1) is a cubic analog of the important Diophantine quadratic equation $x^2 a - y^2 = 1$, Pell's equation and it always has a trivial solution (0, 1). How to find all the solutions if they exist? To get the answer, Delone considered the ring

$$\Sigma = \left\{ z\sqrt[3]{a^2} + x\sqrt[3]{a} + y \mid x, y, z \in \mathbb{Z} \right\}.$$

This ring is obtained by the adjustment of the element $\sqrt[3]{a}$ to the ring of integers $\mathbb{Z}$. Let us consider the element $\varepsilon = x\sqrt[3]{a} + y \in \Sigma$ and assume that (x, y) is an integer solution of (1). Then, it is easy to write down the inverse element ε^{-1}:

$$\varepsilon^{-1} = x^2\sqrt[3]{a^2} - xy\sqrt[3]{a} + y^2.$$

Thus, any solution of (1) corresponds to some unit (invertible element) ε of the ring Σ, and this unit is *binomial* in the sense that it contains no summand like $z\sqrt[3]{a^2}$.

Thus, the question of the existence and finding of nontrivial solutions for (1) reduces to the question of the existence and finding of binomial units in the ring Σ. The celebrated Dirichlet theorem on units implies that any unit $\varepsilon \in \Sigma$ (regardless of whether it is binomial or trinomial) is a power of some so-called *fundamental* unit ε_0: $\varepsilon = \varepsilon_0^m$. Moreover, one can suppose that if $0 < \varepsilon < 1$, then the degree of m is positive.

Using witty arguments, Delone showed that if the fundamental unit ε_0 is binomial, then none of its positive powers ε_0^m, $m \neq 1$, is a binomial unit. Then, he investigated units of the form $\varepsilon = \varepsilon_0^m$ under the assumption that the fundamental unit is trinomial: $\varepsilon_0 = z_0\sqrt[3]{a^2} + x_0\sqrt[3]{a} + y_0$. At this stage, which required even more ingenuity, Delone proved that if the fundamental unit ε_0 is trinomial, then any unit $\varepsilon \in \Sigma$ (which, by Dirichlet's theorem, is a power of ε_0^m) is also trinomial. Therefore, if a unit in the ring Σ is binomial then it is the binomial fundamental unit.

Thus, Delone has obtained the final result for equation (1). In addition to the solution (0, 1), the equation can have at most one integer root. To get it, we need to find the fundamental unit ε_0 in the ring Σ. If it is trinomial then there are no non-trivial solutions. If ε_0 is binomial: $\varepsilon_0 = x\sqrt[3]{a} + y$ then it gives the only non-trivial solution (x, y).

In 1897, Voronoy built an efficient algorithm for finding the fundamental unit in the cubic case. At that time, this result made a strong impression on his scientific adviser A.A. Markov. The Voronoy and Delone families were friends and, certainly, Boris knew very well about this breakthrough. Moreover, it was the work of Voronoy that prompted Boris Delone to research, due to which the question of solving the cubic analog of Pell's equation was completely closed.

Encouraged by success, Delone passed to a more general problem: to find the roots of the equation

$$f(x, y) = 1, \tag{2}$$

where $f(x, y) = x^3 + ax^2y + bxy^2 + cy^3$ is a cubic form in two variables with *negative discriminant.* The negativity of the discriminant means that the equation

$$t^3 + at^2 + bt + c = 0 \tag{3}$$

associated with the form $f(x, y)$ has only one real root ρ.

It is natural to assume that ρ is irrational, otherwise the form $f(x, y)$ would be decomposable and finding the root of (2) would be reduced to solving Diophantine equations of degrees one and two.

In this general case, Delone again considered the ring $\Sigma = \mathbb{Z}[\rho]$ of algebraic numbers $x + y\rho + z\rho^2$, and reduced the problem of finding solutions of equation (2) to the problem of finding binomial units of the form $x + y\rho$ in the ring Σ. As a result, he proved the following fundamental theorem.

For any cubic Diophantine equation (2) *with negative discriminant, the number of integer roots does not exceed five; only one specific equation has exactly five solutions; two more equations have four solutions; all other equations of the form* (2) *have no more than three solutions.*

Delone considered several equations of the form (2) and could find all solutions for each. To do it, he invented the so-called "boosting algorithm." Despite the ingenuity of this method and the possibility of applying it to any equation (2), Delone did not give any upper bound for the number of steps that guarantee finding all solutions.

The question of an effective upper bound for a very wide class of Diophantine equations of degree above two was solved half a century later by English mathematician Alan Baker (Fields Medal, 1970). Consider the Diophantine equation $f(x, y) = m$, where $m > 0$ is an integer, $f(x, y)$ is an irreducible binary form with integer coefficients of degree $n \geq 3$. Baker showed that there exists a constant C depending only on m and the coefficients of the form $f(x, y)$

such that for any solution (x_0, y_0) of the Diophantine equation $f(x, y) = m$ the inequality $\max(|x_0|, |y_0|) < C$ holds.[2]

It follows from Baker's work that it is theoretically possible to find all solutions of an ample class of Diophantine equations, including the cubic equations studied by Delone. However, this circumstance does not detract from the importance of Delone's work, in which the nature of solutions of indefinite cubic equations is revealed. The closest student of Boris Delone, Dmitry Konstantinovich Faddeev, described this work in the following words:

> In terms of the specificity of analysis, simplicity and clarity, the series of works by B.N. Delone devoted to indefinite equations is exceptional in mathematics of the 20th century with its often cumbersome apparatus and abstract constructions. In its style, this cycle is close to the best examples of the classical works of Gauss and Chebyshev.

Nikolai Dolbilin

[2] More precisely, Alan Baker showed the following. Let f be a homogeneous polynomial with integer coefficients f in x, y of degree $n > 2$, irreducible over $\mathbb{Q}$, and $m \neq 0$ be an integer. Then, any solution (x_0, y_0) of the Diophantine equation $f(x, y) = m$ satisfies, for any $\kappa > n + 1$, the inequality

$$\max(|x_0|, |y_0|) < Ce^{\log(m)^\kappa},$$

where C depends only on f and κ.

Abram Samoilovitch Besicovitch (1891–1970)

Besicovitch's scientific legacy consists of more than 130 papers on the theory of quasiperiodic functions, topology of the plane, measure theory, Hausdorff measure, etc. His joint work with Harald Bohr resulted in the monograph *Almost Periodic Functions* (authored by Besicovitch alone), which won the Adams Prize in 1930, and the quasiperiodic functions he introduced were called Besicovitch functions.

Abram Besicovitch's development of Hausdorff's results in the dimensionality of sets had a great resonance in modern mathematics. As Benoit Mandelbrot put it, if Hausdorff can be called the father of non-standard dimensionality, then Besicovitch undoubtedly earned the title of its mother. Besicovitch also solved the famous "Kakeya needle problem" (what is a minimum area of a region D in the plane, in which a needle of unit length can be turned through 360°) posed by the Japanese mathematician Soichi Kakeya. Besicovitch showed that this figure could have an arbitrarily small area.

Abram Besicovitch was born in Berdyansk in the Taurian governance of the Russian Empire to a large family of a Karaite goldsmith. After graduating from Berdyansk Gymnasium (1908), he studied at the Mathematics Department of St. Petersburg University.

There were a lot of advanced and enterprising pupils in the course of 1908–1912, e.g., Vladimir Smirnov, Nikolai Rose, Abram Besicovitch, Jacob Tamarkin, Alexander Friedmann, Jacob (James) Shohat, and others. They were active in the scientific circle of the Department of Pure Mathematics, headed by Andrei Markov and Yakov (James) Uspensky, went to the "home seminar" of the Austrian physicist Paul Ehrenfest, who was working in Russia at that time. At this seminar young students learned about the latest achievements in physics. Immediately after university, and possibly even during his senior year, Abram Besicovitch took part in another mathematical circle, organized by Alexander Friedmann, Jacob Tamarkin, and Alexander Gavrilov. Members of that "scientific circle without a leader" worked according to their own program, studying not only classical works but also the then-new areas of mathematical analysis, which were outside the interests of the older generation of St. Petersburg mathematicians.

After graduation, on the petition of academicians Andrei Markov and Vladimir Steklov, Besicovitch was retained at the university to prepare for professorship (1912). He passed his Master's examinations in 1915, but he was never awarded a Master's degree. The same year, his first research article *New Proof of the Theorem on the Limit of the Probability in the General Case of Independent Trials* appeared in the *Proceedings of the Academy of Sciences.* In 1916, Abram Besicovitch converted to Orthodoxy in order to get married. The defense of his thesis (reviewers were Grigory Fichtenholz and Jacob Tamarkin), scheduled for December 31, 1917, did not take place, most likely due to the revolutionary upheaval in Russia.

Young Abram Besicovitch.

In 1917, Abram Besicovitch, then a privatdozent of Petersburg University, together with a group of young mathematicians, was sent to Perm University, where he worked as an extraordinary professor in the Department of Mathematics of the Physics and Mathematics Faculty. Young professors from Petersburg brought the spirit of scientific inquiry of their alma mater to the Permian land: in 1918, they had already organized the Permian Society of Physics and Mathematics and published *The Journal of Physics and Mathematics Society at the Perm State University*, where they printed their papers.

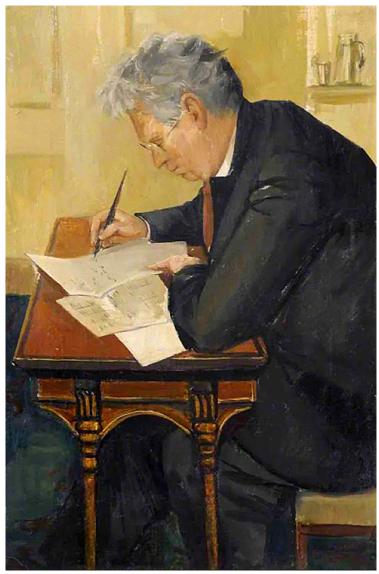
During his time in Perm, Abram Besicovitch was much engaged in public education in the Perm region, being a representative of the university on the city committee of public education. In June 1919, he was appointed rector, and in October of the same year, dean of the Physics and Mathematics Faculty; he also organized a *rabfac.*[1] He performed the rector's duties for less than half a year, albeit in the very complicated circumstances of the civil war. When Kolchak's army retreated, and the Red Army occupied the city, the university was subjected to some ravaging. However, the young 29-year-old rector skillfully and effectively organized the rescue of the university's books and other scientific valuables.

[1] Literally *workers' faculty*: an educational establishment set up to prepare peasants and workers for higher education.

"The only person who thought sensibly and saved what was left was B., apparently a disciple of Markov not only in the field of mathematics but also in the field of decisive, exact, and definite actions," wrote Alexander Friedmann to Academician Steklov (Perm, August 1919).

He was a dean for only a year, as he was sent on a scientific trip abroad in 1920 by the Perm University Council. He managed to reach Petrograd only, where he was admitted to the position of a privatdozent at the university and then of professor at the Pedagogical Institute. The consequences of the civil war, devastation, and half-starved existence did not favor science. Yet it was during this period that Abram Besicovitch fruitfully worked on the problems of differentiability of continuous functions, supplementing the results of Weierstrass and Denjoy, and the problem of a relationship between the maximum and minimum of the absolute value of an entire function of order less than 1, proving one of the important theorems of John Edensor Littlewood.

With the help of Paul Ehrenfest, who sent his works to Harald Bohr in Denmark, Johannes van der Corput in the Netherlands, and J. E. Littlewood in England, Besicovitch was awarded a Rockefeller Foundation scholarship for scientific work abroad in November 1924. The Soviet authorities did not permit him to leave the country, so he crossed the border illegally (Finnish or Latvian, according to different sources) and went to Copenhagen. There, he carried out research in quasiperiodic functions under Bohr for about a year and then moved to Oxford to G.H. Hardy, who arranged for him to give lectures at the University of Liverpool. From 1927, Abram Besicovitch lived and worked in Cambridge, holding the position of University Lecturer, then became a member of Trinity College, and from 1950, he chaired the Department of Mathematics. He died in Cambridge on 2 November 1970.



Besicovitch's portrait from Trinity College gallery.

For his distinguished works in almost periodic functions, measure theory, integration theory, and many other fields of function theory, Abram Besicovitch was elected a Fellow of the Royal Society[2] in 1934, in 1950, awarded the De Morgan Medal, the highest award of the London Mathematical Society, and the Royal Society Sylvester Medal in 1952.

[2] The Royal Society of London for Improving Natural Knowledge.

Calling himself an expert in the "pathology of mathematics," Besicovitch ran a weekly column on "moot points" in the Cambridge University Gazette for many years, to the delight and benefit of students, carefully reading and annotating their solutions.

Besicovitch was a talented mathematician and an equally talented teacher, charming and witty. Russian "Bessi," as his students affectionately called him (and the language of his lectures was called "Bessic English") was forgiven for his bad English, with wrong phrases and no articles, for just one joke: "Gentlemen, 50 million Englishmen speak English like you, but 500 million Russians speak English like me." According to the recollections of his students, many of whom became famous scientists, e.g., Hermann Bondi, Joseph Gillis, Oliver Aberth, and others, there were legends about his lectures in Cambridge, and paradoxical problems he gave to students were passed on by word of mouth. For example: "In an enclosed circus there are a hungry lion and a Christian, both have the same maximum speed. What tactics should the Christian use to avoid being caught by the lion? And how should the lion move to get his breakfast?"[3] Besicovitch was a master of complex constructions that could reveal paradoxical truths. He did not strive for abstractions and generalizations; he was a problem solver rather than a system builder, but his contribution to mathematics is worthy of respect and deserves to be honored in the scientific world.

Natalia Lokot

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[3] The problem itself was invented by the German-British mathematician Richard Rado in the 1930s.

Besicovitch projection theorem and Besicovitch sets

Besicovitch founded geometric measure theory. But before I go to that I shall discuss one of his earliest results from 1919. This is on Riemann integration. It has had and continues to have a huge impact on modern harmonic analysis and other topics. One might wonder how something on Riemann integration a hundred years ago, when Lebesgue integration had already surpassed Riemann integration, could have such an impact today. In fact it does not, but the method Besicovitch introduced does.

The question was whether for any Riemann integrable function f in the plane one could find a rotation ρ so that the function $y \mapsto f(\rho(x, y))$ would be Riemann integrable for all x. Besicovitch showed that one cannot always do this. He constructed a compact set B in the plane of Lebesgue measure zero that contains a line segment of unit length in every direction. Then the characteristic function of the subset of B consisting of the points with at least one rational coordinate serves as a counter-example. The sets like B nowadays are called Besicovitch or Kakeya sets.

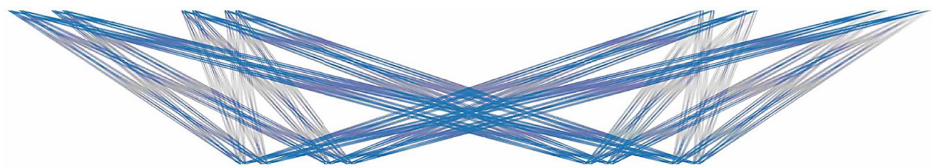
Besicovitch's paper was published in the Perm journal and probably did not draw much attention. In 1917 Kakeya asked the following question, which Besicovitch obviously was unaware of: What is the smallest area of a plane domain where one can continuously turn around a unit line segment? When Besicovitch later heard about this question he republished his construction in 1928 in *Mathematische Zeitschrift*. His method tells us that one can do this in a domain of an arbitrarily small area.

The construction is technically complicated but uses only elementary geometry. Later on, it has been modified and simplified by many people. Perhaps the most elegant way to do this is by Besicovitch himself from the 1960s. Before going into this I explain how he founded geometric measure theory in the 1920s. At the beginning of the last century, Lebesgue developed his measure theory. One aspect of this was to introduce an outer measure that gives a non-negative value to any subset of the Euclidean n-space $\mathbb{R}^n$ and when applied to nice sets, it gives the volume, agreeing with any other reasonable definition.

In 1914 Carathéodory constructed for every integer $k = 1, \ldots, n-1$, an outer measure $\mathcal{H}^k$ which gives the k-dimensional area for smooth k-dimensional surfaces and other k-dimensional objects which have the k-dimensional area defined in some other natural way. Today, this measure is called the Hausdorff measure since in 1919, Hausdorff showed that the construction can be given for non-integral values of k and applied to Cantor sets.

For $k=1$ we have the length measure $\mathcal{H}^1$ and we can ask what is there to say about the structure of sets with positive and finite length. Obvious examples of such sets are rectifiable curves and their subsets of positive length. But there also are many fractal examples. For example, let C_1 be the standard Cantor set in [0, 1] of Hausdorff dimension 1/2 obtained by first deleting from [0, 1] the middle interval [1/4, 3/4], then deleting from the remaining intervals of length 1/4 the middle intervals of length 1/8, and so on. Then $C = C_1 \times C_1$ has a positive and finite length and it meets every rectifiable curve in zero length.

This is rather easy to show: any rectifiable curve has a tangent at almost all of its points but C does not have any tangents at all. Hence they can only intersect in a set of length zero. So what in general could one say about sets with positive and finite length when such completely different examples exist? Besicovitch showed that any set with finite length can be decomposed, uniquely up to length zero, as a union of a *regular* set, whose properties are similar to those of rectifiable curves, and of an *irregular* set, whose geometric properties are completely opposite and resemble those of the above Cantor set C.



An approximation to the Besicovitch type set that contains needles in all the directions between (1, 1) and (−1, 1).

Besicovitch proved that regular, and hence also irregular, sets can be characterized by many geometric properties such as covering with countably many rectifiable curves up to sets of measure zero, having almost everywhere tangents, defined in a measure-theoretic way, and with analogs of Lebesgue density theorem.

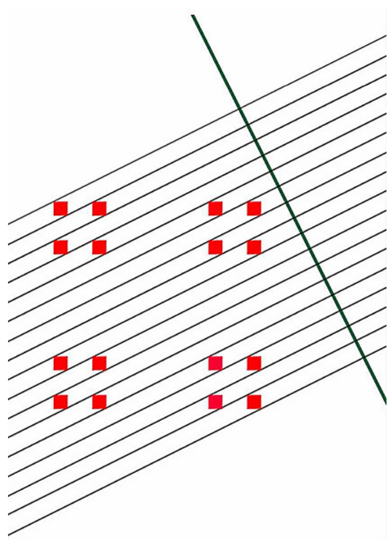
Another especially interesting characterization was in terms of orthogonal projections. Besicovitch proved that a set with finite length is irregular if and only if its projection on almost every line through the origin has length zero. This is the Besicovitch projection theorem:

Theorem. *Let* $E \subset \mathbb{R}^2$ *be* $\mathcal{H}^1$ *measurable with* $\mathcal{H}^1(E) < \infty$. *Then* $\mathcal{H}^1(\Gamma \cap E) = 0$ *for every rectifiable curve* Γ *if and only* $\mathcal{H}^1(P_e(E)) = 0$ *for almost all unit vectors* $e \in S^1$, *where* P_e *is the orthogonal projection onto the line* $\{te : t \in \mathbb{R}\}$.

The essential part is the 'only if' part, the other direction is easy. So one needs to show that irregular sets project to zero length in almost all directions. Besicovitch proved this in 1939 with an ingenious argument. Except for technical modifications, it is still the only known proof.

Let us go back to Besicovitch sets. In 1964, 45 years after his first construction and 25 years after the projection theorem, Besicovitch found a beautiful connection between these two. Using the duality between points and lines in the plane and his projection theorem he proved even more: there exist Borel sets of measure zero which contain a full line in every direction. He studied more generally regular and irregular line sets and their relations to corresponding point sets using the polar correspondence between points and lines. It is slightly simpler to explain the idea for the construction of Besicovitch sets with the following parametrization of lines. For $(a, b) \in \mathbb{R}^2$ denote by $l(a, b)$ the line $y = ax + b$.

For $A \subset \mathbb{R}^2$ the union $l(A) := \cup_{(a,b)\in A} l(a, b)$ contains a line with every slope $a \in [0, 1]$ if the projection $\pi(A)$ on the first axis contains $[0, 1]$. Taking the union of four rotated copies we get a set that contains a line in every direction. Hence it suffices to find A such that $\pi(A)$ contains an interval and $l(A)$ has zero area. Now $l(A) = \{(x, y) : y = ax + b, (a, b) \in A\}$, whence $l(A) \cap \{x = t\} = \{t\} \times \pi_t(A)$ where $\pi_t(a, b) = ta + b$. The map π_t is essentially a projection and the Besicovitch projection theorem tells us that $\mathcal{H}^1(\pi_t(A)) = 0$ for almost all t if A is irregular with $\mathcal{H}^1(A) < \infty$. Then A would have zero area by Fubini's theorem. So all that is left to do is to find an irregular set with finite length with one projection being an interval. For example, the Cantor set C above serves for this.



The set C (square of the Cantor set) and its projection onto the line $y = -2x$.

It is amazing how much Besicovich's work described above has influenced and continues to influence several areas of completely different character. Let us first look at rectifiable, i.e. regular, sets. Federer generalized in the 1940s most of Besicovitch's theory to higher dimensions and introduced the terminology rectifiable and purely unrectifiable instead of regular and irregular, commonly used today. In particular, he proved the projection theorem relying heavily on Besicovitch's planar proof.

Later, rectifiability and the Besicovitch–Federer projection theorem played a fundamental role in the geometric calculus of variations. In 1960, Federer and Fleming developed the geometric theory of currents, which provides a very general and convenient setting for the Plateau problem: to find and describe k-dimensional surfaces with minimal area and a given boundary surface. The minimizing currents exist by easy weak compactness arguments, but they first exist only as very general objects (distributions). A basic theorem of Federer and Fleming was the compactness theorem for rectifiable currents giving a rectifiable structure for the minimal currents. The proof of this was based on the Besicovitch–Federer projection theorem.

The role of Besicovitch sets in modern harmonic analysis is even more amazing. The methods related to them are usually called the Kakeya methods. In 1971 Fefferman used them to solve the multiplier problem for the ball. For a ball B in $\mathbb{R}^n$, $n \geq 2$, define the Fourier multiplier operator T_B by $\widehat{T_B f} = \chi_B \widehat{f}$. By the Plancherel theorem, it is bounded in L^2. Fefferman showed that it is unbounded in L^p for all $1 < p < \infty$ except $p = 2$.

Using again the duality method, Davies proved in 1971 that Besicovitch sets in the plane have the maximal Hausdorff dimension two. It is conjectured (called Kakeya conjecture) that the same is true in higher dimensions, that is, Besicovitch sets in $\mathbb{R}^n$ should have Hausdorff dimension n. There are partial results by many people but it is open for $n \geq 3$. This is interesting because if it is false then many important conjectures in harmonic analysis are false. For example, Stein's restriction conjecture, which asks whether for smooth functions f,

$$\|\widehat{f}\|_{L^p(S^{n-1})} \leq C\|f\|_{L^p(\mathbb{R}^n)} \text{ for } 1 \leq p < 2n/(n+1).$$

If the Kakeya conjecture turns out to be true, it does not prove any of these conjectures. But there is good hope that it might lead to new progress because Bourgain and many others since the 1990s have applied Kakeya methods to prove restriction estimates.

The following list of references only contains some books where the above topics are discussed and many more references can be found.

I would like to thank Dmitriy Stolyarov for inviting me to write this essay and for his help with many comments and providing the pictures.

Pertti Mattila

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Ivan Matveevich Vinogradov (1891–1983)

I.M. Vinogradov contributed many new and important results and methods to number theory. In 1934, for example, he devised a new method for handling exponential sums (the so-called Vinogradov method), which allows for significant improvements in estimates in many problems in analytic number theory. In 1937, he solved Goldbach's weak conjecture (that any odd number greater than 5 can be expressed as the sum of three primes) for sufficiently large odd numbers. He became an honorary member of many foreign academies and was the first person in the USSR to receive the Stalin Prize. He was twice awarded the Order of the Hero of Socialist Labor and received the Lomonosov Gold Medal — the most prestigious award of the Academy of Sciences of the Soviet Union. In the years 1932–1941 and 1944–1983, Vinogradov served as director of the Steklov Institute.

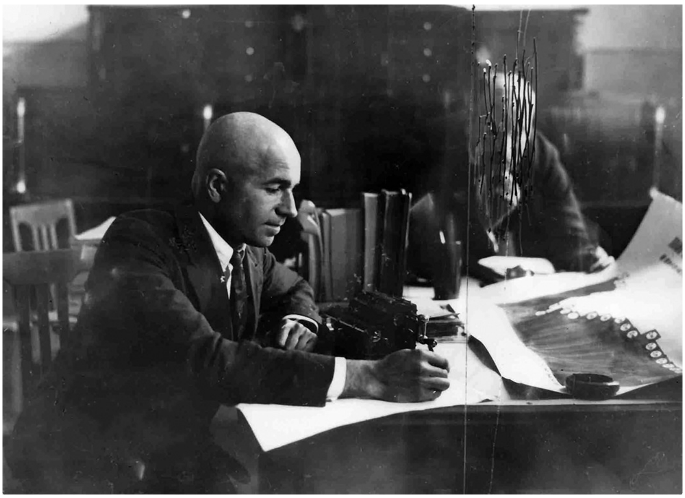
I.M. Vinogradov was born in 1891 in the churchyard of Milolyub, county of Velikie Luki, into a priest's family. After finishing Realschule (secondary school) in his home county, Vinogradov was admitted to the Department of Mathematics at St. Petersburg University in 1910. There, he wrote his master's thesis on the distribution of quadratic residues and nonresidues, under the supervision of J.V. Uspensky. From 1915, Vinogradov remained at the university to prepare for an academic career.

In 1920 he became a professor at the university in Perm, and later returned to St. Petersburg. Based on the course he taught, Vinogradov wrote *Elements of Number Theory*, a book that has been published multiple times in many languages. This book was a starting point for many famous Soviet mathematicians. In 1927, Vinogradov found a new proof of Waring's problem (that any natural number is a sum of a fixed number of n-th powers) by extending the Hardy–Littlewood circle method for finite sums. In 1929 he became a Full Member of the Academy of Sciences of the Soviet Union.

He was director of the Institute for Demographic Research in Leningrad (1930–1932) and of the Institute of Physics and Mathematics (1932–1934).

After the latter split into two parts in 1934, Vinogradov became director of the mathematics section (the Steklov Mathematical Institute). He moved to Moscow, along with the institute, in 1934.

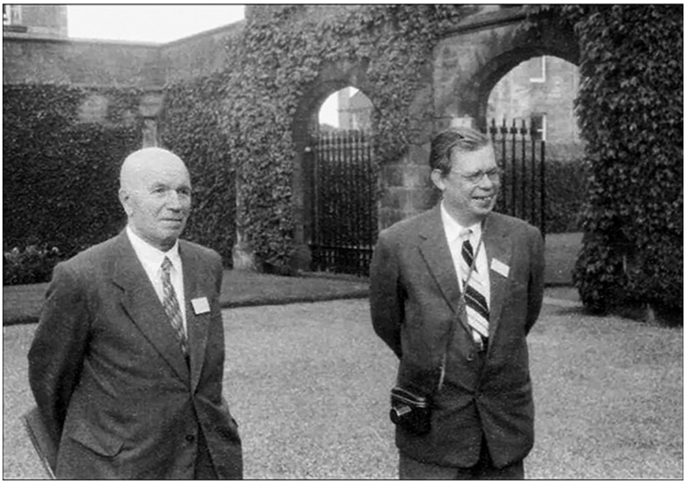
Managing the institute was Vinogradov's main focus in life; he put a lot of effort into it, remaining director until 1983. The only interruption in his position was during WW2, in the years 1941–1944, during which he was replaced by Sobolev. In concurrence with the unofficial policy of the Communist Party, starting in the 1950s, Vinogradov refused to hire Jews and anyone against the party line. Vinogradov had never been a member of the Communist Party, however (even though such membership was almost required for someone of his rank).



November 1931. I.M. Vinogradov, director of the Institute for Demographic Research in Leningrad at that time. On the right: presumably N.I. Bukharin (executed in 1938). The faces of sentenced and persecuted persons would be scratched out in photos in those years.

Vinogradov was a strong-willed director, but it was possible to sway him from time to time. According to I.R. Shafarevich, Vinogradov would initially refuse any request he received as director, even those he obviously supported. Any accusations or complaints that were written by the institute's staff would not be sent anywhere; instead, Vinogradov stashed them in a briefcase, which he asked to be burnt before he died. In 1955, he signed the famous "Letter of the Three Hundred" in support of Soviet geneticists against T.D. Lysenko's team.

Vinogradov was very strong from his youth. In Realschule, he joined an acrobatics club, where he did strength training. Remembering his childhood, he later said: "My first stadium was the forest, first teachers — the village boys: tanned, agile, tough... I would spend all day in the forest or by the pond, where I built my first raft and set sail when I was five..."



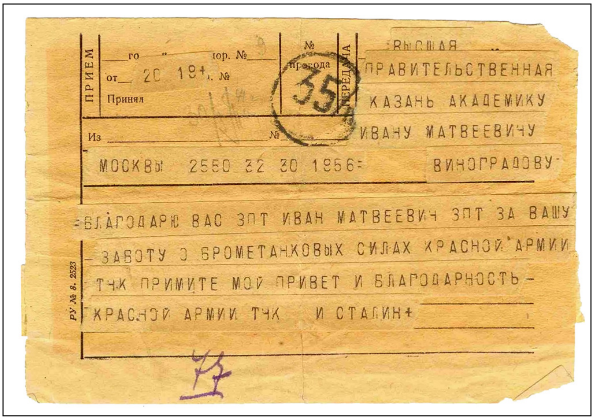
I.M. Vinogradov and N.N. Bogolyubov, August 1958. Edinburg, ICM.

A 90-year-old Vinogradov reminisced about a youthful joke of his — the grand piano story. It took place in England, in 1946, at a reception organized by the London Mathematical Society. Vinogradov (who barely spoke English) was standing alone by the window in a huge room full of scientists discussing mathematical problems. In the middle of it stood a large concert grand piano. Vinogradov climbed under it, lifted it a little, and carried it off a certain distance in the silenced room. Then, applause rang out. After that, some mathematicians wanted to demonstrate their strength as well, but who could compete with Vinogradov? To justify himself in front of his foreign colleagues, Vinogradov pretended to be a "simple Simon" and said that he just wanted to find out the brand of the piano.

Vinogradov approached everything with great passion — if he was to play chess, then it was 100 games; if it was cards, then it was for a couple of days straight; if he were to go on a hike, then it would be a long and tough one.

In an interview for a chess magazine, he said: "I remember when I was a professor at the Leningrad Polytechnic Institute, I would sometimes offer my students to play a game of chess on exam day. And, you know, the pre-exam tension would diffuse during the match, and it would lighten the mood of both the examiner and the examinee."

Vinogradov's museum in Velikie Luki exhibits a telegram addressed to I.M. Vinogradov by J.V. Stalin. In 1941, Vinogradov donated all of his money (85000 rubles) to the National Defense Fund to build tanks. This was common practice, but very few got a personal telegram of gratitude from Stalin. Vinogradov never married, and never had children; he led an austere life. Year-round, he left for his dacha in the village of Abramtsevo (near Sergiyev Posad) on Thursday and stayed there until Monday. There he had a separate office where, when he wasn't doing physical work, Vinogradov would sit at the desk and work with enviable productivity. He died on March 20, 1983.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A copy of Stalin's telegram to Vinogradov from Vinogradov's museum in Velikie Luki, https://vk.com/museumimv?w=wall-174249572_210.

From Vinogradov's opening speech at the International Conference on Analytical Methods of Number Theory and Analysis, 1981:

> To close, I would like to say a couple of words that could be helpful to those who want to dedicate themselves to studying mathematics. You have to try to solve important problems, without regard to their difficulty. Their solutions will go down in the history of science and be useful to many. That is what our predecessors did. You mustn't get carried away with simple and unnecessary problems just because they require little effort. Scientists who do this can easily set their students on the same wrong path. Having chosen a worthy topic, you should create a work plan and never leave it, as long as there is the tiniest hope of success. It is important to know the works of the classics — their ideas can prove to be a deciding step toward your own success.

Nikita Kalinin

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Vinogradov's work on exponential sums

I.M. Vinogradov set records for deep questions in analytic number theory by pioneering methods involving exponential sums. These ideas continue to stimulate research in number theory today. Let us look at four problems that Vinogradov's work on sums impacted.

Quadratic nonresidues. Given a prime p, an integer n, $1 \leq n < p$, is a *quadratic residue* if there is some integer x such that $n \equiv x^2 \pmod{p}$. Otherwise, n is a *quadratic nonresidue*. There are $(p-1)/2$ quadratic residues and $(p-1)/2$ quadratic nonresidues modulo p.

What is the smallest quadratic residue modulo p? It is always 1. What is the smallest quadratic nonresidue, call it n_p, modulo p? This question has connections to deep open questions in analytic number theory.

An elementary argument reveals that for each prime p, n_p must itself be prime. How does n_p behave as $p \to \infty$? It is known that the sequence n_p is unbounded when p grows. In fact, given any number N, there are infinitely many primes p such that $1, 2, 3, \ldots, N$ are all quadratic residues modulo p. This can be proved using quadratic reciprocity, the Chinese Remainder Theorem, and Dirichlet's theorem on the infinitude of primes in arithmetic progressions. Nevertheless, Vinogradov conjectured n_p cannot grow much: for every $\varepsilon > 0$ there is a constant C_ε such that $n_p \leq C_\varepsilon p^\varepsilon$ for all primes p. This remains open.

Fermat, Euler, Lagrange, Legendre, and Gauss all studied quadratic residues. Legendre introduced a symbol (n/p) that takes the value $+1$ if n is a quadratic residue modulo p, -1 if n is a quadratic nonresidue modulo p (and 0 if $p|n$). This is an example of a Dirichlet character χ_q modulo q, and thus Vinogradov's conjecture fits into an important landscape of questions about character sums. For any Dirichlet character χ_q it is interesting to study for $1 \leq H \leq q$ the sums

$$S(\chi_q, H) = \sum_{1 \leq n \leq H} \chi_q(n).$$

We can seethat n_p is the smallest integer $H \geq 1$ such that $S((n/p), H) < H$.

In 1918, Vinogradov proved (concurrently with Pólya) that

$$|S(\chi_q, H)| \leq C q^{1/2} \log q$$

for any non-principal Dirichlet character modulo q. The key idea was to use a Fourier expansion modulo q, thus introducing exponential-type sums to the problem. Moreover, Vinogradov developed a trick to upgrade the consequence for n_p, showing that for any H such that $S((n/p), H) < H$, in fact $n_p \leq H^{1/\sqrt{e}+\varepsilon}$ for any $\varepsilon > 0$. Thus Vinogradov showed $n_p \leq p^{1/(2\sqrt{e})+\varepsilon}$.

In 1942, Linnik showed that Vinogradov's conjecture holds if the Generalized Riemann Hypothesis is true. (And, assuming GRH, there is a deterministic polynomial-time algorithm to find n_p.) Linnik also showed unconditionally that for any $\varepsilon > 0$, the number of primes p with $N^\varepsilon < p \leq N$ with $n_p > p^\varepsilon$ is bounded independently of N. Thus the primes for which Vinogradov's conjecture could fail must be density zero among all primes. Today, the best known bound is $n_p \leq p^{1/(4\sqrt{e})+\varepsilon}$. This combines Vinogradov's trick with a bound for $S((n/p), H)$ when $H \approx p^{1/4+\varepsilon}$, due to Burgess in 1957.

Vinogradov's conjecture remains of great interest because of its connections to character sums, Dirichlet L-functions, and their associated Riemann Hypotheses, and even (as Tao observed in 2015) to sieve theory, which is used to detect bounded gaps between primes, for example.

Waring's problem. In 1770, Waring asserted that for each $k \geq 2$, there exists an $s = s_*(k)$ such that every integer $N \geq 1$ may be expressed as $N = x_1^k + \cdots + x_s^k$ with integers $x_i \geq 0$. Hilbert proved this assertion in 1909. (We know that $s_*(k) \geq 2^k + \lfloor (3/2)^k \rfloor - 2$, but whether this is optimal in general is open.) A different question is currently of central interest: given (sufficiently large) N, *how many* solutions $x_i \geq 1$ to this equation are there?

The circle method, initially developed by Hardy, Ramanujan, and Littlewood in the 1920's, expresses the number $r_{s,k}(N)$ of solutions as the integral

$$\sum_{1 \leq x_1, \ldots, x_s \leq N^{1/k}} \int_0^1 e^{2\pi i \alpha (x_1^k + \cdots + x_s^k - N)} d\alpha$$

and then extracts a main term from the "major arcs" (intervals centered at rational numbers a/q with q sufficiently small), and shows that the remaining "minor arcs" contribute a smaller remainder term. As long as s is sufficiently large relative to k, this method works well, and shows that $r_{s,k}(N)$ is asymptotic to $c_{s,k}(N) N^{s/k-1}$ as $N \to \infty$, where $c_{s,k}(N) > 0$ for $s \geq \max\{5, k+2\}$.

What is the least number of variables s such that we can accurately count $r_{s,k}(N)$ in this way? Initially, Hardy and Littlewood required $s \approx k2^{k-1}$. Conjecturally, it is thought that taking $s \approx k$ will be the limit of these methods. The key is bounding exponential sums of the form $S(\boldsymbol{\alpha}, X) = \sum_{1 \leq x \leq X} e(\alpha_1 x + \cdots + \alpha_k x^k)$. In the 1930s, Vinogradov started a program for bounding $S(\boldsymbol{\alpha}, X)$ via bounding the mean-value

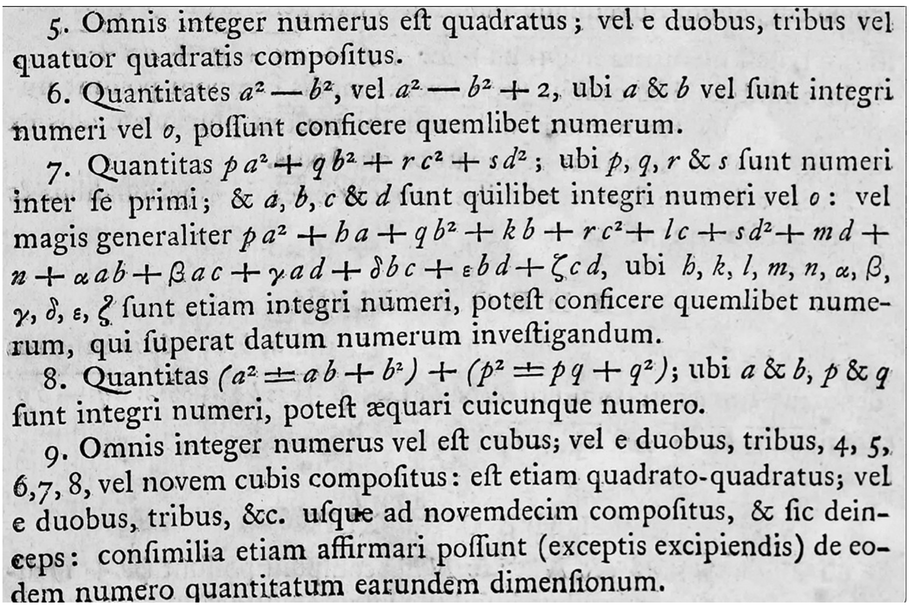
$$J_{s,k}(X) = \int_{(0,1]^k} |S(\boldsymbol{\alpha}, X)|^{2s} d\boldsymbol{\alpha}.$$

Vinogradov outlined the *Main Conjecture*: $J_{s,k}(X) \ll X^{\varepsilon}(X^{s} + X^{2s-\frac{1}{2}k(k+1)})$ for all integers $s, k \geq 1$. Vinogradov made celebrated progress toward this conjecture, and showed how to extract upper bounds for $S(\boldsymbol{\alpha}, X)$ from $J_{s,k}(X)$. In 1935 he set new records in Waring's problem (showing $s \approx 10k^2 \log k$ suffices).

The Main Conjecture in Vinogradov's program for $J_{s,k}(X)$ remained open for 80 years, despite much attention. Wooley ($k = 3$) and Bourgain, Demeter, Guth ($k \geq 4$) finally polished it off with breakthrough papers published in 2016. (Consequently, $s \approx k^2$ suffices in Waring's problem.) Moreover, their methods (efficient congruencing and decoupling) have opened up exciting new questions at the intersection of number theory and harmonic analysis.

Riemann zeta function. In 1859, Riemann initiated the study of the zeta function $\zeta(s) = \sum_{n \geq 1} n^{-s}$ for a complex variable s. His work established that the key to counting the number $\pi(x)$ of primes $p \leq x$ is to understand the zeroes $s = \sigma + it$ of $\zeta(s)$ in the critical strip $0 < \sigma < 1$. In fact, for each $0 < \delta \leq 1/2$, the statement $\zeta(s) \neq 0$ for all $\sigma > 1 - \delta$ is equivalent to $\pi(x) = Li(x) + O(x^{1-\delta+\varepsilon})$ for all $\varepsilon > 0$.

To prove a zero-free region for $\zeta(s)$, Vinogradov studied partial sums of the form $\sum_{N<n<M} n^{-it}$. By writing $n^{it} = e^{it \log n}$ and taking a truncated Taylor expansion of the logarithm, once again the key is to study exponential sums


5. Omnis integer numerus eſt quadratus; vel e duobus, tribus vel quatuor quadratis compoſitus.

6. Quantitates $a^2 - b^2$, vel $a^2 - b^2 + 2$, ubi a & b vel ſunt integri numeri vel o, poſſunt conficere quemlibet numerum.

7. Quantitas $pa^2 + qb^2 + rc^2 + sd^2$; ubi p, q, r & s ſunt numeri inter ſe primi; & a, b, c & d ſunt quilibet integri numeri vel o: vel magis generaliter $pa^2 + ha + qb^2 + kb + rc^2 + lc + sd^2 + md + n + \alpha ab + \beta ac + \gamma ad + \delta bc + \varepsilon bd + \zeta cd$, ubi h, k, l, m, n, α, β, γ, δ, ε, ζ ſunt etiam integri numeri, poteſt conficere quemlibet numerum, qui ſuperat datum numerum inveſtigandum.

8. Quantitas $(a^2 \pm ab + b^2) + (p^2 \pm pq + q^2)$; ubi a & b, p & q ſunt integri numeri, poteſt æquari cuicunque numero.

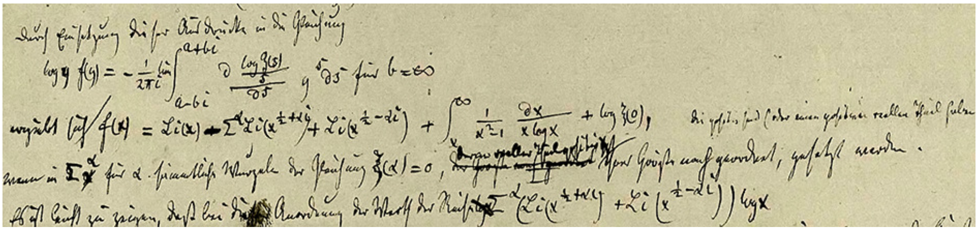
9. Omnis integer numerus vel eſt cubus; vel e duobus, tribus, 4, 5, 6, 7, 8, vel novem cubis compoſitus: eſt etiam quadrato-quadratus; vel e duobus, tribus, &c. uſque ad novemdecim compoſitus, & ſic deinceps: conſimilia etiam affirmari poſſunt (exceptis excipiendis) de eodem numero quantitatum earundem dimenſionum.

Waring's assertions appear in (5) and (9) of Theor. XLVII in his *Meditationes Algebraicae.*

like $S(\boldsymbol{\alpha}, X)$. The resulting Vinogradov–Korobov zero-free region, still the best type known today, shows $\zeta(s) \neq 0$ for $\sigma \geq 1 - C(\log t)^{-2/3}(\log\log t)^{-1/3}$, for all $t \geq 3$.

Ternary Goldbach problem. In the 1740s, correspondence between Goldbach and Euler initiated *Goldbach's problem* (every even $N > 2$ is a sum of two primes) and the *ternary Goldbach problem* (every odd $N > 5$ is a sum of three primes). In 1923 Hardy and Littlewood proved the ternary Goldbach problem for all sufficiently large N, assuming the Generalized Riemann Hypothesis. In 1937, Vinogradov proved this directly and unconditionally, and even counted the number of such representations, via a sophisticated understanding of exponential sums like $S(\boldsymbol{\alpha}, X)$ with x restricted to prime numbers. In 2013, Helfgott resolved the ternary Goldbach problem for all $N > 5$. Goldbach's problem remains tantalizingly open.

Lillian B. Pierce



In Riemann's 1859 manuscript, we see a sum $\sum^{\alpha}$ that corresponds to summing over the zeroes of the zeta function $\zeta(s)$. (Cod. Ms. B. Riemann 3: folio 19r–20r in the Göttingen collection).

Vladimir Alexandrovich Fock (1898–1974)

V.A. Fock was a theoretical physicist, a professor at Leningrad State University, and a member of the Academy of Sciences of the Soviet Union (1939). He worked in quantum mechanics, quantum field theory, diffraction theory, and gravitation. His most famous achievements were the discovery of the dynamic O(4) symmetry of hydrogen atoms, his approach to second quantization (Fock space and the Klein–Gordon equation), approximate calculations in multi-electron atom theory (Hartree–Fock method), and the method of parabolic calculation in diffraction theory. Fock also invented the so-called Fock–Schwinger gauge in Quantum Electrodynamics which was widely used in Quantum Chromodynamics in the 1980s–1990s.

Vladimir Alexandrovich Fock was born on December 22 (10), 1898 in St. Petersburg. His father, A.A. Fock, was an inspector in the Foresters' Corps (today, he would have been called a forest ecologist). His great-grandfather, Nikolai Antonovich Fock, moved from Holstein, Germany, to Russia at the beginning of the 19th century to work as a hydraulic engineer. Fock's other great-grandfather, Nikolai Mikheevich Arkhangelsky, was the founder of the Department of Physics and Mathematics at Kharkov University.

In 1916, V.A. Fock attended Petrograd University[1] but he decided to volunteer for the front lines because he thought that his progressive hearing loss would get in the way of his studies. After five months of courses at the Constantin artillery school, he became an ensign[2] and in September 1917 ended up on the Romanian front, at which point utter chaos ruled. In February 1918, his brigade was disbanded and everyone was advised to get back home on their

[1] Petrograd University and Leningrad State University are one and the same. St. Petersburg was renamed Petrograd in 1914 at the start of World War I because 'St. Petersburg' sounded German. After Lenin's death in 1924, the city was renamed Leningrad.

[2] An ensign is a junior officer, usually the lowest commissioned rank in military services. An ensign is ranked just below a lieutenant.

own. In St. Petersburg, Vladimir Alexandrovich worked as an accountant for some time under the supervision of Mikhail Chekhov, the writer's brother (since that time the book *Textbook on Double-Entry Bookkeeping* was kept in Vladimir Alexandrovich's home library).

Since his family had moved away, Fock lived alone in his apartment; his address was 22, Line 9 on Vasilyevsky Island. He was forced to share his apartment with some perpetually drunk sailors who beat him and stole his belongings, so he sued them, and the sailors were evicted. Fock remembered this incident throughout his whole life as the main argument supporting his loyalty to the Soviet government that had "protected" him.

In early 1919, V.A. Fock followed D.S. Rozhdestvensky's advice to work at the newly-founded Optical Institute and returned to his university studies. Rozhdestvensky even arranged some food rations for him, which, along with the help of his close friends, saved Fock from starving. As a former officer, he was subject to being conscripted into the army, but he managed to evade it due to his advancing deafness. Classes at the university were so small that they were almost individualized instruction, so his deafness caused much fewer problems than before. Vladimir Alexandrovich would talk about his teachers more often than others: D.S. Rozhdestvensky, of course, but also Yu.A. Krutkov, V.R. Bursian, V.S. Frederik, and A.A. Friedmann. In 1922, V.A. Fock graduated from university, but stayed there in "preparation for a professor's position," equivalent to today's postgraduate studies. One of D.S. Rozhestvensky's lectures sparked Vladimir Alexandrovich's interest in quantum mechanics, which became his main subject of thought for many years. Parallel to that, he did some purely applied work as well, mainly on calculations for electric cables. In 1927, he received a Rockefeller Foundation grant upon the recommendation of P. Ehrenfest, which allowed him to work for a year in Göttingen starting in August 1927. Crossing the border did not pose any problems at that time; the only document needed for his future wife, A.V. Lermonotova, to visit him was a note from her housekeeper. In Göttingen, Fock worked primarily with M. Born.

During the next decade, Fock wrote his most outstanding papers on quantum mechanics and field theory. Seeing his friends and colleagues disappearing,[3] Vladimir Alexandrovich immersed himself entirely in science. But on February 5, 1937, he was arrested. His wife called A.N. Krylov, with whom P.L. Kapitsa[4] was staying, and said: "Vladimir Alexandrovich will not dine with you tonight..." The meaning was clear, and P.L. Kapitsa wrote a letter to

[3] This was the period of the Great Purge or the Great Terror (1936–1938), when Stalin attempted to centralize and consolidate power by accusing a wide variety of Soviet citizens, ranging from intelligentsia to peasants, of various political crimes like anti-Soviet agitation or sabotage. These citizens were either executed or sent to gulags (i.e., forced labor camps). Scholars estimate that around 700,000 people lost their lives as a result of the purges.

[4] Pyotr Leonidovich Kapitsa (1894–1984) was a preeminent Soviet physicist, engineer, and Nobel Prize laureate, known for his contributions to low-temperature physics. Among his

Stalin that same day, which fortunately worked. Vladimir Alexandrovich was transported to Moscow and released right there in Yezhov's office.[5] Vladimir Alexandrovich was forever thankful to P.L. Kapitsa for saving him. He wrote many letters to try and help his arrested colleagues and friends himself, but unfortunately, they were unsuccessful.

Despite his involvement in the Academy of Sciences of the Soviet Union (as a corresponding member since 1932 and a Full Member since 1939), his life was very difficult. He did not even have enough money for food and firewood. During the winter it was about 10° in his apartment. He bought the paper in bulk, and later recollected how he had to write some of his main papers on newspaper margins and wrapping paper. In the summer, however, his family rented a house (dacha) in a village near Oranienbaum. In 1939, P.A.M. Dirac visited that village despite it being forbidden to do so since the dacha was

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achievements were the invention of new machines for the liquefaction of gases, and discovering the superfluidity of liquid helium in 1937.

[5] Nikolai Yezhov (1895–1940) was a Soviet secret police official who was the head of the People's Commissariat for Internal Affairs during the peak of the purges, from 1936 to 1938.

officially located in the border zone. News of the imminent war got to Vladimir Alexandrovich while he was on vacation in Kislovodsk, where he wrote letters to his family saying that they should not move anywhere since the war would soon be over. Still, he immediately returned to Leningrad. In September 1941, he and his family, under threat of arrest, were evacuated on an airplane accompanied by five fighter jets first to Moscow, and then to Yelabuga. In Yelabuga, he did work on radiolocation, for which he and his family were called to Moscow in the summer of 1943 and given an apartment near the Kremlin with food rations as compensation. His daughter, Natalya Vladimirovna, said that she had never eaten better than in Moscow in 1943.

In 1944, V.A. Fock was offered a position as the head of the theoretical physics department at Moscow State University. However, having found out that his views contradicted those of the faculty's management and that his signature was forged on certain documents, Vladimir Alexandrovich retired from the position after a few months. In addition to those reasons, Fock also felt hostility towards D.I. Ivanenko, whom he blamed for the arrest and death of M.P. Bronstein.

Since 1945, Vladimir Alexandrovich lived and worked in Moscow and Leningrad. He even managed to get a double residence permit in his passport. He spent the summer and weekends at his datcha in Komarovo.

Beginning in the 1930s, Vladimir Alexandrovich tried to participate in philosophical discussions with radical Marxists, believing it was his duty to protect science from their attacks. Later, some people who were unaware of Fock's contributions to theoretical physics thought of him as a Marxist philosopher and invited him to appropriate conferences. Towards the end of his life, he admitted that unfortunately his health didn't allow him to do anything more substantial than that.

Despite his loyalty to the state, Vladimir Alexandrovich tried to stay away from official matters: he did not sign letters condemning colleagues and did not participate in hearings. He had a famous phrase that he would say quite loudly (he could not speak in quiet voice because of his deafness) in response to a diatribe: "Cowardice does not affect the chances of imprisonment." From 1954 onwards, he was allowed to travel abroad; many remember his phrase: "I am not your serf!" which he said in response to the demand to hand over the money he had earned abroad. However in 1969, his lack of opposition to yet another conference on gravitation in Israel was not forgiven, and (due to A.Z. Petrov's tip-off to the authorities) he was banned from traveling abroad.

In his youth, Vladimir Alexandrovich went to church but, like most people in his circle, he began to view any religiousness very negatively by adulthood.

Vladimir Alexandrovich died on December 27, 1974, in Leningrad and is buried at the Komarovo cemetery.

Vladimir Fock

Nikolai Evgrafovich Kochin (1900–1944)

Nikolai Evgrafovich Kochin was an outstanding scientist in hydro- and continuum mechanics and a Full Member of the Academy of Sciences of the Soviet Union (USSR AS). He is famous for solving important problems in meteorology, studying surfaces of discontinuity in a compressible liquid, and solving the classic problem about the breakdown of discontinuities in an ideal gas. N. Kochin also solved the problem about the circumfluence of a thin wing, developed the theory of the irregular movement of bodies under the surface of an ideal liquid, obtained a new solution for the Cauchy–Poisson problem about waves on water, solved the plane problem of underwater wings, and got formulas for calculating the friction of a ship, taking into account interactions between the ship hull and the water. He was one of the authors of the remarkable, two-volume textbook *Theoretical Hydromechanics.*

Nikolai Evgrafovich Kochin was born on May 6th, 1900, in St. Petersburg. His father, Evgraf Samoylovich, was a clerk in a small textile mill, and his mother, Yelizaveta Nikolaevna (maiden name — Komarova) was a peasant. When Kochin was eight years old, he was admitted directly into the second year of primary school. After graduation, he attended the First Classical St. Petersburg Gymnasium. Nikolai stood out due to his success in almost every subject. After graduating from the gymnasium (during this time it was called the First Gymnasium of Petrograd) in 1918, he enrolled in the mathematical section of the Physics and Mathematics Department at Petrograd University. During that same year, he was drafted into the Red Army. In October of 1919, Kochin fought in battles near Yamburg and miraculously survived. In 1920, he was assigned to join the Petrograd Technical Artillery School of the Red Army, where he served in telephone communication services and simultaneously attended lectures at Petrograd University. In 1922, he returned to his university studies full-time and graduated in 1923.

N. Kochin's hydrodynamics teacher was A.A. Friedmann (author of the famous cosmological solution to Einstein's equation describing three scenarios of the evolution of the Universe). In 1922, before graduating from university, N.Ye. Kochin started working in the mathematical bureau of the Central Geophysical Observatory founded by A.A. Friedmann, and he was employed there until he moved to Moscow in 1935. He held several positions, from computist to professor, and director of the Institute of Theoretical Meteorology. At the same time, N. Kochin taught at the Naval Academy, the Mining Institute, and Leningrad State University. In 1933, he was already a full professor, and in 1936 he obtained a doctoral degree in physical and mathematical sciences.

A.A. Friedmann posed a problem whose solution made N. Kochin immediately famous in meteorology. Kochin found the solution to hydrodynamic equations of compressible fluids, taking into consideration the rotation of the Earth, represented as a model of cylindrical vortices. This model significantly generalized the most comprehensive model of the time proposed by A.A. Friedmann. The results, including the trajectory of gas particles, closely matched existing observations. During his time at the Central Geophysical Observatory, N. Kochin posed and solved some of the most important problems in meteorology, studied the surfaces of discontinuities in a compressible fluid, and solved the classic problem of the breakdown of discontinuities in an ideal gas. His main objects of study, however, were surfaces that separate currents or masses of air with different parameters in the atmosphere. These parameters define important atmospheric phenomena and display tangential discontinuities, whose properties are significantly impacted by the Coriolis force. He explored the problem concerning the linear stability of such currents, which in and of itself is very difficult. N. Kochin's work includes an analysis of possible implications of the movement of fronts on the Earth's surface, which can be done using his research on the discontinuity surfaces.

N. Kochin also obtained fundamental results in the theory of general atmospheric circulation. He was the first to notice the effect of the atmosphere's air friction, which used to be ignored. He estimated the thickness of a bordering layer on the Earth's surface, in which the Coriolis force plays an integral role. N. Kochin was the first to build a model for a humid cyclone. These and N. Kochin's other works built the foundation for further studies in meteorology.

In 1925, Nikolai Evgrafovich married Pelageya Yakovlevna Polubarinova (1899–1999), whom he had met earlier due to their mutual interest in science. They had two daughters.

Parallel to his scientific research, N. Kochin worked at Leningrad University, where he taught classes on mathematics and mechanics, including postgraduate courses. In 1932, he also started working at the Institute of Physics and Mathematics at the USSR Academy of Sciences. After it split into separate physical and mathematical institutes, N. Kochin moved to Moscow in 1935 to work as the head of the mechanics department at Steklov Institute. From 1935 to 1938, he also worked at the Central Aerohydrodynamic Institute (CAHI), which at the time focused on problems relating to aviation, a rapidly developing field during this time, and the movement of ships, including submarines. N. Kochin actively participated in researching new problems, which he solved using the theory of ideal incompressible liquids; this theory includes the theory of waves on water and the irregular motion of bodies under a liquid's surface. Kochin studied these problems with the help of his new method, using what is now called "functions of general circulation." N. Kochin also solved the problem dealing with the circumference of a thin wing, got a new solution to the Cauchy–Poisson problem for waves on water, studied the torsional vibration of piston engine crankshafts, and investigated the possible forms of balloon cables in the wind that changes with altitude.

Nikolai Kochin with Pelageya Kochina-Polubarinova.

After moving to Moscow, N. Kochin taught at Moscow University. First, he taught postgraduate courses on wave theory, and from 1938 until the end of his life, he was head of the hydromechanics department. From 1938 to 1940, he was a scientific secretary at the Moscow Mathematical Society. In 1939, he was elected a Full Member of the Academy of Sciences of the Soviet Union, skipping the corresponding member stage. That same year, the Institute of Mechanics at the USSR AS was created, where N. Kochin worked as a department head. At the beginning of the Second World War, N. Kochin moved to Kazan along with the Institute of Mechanics.

N. Kochin's work during the years preceding the war was a great help in solving military defense problems, especially his method of solving the plane problem of an underwater wing and his formulas for calculating the friction of a ship, taking into account interactions between the ship hull and the water.

During World War II, Kochin developed and solved a set of problems on the theory of circular wings, which allowed him to calculate the forces that acted on an airplane's wings during flight.

Apart from individual scientific works, N.Ye. Kochin wrote books that included more general results that laid the foundations for a new era of meteorology and hydromechanics. His numerous results in meteorology were presented in the two-volume book *Dynamic Meteorology* (1935). He wrote the textbook *Vector Calculus and Fundamentals of Tensor Calculus*, which was reprinted many times, and, finally, the two-volume textbook *Theoretical Hydromechanics* [1] with A.I. Kibel and N.V. Rose, that students studying hydrodynamics in Russia continue to use even nowadays.

At the start of 1943, Nikolai Evgrafovich was diagnosed with bone cancer. N. Kochin was hospitalized and his leg was amputated. After the operation, he got better, and Nikolai Evgrafovich worked hard and vigorously once again at the Institute of Mechanics and at Moscow University. In August of 1944, the disease returned, however, and Nikolai Evgrafovich died on December 31st, 1944, at the age of 43.

N. Kochin's major papers are included in his two-volume collected works [2]. The full list of the over 100 scientific works that Kochin published can be found in his biography [3]. An analysis of the development of N. Kochin's ideas and methods, as well as very interesting stories of scientists who worked with N. Kochin or knew him personally, are included in the materials of the conference [4] that was held in honor of Kochin's 80th birthday.

The enormous mass of work that N. Kochin completed during his very short life is astonishing. His results, as well as his conceptual ideas, defined hydro-aerodynamics and meteorology for many years and continue to do so today.

Andrey Il'ichev and Andrey Kulikovskii

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Decay problem for an arbitrary discontinuity

The decay problem for an arbitrary discontinuity is the problem of constructing an analytic solution for nonstationary equations of the continuous medium mechanics as applied to the decay of an arbitrary discontinuity. It often arises in a wide range of problems in continuum mechanics, where discontinuities are formed in a continuous flow, as well as in problems where discontinuities initially occur in the distribution of the medium physical parameters. This problem was solved by N.E. Kochin in 1924–1925 and has been playing an important role not only in gas dynamics, but also in the entire mechanics of continuous media.

The problem is posed as follows. For $t = 0$, in the space (x, y, z) the domain for $x < 0$ is filled with a homogeneous gas with the following parameters: the velocity v (vector with components v_x, v_y, v_z), the pressure p, the density ρ, and the specific internal energy e (depending on the density and pressure): v_0, p_0, ρ_0, e_0. The domain $x > 0$ is filled with a homogeneous gas with other parameters v_1, p_1, ρ_1, e_1. The relations between these parameters in principle can be arbitrary. The problem is to find the gas motion for $t > 0$. The gas is assumed to be inviscid and non-heat conducting, as well as perfect, that is, it is an ordinary rarefied gas with internal energy $e = \dfrac{1}{\gamma - 1}\dfrac{p}{\rho}$, $\gamma = const$. If the initial data (the relations between the parameters v_0, p_0, ρ_0, e_0 and v_1, p_1, ρ_1, e_1) are arbitrary, then in the course of time this discontinuity, generally speaking, ceases to exist and must decay.

The motion generated by the decay of an arbitrary discontinuity is self-similar, i.e., the dimensionless parameters v/a_0, p/p_0, ρ/ρ_0, e/e_0 are functions of the variable $x/(a_0 t)$, ($a = \sqrt{\dfrac{dp}{d\rho}}$ is the speed of sound) and of the constant parameters v_0/a_0, v_1/a_0, p_1/p_0, ρ_1/ρ_0, e_1/e_0. The x/t-dependent solutions are well known. These are centered Riemann waves and shock waves [1]. To these nontrivial solutions of the gas dynamics equations, propagating through the gas, one should add tangential discontinuities, on which the following continuity conditions for the pressure and the normal component of the gas velocity should be satisfied:

$$[p] = 0, \quad [v_x] = 0. \tag{1}$$

Square brackets above denote the change (jump) of the quantity enclosed in brackets when passing through the discontinuity. A tangential decay does not impose restrictions on $[v_y]$ and $[v_z]$. If $[v_y]=0$ and $[v_z]=0$, then the decay is called contact.

In Riemann waves and shock waves, ρ, v_x, and p are variables and the changes in these quantities are interdependent, hence, one may consider the changes in ρ and v_x to be functions of the change in p. The components v_y and v_z do not change in the waves under consideration, therefore, at the place of the initial discontinuity in the gas, one still has a tangential discontinuity with unchanged (coinciding with initial) discontinuities $[v_y]$ and $[v_z]$.

Thus, the solution to the problem of the decay of an arbitrary discontinuity should consist of centered Riemann waves and shock waves, selected in such a way that the tangential discontinuity remains at the site of the initial discontinuity, where equalities (1) are satisfied.

Firstly, N.E. Kochin uses the fact that, for the Riemann waves, the values of x/t corresponding to the wave are characteristics of the gas dynamics equations. Secondly, he uses the fact that the characteristics behind the shock wave follow the shock wave. This allows him to conclude that only one wave, either a shock or a centered Riemann wave, can propagate in each direction from the tangential discontinuity in the gas. Indeed, the first shock wave propagates through the gas behind the wave at a speed lower than the speed of sound, while the second shock wave moves through the gas in front of it at a speed greater than the speed of sound; the Riemann wave propagates at the speed of sound. This is not possible for a self-similar movement. If a centered Riemann wave propagates through the gas, then its trailing front moves through the gas at the speed of sound, and the shock wave following it moves at a speed greater than the speed of sound (along the gas particles in front of the wave). For self-similar movements, this situation is not possible. If a Riemann wave propagates behind a Riemann wave, then the width of the homogeneous zone separating these two waves remains constant, since the speeds of the Riemann waves are equal to each other; this is also impossible for a self-similar motion. The above arguments provide the following result.

Theorem ([2]). *For the decay of an arbitrary discontinuity, only the following three different wave configurations are possible:*

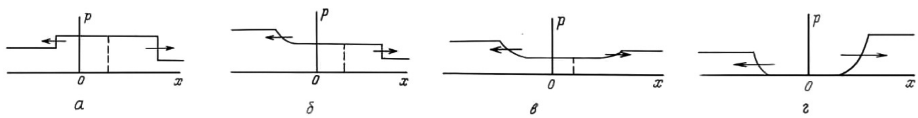
1. *tangential discontinuity takes place and a single shock wave propagates in each direction from it;*
2. *on one side of the tangential discontinuity, a shock wave propagates; on the other side, a centered Riemann wave propagates;*
3. *in both directions from the tangential discontinuity, one centered Riemann wave propagates.*

In all the cases, in the interval between the diverging waves, a domain is formed, filled with a gas with constant pressure and velocity values, including the tangential discontinuity surface. In the general case, the gas density is discontinuous on this surface.

To find the parameters of the diverging waves, one investigates the changes of the quantities v_x and p in the Riemann wave, where p is decreasing (after the passage of the Riemann wave, the pressure p is smaller than before its passage) and the changes of these quantities in the shock waves, where p is increasing. On the plane v_x, p, such relations imply that the values of v_x and p after the passage of a wave (Riemann or shock) belong to some curve passing through the point corresponding to the initial values of v_x and p. This argument applies to each side of the initial arbitrary discontinuity under consideration. Since the constructed curves intersect, conditions (1) are fulfilled at the tangential discontinuity; also, we find the values of v_x and p in its neighborhood and, thus, we determine the type and intensity of waves propagating in each side.

For discontinuity decay with the formation of two Riemann waves, the separation of one gas mass from another is possible. This happens if the absolute value of the difference between the normal initial velocities is sufficiently large. Expanding gases cannot fill the domain formed via scattering and a vacuum zone is formed between the leading fronts of the expanding gases.

The above arguments show that at a time different from the initial one, we have a solution of one of the types indicated below.



Four possible cases of the decay for an arbitrary discontinuity. The dashed line indicates the tangential discontinuity; a) an arbitrary discontinuity splits into two shock waves propagating in both directions from the tangential discontinuity; б) split into a centered Riemann wave propagating to the left and a shock wave propagating to the right of the tangential discontinuity; в) splits into two centered Riemann waves propagating in both directions from the tangential discontinuity; г) formation of an intermediate vacuum zone in case в).

N.E. Kochin had given criteria for the formation of possible solutions and had presented formulas describing them. The decay of an arbitrary discontinuity into two shock waves occurs, for example, if the initial values of the parameters differ only in the direction of the velocity, and the velocities are directed towards the interface (the problem on symmetric collision of two masses of gas). The decay into a shock wave and a Riemann wave occurs, for example, if the gases on both sides of the discontinuity are initially at rest, but have different pressures (the pressure equalization problem). The decay of an arbitrary discontinuity into two Riemann waves occurs, for example, if, with the same initial pressure and density, the initial velocities are directed outwards from the discontinuity surface (the problem of the expansion of two gas masses).

To solve the decay problem for an arbitrary discontinuity it is not enough to solve the problem with initial data; also, one should solve the collision problems for shock waves under their oncoming and concurrent motion, as well as of the collision problem for a shock wave with a tangential discontinuity.

Let us mention the further development of N.E. Kochin's ideas contained in his papers. First of all, these are papers related to one-dimensional motions of media with more complex, in comparison with a perfect gas, equations for the internal energy e in terms of p and ρ. In this case, in each direction from the tangential discontinuity, a certain system of waves, a combination of discontinuities and Riemann waves, can propagate. To solve such problems, the concept of a generalized shock adiabat was introduced in [3].

It is known that, in certain cases, the solution to the problem of the decay of the initial discontinuity may be non-unique. In this case, to guarantee the uniqueness, one introduces additional criteria for the selection of "admissible" shock waves. The requirement for the existence of a structure in the shock wave is an example of such a criterion.

The problem of the decay of an arbitrary discontinuity was also studied in other models of a continuous medium, in particular, in magneto-hydrodynamics and the nonlinear theory of elasticity. For example, in the case of magneto-hydrodynamics, to satisfy the conditions on the contact discontinuity remaining from the initial discontinuity, it is necessary to dispose of the amplitudes of three waves of different types propagating in each direction. We can say that according to the generally accepted concepts in continuum mechanics, each new model of a continuous medium with hyperbolic equations must be checked for the solvability and uniqueness of a solution to the problem of the decay of an arbitrary discontinuity.

It is also important to observe that many numerical methods (Godunov-type methods) are based directly on the solutions of the problems on the decay of an arbitrary discontinuity on each face of the partition of the domains where the solution is constructed.

Thus, the formulation and solution by N.E. Kochin of the problem of the decay of an arbitrary discontinuity in gas had a great influence on the further development of continuum mechanics.

Andrey Il'ichev and Andrey Kulikovskii

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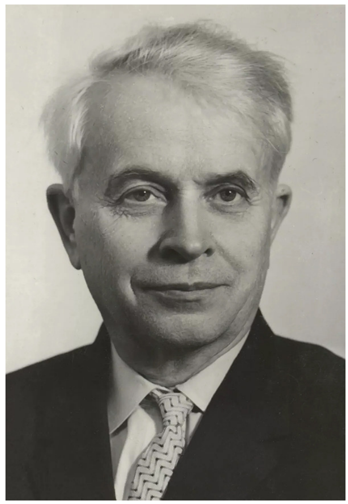
Post-war mathematics

Andrei Andreyevich Markov, Jr. (1903–1979)

Andrei Markov, Jr. is known for his results in theoretical physics, celestial mechanics, dynamical systems, braid groups, topological algebra, logic, and cryptography.

He proved that the equality problem in associative systems is undecidable. Also, Markov proved that no algorithm could decide whether two four-dimensional manifold presented as symplicial complexes are homeomorphic. He proved the theorem of a family of mappings having a common fixed point (Markov–Kakutani fixed-point theorem). He is the author of the Markov principle ("Leningrad principle" in constructive mathematics, a weakened version of the double negation) and introduced the notion of the normal algorithm, also known as the Markov algorithm.



Together with Nikolai Shanin and Gregory Tseytin, he created the school of constructive mathematics and logic in the USSR. He was head of the Geometry department at Leningrad State University from 1936 to 1953 and the director of LOMI[1] from 1942 to 1953. In 1953, he became a corresponding member of the Academy of Sciences. Since 1954 he lived in Moscow and was the head of the Department of Mathematical Logic at Moscow State University (1959–1979).

On 9 September 1903, in the family of the famous mathematician Andrei Andreyevich Markov (1856–1922) and Maria Ivanovna Markova (née Valvateva), after 20 years of marriage, their long-awaited son Andrei was born. His parents would take the ailing boy to the fashionable German resort of Baden-Baden in the summer and to Italy or the south of Switzerland in the autumn.

[1] Leningrad (St. Petersburg) Department of Steklov Mathematical Institute of the Russian Academy of Sciences.

Like most of the boys who lived in the famous apartment building[2] of the Academy of Sciences, Andrei studied at the 8th St. Petersburg Gymnasium for Boys (8, 9th Line of Vasilievskiy Ostrov). The boy's psyche was bound to be affected by his father's protracted (from 1912) departure from Orthodoxy and his style of communicating with those around him, unapologetic and not tolerating contradiction.[3] In 1919, Andrei Markov, after graduating from the "common labor school" (as gymnasiums came to be called then), entered the Physics and Mathematics Department of Petrograd University (Physics Division). After graduating from the university in 1924, he became a research fellow at the State Institute of Physics and Technology, and in 1925 he entered the postgraduate school of the Astronomical Institute [1].

Markov's first paper, published in 1927 in *Zeitschrift für Physik*, discussed the minimality property of Schrödinger wave groups. For the remainder of his postgraduate studies (up to 1928) Andrei Markov had been trying to find a general solution to the famous three-body problem of celestial mechanics, but he had found a solution only for a particular case [2]. The result was an article [3] published in 1929 in the *Journal of the Leningrad Physical and Mathematical Society*, which dealt with simple collisions in the general three-body problem, and the subject of the work was a study of the motion of a body not involved in collisions.

In 1930 Markov participated in the 1st All-Union Mathematical Congress in Kharkov and gave two talks: "On almost-periodic mappings" and "Proof of the theorem on calculability of a module of almost-periodic mappings." Earlier, in 1927, while he was still a postgraduate student, Andrei Markov also attended the All-Russian Congress of Mathematicians in Moscow, however, he did not give any talks then.

In 1933, Andrei Markov began teaching at Leningrad State University, which was named after A.S. Bubnov. In June 1934, the 2nd All-Union Mathematical Congress was held in Leningrad. At this congress, Markov made **six** contributions (more than anyone else): two in the "Topology" section, "On isotopy of compact sets in Euclidean spaces" and "On some spaces of finite dimension," two in the "Analysis II" section, "Almost periodicity and harmonizability" and a critical review of the mathematical content of the 1932 monograph of Nikolai Krylov and Nikolai Bogolyubov "Investigation of longitudinal stability of an airplane" in "On the theory of stationary oscillatory processes by academician N.M. Krylov and Dr. N.N. Bogolyubov," and one paper each in "Analysis I" and "Approximated Calculations" sections each,

[2] Residential House for academicians of the Academy of Sciences and their families on the Lieutenant Schmidt embankment (formerly Nikolaevskaya) No. 1/2. on the corner with the 7th line of Vasilyevsky Island.

[3] So S. Kovalevskaya in one of her letters to Mittag-Leffler complains: "Markov publicly spoke about my work on rotation that it was full of gross errors! When he was asked to show at least one, he brazenly replied that he did not want to do so."

"Arithmetical characterization of trigonometric polynomials" and "On devices facilitating the construction of functional rectilinear scale", [4].

The polemical reply by Nikolai Krylov and Nikolai Bogolyubov to Andrei Markov's criticism did not prevent Markov from being awarded the degree of Doctor of Physical and Mathematical Sciences (without a defense procedure) in 1935 and the title of Professor in 1936. That same year (1936) he also became the head of the Geometry department at the Faculty of Mathematics and Mechanics of Leningrad State University.

In 1936, the journal Recueil Mathématique (Математический сборник) published an article by Andrei Markov, Über die freie Äquivalenz der geschlossenen Zöpfe ("On the free equivalent of free braids"). The complete solution [5] to the problem was given in 1939 by Noah Moiseevich Vainberg (born 1914, killed at the front in 1942, [6]), a postgraduate student of Markov. The latter referred to this subject once more in 1945 in his monograph *Foundations of Algebraic Braids Theory*, [7].

Andrei Markov's paper "On mean values and exterior densities," published in 1938, played an important role in the development of the theory of measure, [8]. It established the possibility of constructing a general topological theory of measure and integration in normal spaces.

In the paper [9], "To the definition of the concept of the complex" (1939) Markov proposed his definition of "a finite Euclidean symplectic complex which makes it obvious that small displacements of vertices result in a Euclidean symplectic complex." In his papers [10, 11] on free topological groups Andrei Markov pointed out a certain general way to construct topological groups.

Since 1948, Markov had become increasingly interested in the problems of constructive mathematics and the general theory of algorithms (Markov used the term "algorifm"). In this theory, he saw not only "powerful technical, but also rich general logical possibilities." As a result, he "developed the notion of a normal algorifm, which turned out to be very convenient for a lot of purposes," as well as "significantly enhanced fundamentally important S.C. Kleene's theorem of representation of partially recursive functions through primitive recursive ones" and obtained "very interesting results concerning the complexity of normal algorifms that calculate Boolean functions", [12].

As early as 1946 Andrei Markov proved the algorithmic undecidability of a number of algebra problems [13]. In 1958, pursuing the topic of undecidability, he published two important papers in the Proceedings of the USSR Academy of Sciences, [14, 15]: "The Insolubility of the Homeomorphism Problem" and "On the Insolubility of Some Problems in Topology."

In 1947 in his review of topology in the book "Mathematics in the USSR for thirty years 1917–1947" Markov devoted two pages to the results from the unpublished manuscript "Cardinality in Topology" by his postgraduate pupil

Mikhail Perelman[4] who died in 1942. It should be mentioned that in 1953 Markov moved to Moscow and started working in the Computation Center of the Academy of Sciences, where he created the Laboratory of Mathematical Logic and Structure of Machines, which he headed for about 20 years. He also worked at the Steklov Mathematical Institute at Moscow until 1972. From 1959 to the end of his life, Andrei Markov held the chair of Mathematical Logic at the Faculty of Mechanics and Mathematics, Moscow State University. Among his Moscow students we should mention Albert Dragalin (1941–1998), Nikolai Nagornyi (1928–2007), and Boris Kushner (1941–2019), who, together with his Leningrad students Nikolai Shanin (1919–2011) and Gregory Tseytin (born 1936), continued Markov's research in constructive mathematics.

Research Institute of Mathematics and Mechanics of A.S. Bubnov Leningrad State University, 1934.

Workers of the world, unite!

Letter of commendation

To be given to the Member of the Research Institute of Mathematics and Mechanics of A.S. Bubnov Leningrad State University

A.A. Markov

For fruitful theoretical work, particularly for his research of n-dimensional vector spaces, and for his leadership of the topological circle.

Director: A.R. Kulischer
Party Secretary: Lerman.
Labor organizer: N. [illegible]

[A translation of the illustration on the left]

Comments about the letter of commendation

At the time the Institute bore the name of Andrei Bubnov, the People's Commissar for Education then. Three years later, he was arrested for 'anti-Soviet terrorist activities' and executed. In the same year, 1937, the institute's first director, Alexander Kulisher, also lost his job and was replaced by Vladimir Smirnov, whose name the Institute bears today. The fate of the former is unclear: according to some reports, he died in a wave of the Great Terror of 1937–38, while others suggest he got off lightly — by the standards of the time — being expelled from the Communist Party and exiled to Kirov, where he taught mathematics until at least 1944.

[4] The son of Yakov Perelman, who was a Russian and Soviet science writer and author of many popular science books.

The letter of commendation itself is designed in the traditional 'state of workers and peasants' style familiar to all generations of Soviet citizens, with the workers against the backdrop of factories and smoking chimneys (an omnipresent symbol of the 'workers' part at the time) and the peasants onthebackground of combine harvesters, sacks of grain, and sprouting crops. That last element, along with a hammer, was also a part of the coat of arms of the USSR.

However, there is still a portrait of Lenin at the top: after the Second World War and until the mid-1950s, such documents usually contained a portrait of Stalin alone. Not for nothing did the official de-Stalinisation during the so-called Thaw[5] take place under the slogan of "a return to Leninist norms."

In 1953 Andrei Markov was elected a corresponding member of the Academy of Sciences of the USSR. In the same year he became a member of the Communist Party. Still, fifteen years later, in 1968, he signed the famous "Letter of 99" against the forced placement of the dissident mathematician Alexander Yesenin-Volpin (1924–2016) in a psychiatric hospital. Andrei Andreyevich Markov died on 11 October 1979. In conclusion, I would like to quote two stanzas from his humorous poem "The Demon,"[6] popular among mathematicians in the 1960s [16]:

Sitting and eating spinach only

Was a spirit of denial, a spirit of doubt

And was as gloomy as an exhibit

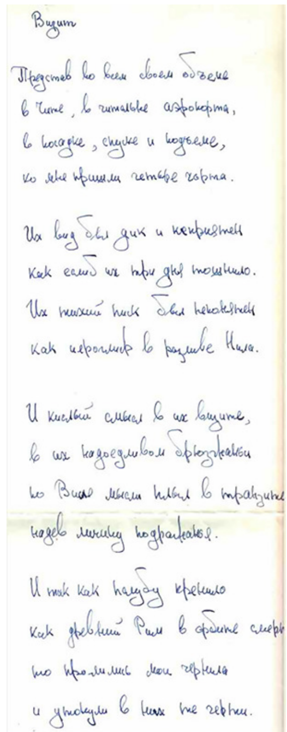
Of the health care museum...

On these strange phenomena

Let bright light be shed:

The spirit of denial, the spirit of doubt

Was sick with the disease of "colitis."



Визит

Представ во всем своем объеме
в Чите, в читальне аэропорта,
в [illegible], скуке и подъеме,
ко мне пришли четыре чорта.

Их вид был дик и неприязнен
как если б их три дня тошнило.
Их тяжкий лик был [illegible]
как иероглиф в разливе Нила.

И кислый смысл в их визите,
в их надоедливом брюзжаньи,
но [illegible] мысли плыл в транзите
надев личину подражанья.

И так как [illegible] [illegible]
как древний Рим в [illegible] [illegible]
то пролились мои чернила
и утонули в них те черти.

Vladimir Odyniec

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[5] The period between the mid-1950s and the mid-1960s when repression and censorship in the Soviet Union were considerably relaxed and many political prisoners were released and acquitted of charges against them.

[6] It has the same name and characteristics as a well-known poem by Mikhail Lermontov, a Russian poet, which is considered to be a masterpiece of European Romantic poetry and is taught in Russian schools.

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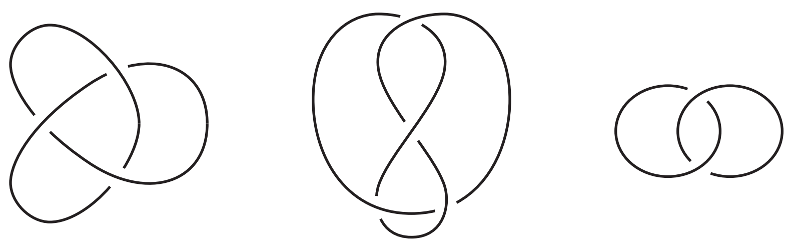
From Markov moves to Markov traces

Andrei Andreyevich Markov Jr., not to be mistaken for his homonymous father, is well known among topologists for an important though isolated result in his mathematical production. This result, which is called Markov's theorem, concerns the classification of knots and links *via* braids and is expressed in terms of so-called Markov moves. Although it dates back to the 1930s, it found an unexpected echo fifty years later with the concept of a Markov trace.

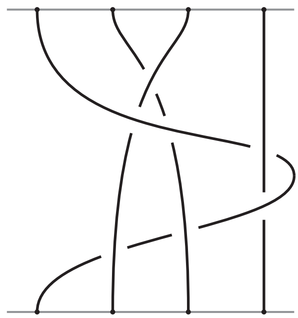
Knots and links. In order to state Markov's theorem, we start with a quick introduction to knots and links. For a mathematician, a *knot* is a closed curve in three-dimensional space without any self-intersection. Two knots are deemed *equivalent* if one can deform continuously one into the other without allowing self-intersections. Already in the 19th century it was known that there are infinitely many knots up to equivalence. Deciding whether two given knots are equivalent or not is a fundamental and difficult question. It is not even an easier task to find out if a given knot is *trivial*, that is equivalent to an "unknotted" knot such as a circle in a plane.

Mathematicians are also interested in links: a *link* is a union of finitely many knots in space without any intersection. As in the case of knots, two links are equivalent if one can deform continuously one into the other without cutting through them. We may similarly ask whether two given links are equivalent or not. The following figure presents the trefoil knot, the figure-eight knot and the Hopf link (from left to right). The number of knots composing a link does not change under deformation; therefore the Hopf link, which is composed of two knots, cannot be equivalent to the other two knots, which themselves are non-equivalent.



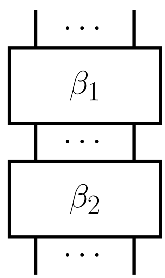
Obviously there are two directions in which one can follow a knot; choosing an orientation of the knot amounts to choosing one of these directions. When we choose an orientation for each of the knots composing a link, we obtain an *oriented link*. Two oriented links are equivalent if there is a deformation from the first one to the second one preserving the orientations.

Braids. One way to produce an oriented link is to close a braid. To visualize what a braid is, put n pegs on a horizontal line and attach a string to each peg. Then let the strings dangle down, braid them ad libitum and attach each bottom end to a lower peg, one of n lying on another horizontal line parallel to the first one. We have thus formed what is called a *braid*. The strings of a braid have a natural orientation which we choose to go from top to bottom. Here is an example of a braid with four strings.



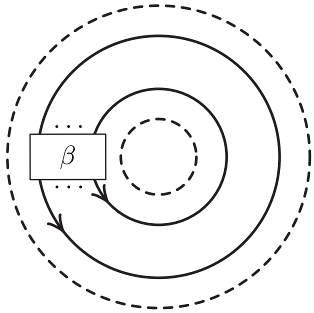
As in the case of knots and links, braids are considered up to deformation. The nice feature about braids with a fixed number of strings is that, up to deformation, they form a group: the product of two braids is obtained by placing one on top of the other and gluing the corresponding ends accordingly. This group (defined by E. Artin in the 1920s) has been extensively studied since.

$\beta_1\beta_2 \quad =$

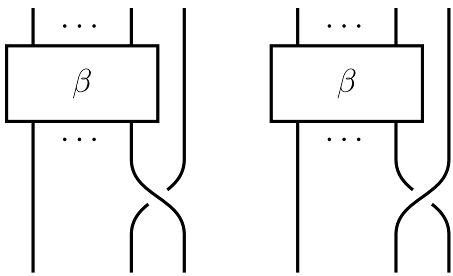


Given a braid β with n strings, connect the upper pegs to the lower ones by n circular arcs as in the following figure. One then obtains a link, which we denote by $\widehat{\beta}$ and call the closure of β. The link inherits the top-to-bottom orientation of the braid.

In the early 1920's the American mathematician James W. Alexander (1888–1971) proved in [1] that any oriented link is equivalent to the closure of some braid. Such a braid is not unique. Indeed, Markov identified three



transformations on braids which, after closing, produce equivalent oriented links. These transformations are called *Markov moves*. The first Markov move uses the group structure mentioned above and replaces any braid β by a conjugate braid, that is one of the form $\beta' = \alpha\beta\alpha^{-1}$, where α, hence β', have the same number of strings as β. The second and the third Markov moves transform a braid β into a braid with an additional string: it consists of adding an extra string to the right of β and braiding it once with the rightmost out-coming string of β; there are two possibilities as shown in the following picture.



Markov's theorem. We say that two braids (possibly with an unequal number of strings) are M-equivalent if they can be related by a finite sequence of Markov moves and their reverses. It is not hard to convince oneself that M-equivalent braids have equivalent closures as oriented links. Markov's theorem states the converse, namely *two braids have equivalent closures if and only if these braids are M-equivalent.*

Markov's theorem appeared in [4], an article in German with a summary in Russian. The text was based on a lecture he gave on September 5, 1935 at the First International Topological Conference, which was held in Moscow and was historically the oldest truly international specialized meeting in topology. Markov gave only a sketch of the proof of his theorem. The first detailed proof appeared 38 years later in Joan Birman's monograph [2].

Markov traces. Markov's theorem resurfaced in the 1980s with the spectacular discovery of a new link invariant by Vaughan Jones (1952–2020; Fields medal, 1990). A *link invariant* is a way to attach to each link a quantity (a number, a polynomial...) which takes equal values for equivalent links. Link invariants enable us to distinguish links that are not equivalent. Previously very few link invariants were known. Triggered by Jones's discovery, Nikolai Yu. Reshetikhin and Vladimir G. Turaev from LOMI (the Leningrad Branch of the Steklov Mathematical Institute) soon came up with infinitely many new link invariants. Their construction is based on Drinfeld and Jimbo's theory of quantum groups (whose origin can in part be traced back to Ludwig D. Faddeev's school). Some of these constructions rely heavily on the following consequence of Markov's theorem: any function defined on braids (with any number of strings) which is invariant under Markov moves gives rise to a link invariant. Such a function is called a *Markov trace*; it very often occurs as the trace map of a suitable family of matrix representations of the braid groups. Numerous significant examples of Markov traces have been constructed in the last decades.

The reader will find further details in the monograph [3] (from which some of the pictures are borrowed).

Christian Kassel

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Dmitry Konstantinovich Faddeev (1907–1989)

Dmitry Faddeev was the founder of the Leningrad algebraic school. He achieved fundamental results in Galois theory. In 1943, independently of his Western colleagues, he defined and studied the cohomology of groups, a subject that he arrived at while creating an apparatus for studying the field immersion problem.

He was born in the town of Yukhnov, Smolensk governance (now Kaluzhskaya County) on the 30th of June, 1907, on his maternal grandfather's estate. Dmitry's paternal grandfather came from peasant stock in the Samara governance, and his surname was created when he was emancipated shortly before the abolition of serfdom.

From an early age, his main pursuits were mathematics and music. In 1923, he entered the Faculty of Physics and Mathematics at Leningrad University, and in 1929 he joined the composition class of Leningrad State Conservatory. Later on, lack of time forced him to leave the Conservatory but his love for music remained, and he played the piano at a quite professional level.

Ivan Vinogradov and Boris Delone, both outstanding mathematicians, were Faddeev's first teachers. He wrote his graduate thesis under the former and conducted his postgraduate studies under the latter. He later said that after graduating from university, it was quite difficult to find a job in his profession, and until 1930 he worked for various employers, including the Weights and Measures Chamber, where he became addicted to smoking because of lengthy interruptions to the observations of the instruments. Later, however, he managed to give up the habit. A curious detail is that one characteristic feature of this time period was a shortage of almost everything, including paper, so he had to conduct his calculations, and they were quite long, on the back of the wallpaper.

From 1930 onwards, Dmitry began teaching at various higher education institutions in Leningrad. At Leningrad University, he taught from 1933 until 1989, the year of his death. In 1935 he defended his thesis, for which he was awarded a higher doctoral degree, bypassing the candidate's[1] degree. In 1937, he became a professor. He was Dean of the Faculty of Mathematics and Mechanics from 1952 to 1954.

Faddeev worked at the Steklov Mathematical Institute of the Academy of Sciences of the USSR, from its inception in 1932 until 1934 (when the Institute moved to Moscow), and then worked at the Leningrad branch (LOMI) from the year it was founded in 1940 until his death. In 1964, Faddeev was elected a corresponding member of the Academy of Sciences. He chaired the Laboratory of Algebraic Methods at LOMI for many years, was president of the Leningrad Mathematical Society, and was the founder and continuous leader of the city-wide algebraic seminar that is now named after him. Together with his wife, Vera Nikolaevna, he was awarded the State Prize in 1981 for their monograph *Computational Methods of Linear Algebra.* On October 20th, 1989, Dmitry Faddeev died and was buried at the Komarovo cemetery.

From the left to the right: L.V. Kantorovitch, I.P.Natanson, D.K. Faddeev, 1938.

Faddeev contributed to many branches of mathematics: the theory of functions, probability theory, geometrical crystallography, and especially to numerical analysis. But at the heart of his work was algebra. He significantly advanced Galois theory and the problem of immersion in Galois theory. His work gave impetus to the development of the theory of representations of

[1] A candidate's degree at that time was equivalent to a Ph.D while a doctoral degree was equivalent to a habilitation.

non-semisimple objects (rings, modules) and integer representations of finite groups.

While dealing with the problem of immersion, he encountered the formalism of the so-called "factor systems," which are omnipresent in that context, and discovered that that formalism was a particular case of a much more general construction. That was how the theory of group cohomology was discovered, independently of and simultaneously with Samuel Eilenberg and Saunders Mac Lane. According to the memoirs of his son, during their evacuation in Kazan in 1943, one evening, his father walked excitedly around the room and exclaimed that he had discovered something remarkable (what he discovered was later called cocycles). His son asked him how many people in the world would understand what he had just done, to which the father replied, "Well, five, maybe."

He was a brilliant teacher and actively developed the concept of teaching. His textbook *Problems in Higher Algebra*, written jointly with Iliya Sominsky, has been reprinted extensively. His textbook *Lectures on Algebra* is still very popular. He and his co-authors also wrote algebra textbooks for schools.

Faddeev's basic approach to teaching and textbook writing was described by him like this: "I believe that abstract concepts should be introduced as we manage to arouse the need for generalization in pupils, or at least if it is possible to illustrate general concepts with more concrete material."

Dmitry Faddeev contributed to the origination of mathematical olympiads for schoolchildren, the first of which occurred in Leningrad in 1934. He was one of the founders of the remarkable boarding school No. 45. Today, it is the D.K. Faddeev Academic Gymnasium at St. Petersburg State University.

His personal qualities deserve particular mention. He was kind and attentive towards his interlocutors, was always as pleased with the success of his colleagues as he was with his own, and was never arrogant. All of these qualities helped facilitate the inviting and creative atmosphere that pervaded the community of mathematicians in Leningrad.

As an extracurricular activity, each year Faddeev had a club for first-year students who wanted to study algebra and number theory seriously, something quite natural back then, however exotic it might seem to some these days. The author of this article was one of the participants of the 1963 club.

Academician Igor Shafarevich wrote the following about Dmitry Faddeev's contribution to the world of mathematics:

> Knowing how easily Dmitry Konstantinovich gives away his ideas, how little he is inclined to emphasize his personal contribution, and how much effort he is prepared to spend on discussing the work of his students and colleagues, one could predict that his influence on the development of mathematics would not be as clearly visible and as widely recognized as it deserves. Words about Zhukovsky, to whom D.K. Faddeev is also in other respects close in spirit, as described by

> Pushkin, are quite appropriate: 'They would translate him into all languages if he himself did not translate so much," the only difference being that 'translate' should be replaced with, say, 'quote.' Dimitri Konstantinovich's contribution to mathematics seems to us to be rather underestimated...
>
> Dmitry Konstantinovich himself was not saddened by this state of affairs in the slightest. And he was, of course, profoundly right. If the principle "manuscripts do not burn" is true, then mathematical ideas "do not burn." And not just in the sense that future mathematicians or historians of mathematics will reconstruct things again. Much more important is the fact that for Dmitry Konstantinovich only the beauty of mathematical ideas created by him was important, and this beauty will always exist and will bear the imprint of his individuality.

In the 1930s, because of the isolation of Russian mathematics from the West, much effort was spent on research in fields already long-established and unknown only to us, instead of new research. And often, work in such fields was in danger of a heartwarming discovery that turned out to be only the rediscovery of a known result. But it was in exactly such a context that Dmitry Konstantinovich's remarkable traits came into play. He was a rare mathematician in that he was happy to listen to his interlocutor, whatever the latter wanted to tell him. In his reaction to a mathematical result, whether it was his own discovery, the result that an interlocutor obtained, or an old theorem previously unknown to the person he was speaking with, the beauty of the result played the main role.

In 1963, Dmitry Konstantinovich offered me the chance to teach at the newly-formed boarding school No. 45. Sometime later, he came to my class to inspect my teaching. I was a very young man then and told the material with, so to speak, far too much seriousness, meticulously proving all the facts, even the obvious ones. After the lesson, Dmitry Konstantinovich publicly praised me and then, when we were alone, said, "It seems to me that you have science prevailing over reason." I remembered this aphorism for the rest of my life, and it greatly influenced my future teaching strategy.

Sergei Vostokov

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Diophantine equations, Galois theory, homological algebra

Dmitry Konstantinovich Faddeev had a very wide range of scientific interests. But certainly, first of all, we remember him as one of the most outstanding algebraists of his time. His research touched almost all sections of algebra. His first results are related to the theory of the Diophantine equations. Faddeev's interest in them was influenced by his teacher Boris Nikolaevich Delone. Already the first attainments were impressive. Dmitry Faddeev was able to broadly expand the class of the third- and fourth-degree equations that could be completely solved.

It is well known that the solutions of binary equations of degree three make a group with respect to natural multiplication. The classical Mordell–Weil theorem claims that this group is finitely generated. However, it is not easy to calculate its rank. In some cases, Faddeev managed to find such estimates of the rank that enable one to get all solutions of the equation. For example, the equations $x^3 + y^3 = A$ were completely solved for all $A \leq 50$. To evaluate the power of this result, let us just mention that before these studies it was only possible to show that for some A the considered equation has only trivial roots.

Let us also mention here one beautiful result concerning the equation $x^4 + Ay^4 = \pm 1$. Faddeev showed that for any integer A the equation has at most one non-trivial root. This unique solution corresponds to the fundamental unit of some purely imaginary algebraic number field of degree four. The root exists only when the fundamental unit is trinomial.[1] These and other results are presented in the remarkable tract, "Irrationalities of the Third Degree", written by Dmitry Faddeev jointly with Boris Delone.

In subsequent years, Diophantine equations continued to attract the attention of Dmitry Faddeev. In one series of papers, he proved the finiteness of the degree zero divisor class group for the Fermat curves of degrees four, five, and seven. This yields very strong corollaries concerning the equations themselves. For instance, it implies that there exists only a finite number of quadratic fields in which the equation $x^4 + y^4 = 1$ has a non-trivial solution, and in each of these fields the number of roots is finite. There were no results of this kind

[1] For details and definitions, please, see the article "Boris Delone and Diophantine equations of degree three" in this volume.

in number theory before. Of course, from today's point of view, when many classical problems in the theory of Diophantine equations are solved, these results are perceived differently, but at the time they were record-breaking.

The second direction, which interested Faddeev in the first years of his scientific career, was Galois theory. He was especially interested in the so-called inverse Galois problem (still not completely solved): to construct an extension of a given field with the prescribed Galois group.

The first Faddeev's results in this area relate to building extensions with small Galois groups, such as the subgroups of the symmetric group on four letters, metacyclic transitive permutation groups of prime degrees, groups of quaternions and quaternion units.[2]

Wherein Faddeev uses a beautiful geometric approach. The desired field is interpreted as a subset of a vector space on which the Galois group action is simple enough. Then, many of the results mentioned here obtain elegant geometric wording. For example, extensions of the rational number field, which have the quaternion Galois group, turn out to be in close connection with the triples of pairwise orthogonal rational vectors in the three-dimensional Euclidean space.

However, the techniques developed by Faddeev for these problems are insufficient for further development. His approach actively uses individual characteristics of groups, and does not allow significant generalizations. The next step here should have been: finding the solution for solvable groups. And the natural approach to this is likely to be the following. To build chains of extensions, each subsequent of which is an extension of the previous one with an abelian Galois group.

This leads to a new problem. How to embed a given Galois extension into a wider field with a given Galois group and with a given epimorphism of this group onto the Galois group of the original extension? This problem, more general than the Galois inverse problem, is called the Galois immersion problem. One of the first Faddeev's papers that started a systematical study of the immersion problem was a remarkable article "Studies in the Geometry of Galois Theory" (1944) written by him again in collaboration with Delone. In the sections of this article, belonging to Dmitry, he formulates the necessary solvability condition of the immersion problem, the so-called *consistency condition*. This condition requires additive solvability of the problem. (The solution is assumed to be a vector space endowed with a group of operators that properly corresponds to the Galois group of the immersible field. The existence of multiplication is not required at all.) He also proves that the consistency is sufficient for immersibility, provided that the immersion problem kernel is the cyclic group of order not divisible by 8. Besides this, in the same work Faddeev proved that if the immersion problem with an abelian kernel has a solution in

[2] Also known as the binary tetrahedral group.

the category of Galois algebras, then there exists also a field solving it. The meaning of these results for further development of research on the Galois inverse problem and the immersion problem can hardly be overestimated.

The consistency condition turned out to be profound and, in a number of cases, very close to a sufficient condition for immersibility. No wonder that Helmut Hasse, an outstanding German algebraist who rediscovered it four years later, tried to prove that the consistency condition is equivalent to immersibility (at least in cases of abelian kernels). However, almost immediately after that D.K. Faddeev and I.R. Shafarevich built examples that refute this conjecture of Hasse. In particular, Faddeev could construct a really simple counterexample, assuming that the kernel of the immmersion problem is a cyclic group of order eight. Later, Dmitry, together with his graduate student R.A. Schmidt, could also find an additional necessary and sufficient immersibility condition for such a kernel.

Faddeev's study on Galois theory had an influence on the further development of this area. Together with his results concerning group cohomology, they became the most essential component in the solution to the Galois inverse problem for solvable groups and number fields (I.R. Shafarevich) and also for the solution to the immersion problem for abelian kernels (A.V. Yakovlev). Long-term research results of Faddeev and other mathematicians in this direction are collected in his monograph *The Immersion Problem in the Galois Theory*, written jointly with V.V. Ishkhanov and B.B. Lurie.

In 1947, in his article "On Factor-systems in Abelian Groups with Operators", Faddeev defined the notion that we now call the cohomology of groups. Simultaneously and independently, it was introduced by Samuel Eilenberg and Saunders Mac Lane. Cohomology groups have become an extremely powerful research tool in various fields of mathematics. It is believable, that Faddeev came to this concept, trying to create an apparatus for the study of the field immersion problem. Many of his subsequent results in homological algebra are clearly inspired by natural constructions in the Galois theory. Indeed, the cohomology turned out to be a perfect tool for studying the immersion problem, constructing fields with solvable groups and solving other problems related to the Galois theory and its applications.

The contribution of D.K. Faddeev to homological algebra is not limited to the definition of cohomology groups. He had other important results on his account. Let us mention the remarkable theorem relating cohomology of a subgroup to cohomology of the whole group with coefficients in the coinduced module.[3]

One should also mention his joint papers with Z.I. Borevich "Theory of homology in groups." I, II, published in 1956 and 1959 *(Bull. of Leningrad State*

[3] In the modern mathematical literature this result is usually addressed as Shapiro's lemma (1961). However, its first proof was published by Beno Eckmann in 1953. Faddeev's proof was, obviously, obtained independently.

Univ.), which played an especially important role for local mathematicians. These papers gave the first systematic review of the group cohomology theory available to the Russian reader. Besides that, the authors not only provide an overview of the theory state, but also present many new results. For example, they give the first construction of minimal projective resolutions for modules over local rings. By means of these resolutions, it was shown that any non-cyclic p-group has infinitely many pairwise non-isomorphic indecomposable p-adic representations.

To study other interesting problems, Faddeev also often used homological methods, in the creation of which he actively participated before. So, he constructed the theory of simple algebras over algebraic function fields and calculated the Brauer group of the field of rational functions over a field of characteristics 0. This result was far ahead of its time. Further development of this study was the celebrated Merkurjev–Suslin theorem, which became one of the brightest attainments of the Leningrad/St. Petersburg Algebraic school created by Dmitry Konstantinovich.

Dmitry Konstantinovich Faddeev was a man of the highest intelligence. In his youth, he studied at the conservatory, and was a great connoisseur of classical music, an excellent pianist. For all of us who knew and loved him, Dmitry Konstantinovich will remain the highest authority forever. We are lucky that we could work and constantly communicate with this wonderful man.

Anatoly Yakovlev

Addendum by the translator (Serge Yagunov). I would also like to say a few words about the many years of pedagogical and organizational activity of Dmitry Konstantinovich. He taught at the Faculty of Mathematics and Mechanics since 1933, being the head of the Department of Algebra in 1944–49, and the dean of the faculty in 1952–54. He was among those, who stood at the origins of the Leningrad/St. Petersburg Mathematical Olympiads for schoolchildren (held since 1934), and was one of the founders of a secondary physics and mathematics school, now bearing his name.

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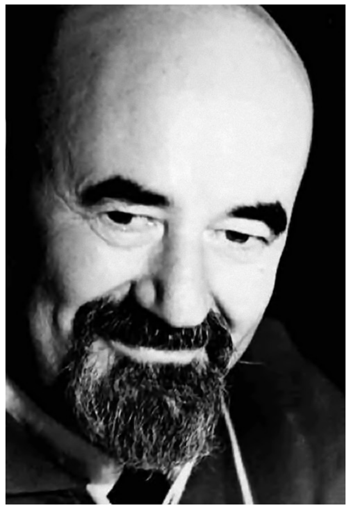
Solomon Grigoryevich Mikhlin (1908–1990)

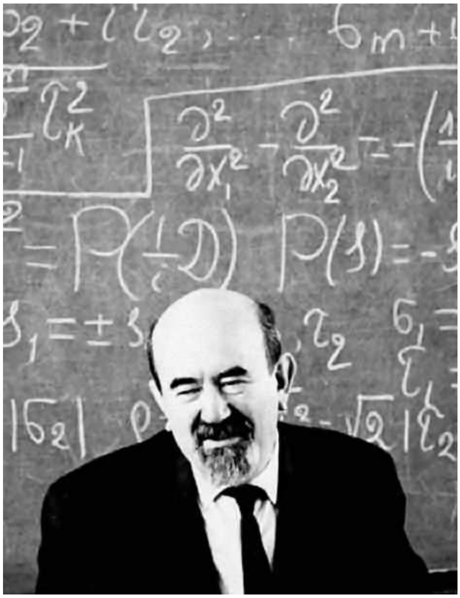
S.G. Mikhlin was born in Kholmech, a Belorussian village, into a Jewish family of modest means. His real name was Zalman Girshevich Mikhlin, and he was the youngest of five children.

Mikhlin graduated from a secondary school in Gomel (Belarus) in 1923 and entered the Leningrad State Pedagogical Institute, named after Herzen,[1] in 1925. In January 1927 he transferred as a second-year student to the Department of Mathematics and Mechanics (MatMekh) of Leningrad State University after passing all his first-year examinations without attending any lectures. Sergey Lvovich Sobolev studied in the same class as Mikhlin. Among their university professors were Nikolai Maximovich Günther and Vladimir Ivanovich Smirnov. The latter became Mikhlin's master's thesis advisor: the topic of the thesis, defended in 1929, was the convergence of double power series.



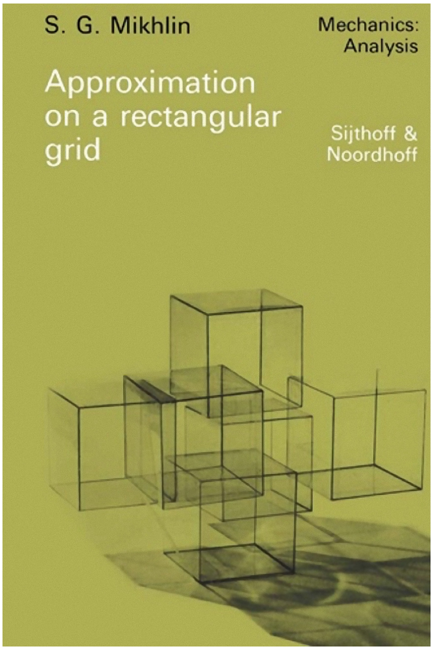
In 1930 Mikhlin started his teaching career, working for short periods at several institutes in Leningrad. In 1932 he obtained a position at the Seismological Institute of the USSR Academy of Sciences, where he worked till 1941. He was awarded the degree of "Doktor nauk" (equivalent to a Doctor of Science) in Mathematics and Physics in 1935, without having to earn the "Kandidat nauk" degree (equivalent to a Ph.D.), and finally in 1937 he was promoted to the rank of professor. During World War II he was a professor at Almaty State University. In 1944 Mikhlin returned to Leningrad State University as a full professor. From 1964 to 1986 he headed the Laboratory of Numerical Methods at the Research Institute of Mathematics and Mechanics at Leningrad State University. From 1986 until his death in 1990 Mikhlin continued working as a senior researcher for this laboratory.

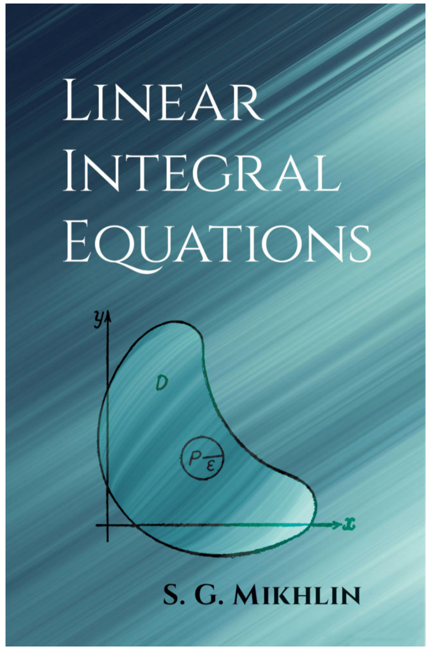
[1] Alexander Herzen (1812–1870) was a writer and important thinker who is known as the "father of Russian socialism." He was one of the originators of agrarian populism. For example, some of his writings inspired and contributed to the emancipation of serfs in 1861 throughout the Russian Empire.



S.G. Mikhlin was the author of more than 220 scientific works, including about 30 excellently written books and textbooks. On a large scale, S.G. divided his research into “works”, each of which consisted of articles. As a rule, each of these “works” resulted in the writing of a book. In each book, he collected and regularized the results of his work, considering it his duty. Mikhlin began his “work” impelled not so much by his own curiosity as by lofty, objective ideas about the usefulness of the corresponding theory for the development of mathematics and its applications. Of course, scientific curiosity played its part too, but so to say, secondarily. The aspect of sportsmanship in mathematics was exceedingly alien to Mikhlin’s creativity.

Mikhlin’s monographs and textbooks are remarkable from a pedagogical point of view, especially those devoted to variational methods and different classes of integral equations. Their style and accessibility to poorly prepared readers made Mikhlin famous in the world of engineers, which was a rare achievement for a mathematician. Almost all his books have been translated into many languages (in particular, into English, German, Chinese, Japanese, and Hungarian) and have had a remarkable influence on many young mathematicians, giving them a solid professional background. The fields that Mikhlin specialized in were: the theory of elasticity and plasticity, the theory of integral operators, numerical methods in mathematical physics, and boundary value problems. As I. Gohberg stated [1]: “Mikhlin considered the theory of singular integral equations his favorite creation. Very soon his results led to pseudodifferential operators, and his notion of the symbol (1936) became a cornerstone of this new theory that revolutionized partial differential



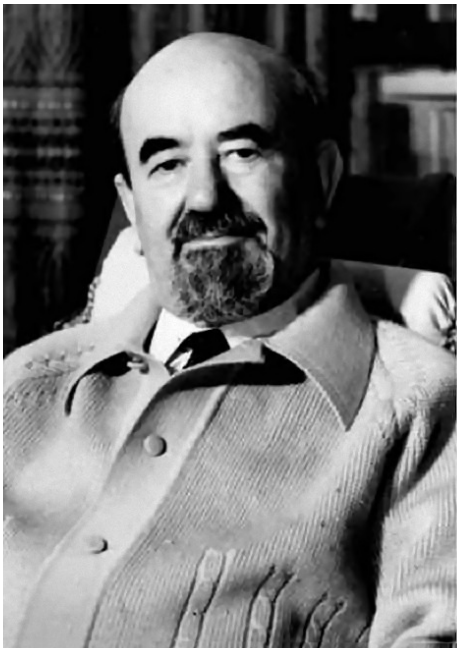


equations." According to G. Fichera [2], Mikhlin was one of the pioneers of modern numerical analysis together with Boris Galerkin, Alexander Ostrowski, John von Neumann, Walter Ritz, and Mauro Picone.

Mikhlin's mathematical idol was the French mathematician Jacques Hadamard. They first met in Moscow in 1934 where S.G. was one of Hadamard's guides on a city tour.

This encounter was a memorable event for the young Mikhlin, who had graduated from Leningrad State University with a master's degree five years earlier and whose first mathematical result was an extension of the Cauchy–Hadamard formula for the radius of convergence to double power series. I remember once in the 1960s, Mikhlin proudly told me that someone found a resemblance between him and the famous French mathematician. Another of Mikhlin's mathematical heroes was his peer S.L. Sobolev. The latter always called Mikhlin by his diminutive name Zyama.

Mikhlin had an innate sense of humor. He roared with laughter at the compositions of the "Oberiuts" (The Society of Real Art, a Leningrad literary group). In the Sixties, these compositions were accessible only through "Samizdat," a Russian abbreviation of the word for self-publishing, a practice that was persecuted by the Soviet regime. He knew the poem "Plish and Plum," translated by D. Kharms from the German poem by Wilhelm Busch, by heart. Mikhlin knew many other texts from memory, including Edward Lear translated by S. Marshak, "The Owl and the Pussycat," "In the Country of the Jumblies," "The Pobble who Has no Toes," and others.



S.G. never went to concerts, saying only that he perceived music as noise. Self-critically, he said that he lacked capabilities for foreign languages, although I happened to hear him speaking German and French.

A convinced atheist, S.G. Mikhlin knew the Pentateuch and, by the way, reproached Thomas Mann for his exceedingly audacious handling of the Torah in *Joseph and His Brothers.* Mikhlin liked neither the latter novel nor M. Bulgakov's *The Master and Margarita.* In general, it was difficult to argue with S.G. on humanitarian themes because of his confidence in his opinion, erudition, and strength of argumentation. His speech was logical and aphoristic.

Mikhlin never prompted answers to poor achievers among the postgraduate students and liked to repeat, after Ilf and Petrov: "The rescue of a drowning man is the drowning man's own job."

S.G. Mikhlin knew that I had grown up without a father, who was killed in the war in 1941, and, I would say, he looked after me in a fatherly way for many years. He often invited me to his place, talked about his life and answered the most diverse questions. It was from him that I heard, still being a student, that Lenin was no less cruel a killer than Stalin, and that concentration camps were first created under Lenin's rule in Soviet Russia. S.G. Mikhlin was referring to the Party and Administration University officials when he told me:

> They just have power, but we have theorems. Therefore, we are stronger!

In 1961 Mikhlin received the State Order of the Badge of Honour. He was awarded the Laurea honoris causa by the Karl-Marx-Stadt (now Chemnitz) Polytechnic University in 1968. He was also elected for membership in two academies, the German Academy of Sciences Leopoldina in 1970 and the Accademia Nazionale dei Lincei in 1981. When he was not allowed to travel to Italy to receive the title, the Italian mathematician G. Fichera and his wife brought the small gold lynx, the badge of an Academician, to Leningrad. They bequeathed it to Mikhlin in his apartment. My wife Tatyana and I were the only guests at this "ceremony."

Solomon Grigoryevich, for his numerous disciples and for his friends, is not only an outstanding mathematician, but also a man of high moral standards and a kind heart. He is a man of great intelligence, profound knowledge, and versatility. His high qualities as a human being and as a scientist are largely appreciated by the mathematical community all over the world and have sparked love for mathematics and scientific research in many young hearts.

Vladimir Maz'ya

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S.G. Mikhlin and multi-dimensional singular integrals

Solomon G. Mikhlin (1908–1990) (S.M.) influenced essentially the development of Analysis and Mathematical Physics for almost half a century. In our article, we aim to give an outline of just one important direction of his creative heritage, namely, the theory of multi-dimensional singular integral operators. This theory, in its turn, became the base for the calculus of pseudodifferential operators, one of the major means in modern mathematical analysis.

The objects of study are integral operators of the form

$$u(x) \mapsto a(x)u(x) + (\mathbf{S}u)(x), \tag{1}$$

$$(\mathbf{S}u)(x) = \int\limits_{\mathbb{R}^d} K(x, y-x)u(y)\,dy, \quad x \in \mathbb{R}^d.$$

Here $K(x, y-x) = \dfrac{f(x,\theta)}{r^d}$, $\theta = \dfrac{y-x}{r}$, $r = |x-y|$. The singularity of the kernel is so strong here that the integral diverges if considered in the usual sense. Therefore, it should be understood in the Cauchy principal value sense. An obvious necessary condition for the latter convergence, even for nice functions u, is

$$\int_{|\theta|=1} f(x,\theta)\,d\theta = 0. \tag{2}$$

In a natural way, by localization, operators (1) can be defined on closed smooth manifolds and also generalized to vector-function $u(x)$, the kernel $K(x, y-x)$ being a square matrix-function.

Integrals in (1) are usually called 'singular'. In one dimension ($d=1$) equations with such integrals appeared at the beginning of XX century in papers by D. Hilbert and H. Poincaré concerning certain boundary problems in PDE and complex analysis. Even in one dimension, the Fredholm theory fails for this kind of equations since operator (1) is not compact.

The one-dimensional case was treated mostly using tools and ideas of complex analysis. The situation is different in the multi-dimensional case where singular integrals appear naturally in the theory of elliptic boundary problems. Here, before Mikhlin's studies, very little was known (F. Tricomi, 1926, 1928, $d=2$, and G. Giraud, 1934–1936, $d>2$, solved equations of a special form).

The actual breakthrough was made by S.G. Mikhlin. In papers in Doklady and Mat. Sbornik, [4, 5] published in 1936, he laid a foundation of the general theory of singular integral equations. In the case d=2 he expanded the function f in the trigonometric series $f(x,\theta)=\sum_{n\neq 0} b_n(x)e^{in\theta}$. Modifying this series, he introduced a certain function, which he called the *symbol* of the operator $a(x)+\mathbf{S}$:

$$\Phi(x,\theta) = a(x) + \sum_{n\neq 0} a_n(x)e^{in\theta}, \tag{3}$$

$$a_n(x) = \frac{2\pi i^n}{n} b_n(x).$$

Being impressed by Mikhlin's paper [4], Giraud [2] wrote: 'An ingenious procedure indicated by S. Mikhlin makes it possible to treat equations with double principal integrals of a very general type.' Giraud generalized Mikhlin's construction and published a formula for the symbol of the singular integral operator on a Euclidean space of arbitrary dimension, using spherical harmonics. Giraud has never published a proof of his formula. A proof was published by S.M. in 1955, twelve years after Giraud's death, for the first time.

The operator is recovered uniquely from its symbol, up to a compact additive term. There exists a correspondence between the sums and products of operators and their symbols, again up to a compact additive term.

Main topics of the study by S.G. Mikhlin in this field were:

— Boundedness conditions for the operator $\mathbf{S}$ in function spaces;

— Solvability analysis for equations of the form

$$(\mathbf{A}u)(x) \equiv a(x)u(x) + (\mathbf{S}u)(x) = v(x), \tag{4}$$

here $a(x)$, $v(x)$ are given matrix-, resp., vector-functions.

In lieu of explicit formulas for solutions of the equation (4) S.M. proposed the procedure of regularization: finding a singular integral operator $\mathbf{R} = b(x) + \tilde{\mathbf{S}}$ such that the operators $\mathbf{AR} - \mathbf{I}$ and $\mathbf{RA} - \mathbf{I}$ are compact in proper spaces. As soon as such a regularizer is found, the singular integral equation is reduced to an equation with a compact operator. The invertibility of the symbol leads to the existence of a regularizer.

The notion of symbol enables one to treat many problems concerning singular integrals. For instance, it was established by S.M. that the boundedness of the symbol, together with certain smoothness, guarantees the boundedness of the operator in $L_2(\mathbb{R}^d)$.

For 15 years S.M. was the only researcher who was working in the theory of multi-dimensional singular integral operators. It was only in 1952 that the fundamental paper by A.P. Calderón and A. Zygmund [1] appeared. They extended M. Riesz' theorem on the L_p-boundedness of the Hilbert transform

$$(Hu)(x) = (\pi i)^{-1} \int_{-\infty}^{\infty} \frac{u(y)}{y-x}\, dy.$$

These authors showed that the multi-dimensional singular operator of convolution type $u \mapsto \dfrac{f(x/|x|)}{|x|^d} * u$ is bounded in $L_p(\mathbb{R}^d)$, $1 < p < \infty$, provided f satisfies (2) and

$$\int_{|\theta|=1} \log(2 + |f(\theta)|)|f(\theta)|\, d\theta < \infty. \tag{5}$$

Further on, Calderón and Zygmund succeeded in extending their results to a wide class of singular integrals of nonconvolutional type. Namely, with a kernel $Q(x, y)$ satisfying $|Q(x, y)| \le C|x - y|^{-d}$ and

$$|Q(x, y + h) - Q(x, y)| + |Q(x + h, y) - Q(x, y)| \le Ch^\alpha |x - y|^{-d-\alpha},$$
$$\alpha \in (0, 1), \quad 2h \le |x - y|, \tag{6}$$

one associates the operator $\mathbf{T}_Q$ defined by the bilinear form

$$\langle \mathbf{T}_Q \varphi, \psi\rangle = \iint \psi(x) Q(x, y) \varphi(y)\, dx\, dy, \tag{7}$$

for functions $\varphi, \psi \in C_0^\infty(\mathbb{R}^d)$ with disjoint supports. If the operator $\mathbf{T}_Q$ extends to a bounded operator in L_2, i.e., for all $g, h \in C_0^\infty$, the estimate $|\langle \mathbf{T}_Q g, h\rangle| \le C\|g\|_{L_2}\|h\|_{L_2}$ holds, then the operator is bounded in all L_p, $1 < p < \infty$. Thus, the boundedness problem is reduced to the single case of $p = 2$. Essential progress in the latter case was made by E. Stein, A. McIntosh, M. Christ, and others. In particular, due to the impressive $T\mathbf{1}$ theorem, the operator with Calderón–Zygmund kernel Q is bounded in L_2 iff both $\mathbf{T}_Q$ and its transposed $\mathbf{T}'_Q$ map just one single function, that equals identically one, into the space BMO (see, e.g., [3] for details and references).

The notion of *index* of an operator, the difference of dimensions of the null spaces of the operator and its adjoint, introduced in F. Noether's works of 1920-s for the one-dimensional case, was investigated by S.M. for the multi-dimensional case. The symbol of a singular operator, as S.M. introduced it, became central in this theory and had numerous applications. S.M. proved that a scalar elliptic singular integral operator in the multi-dimensional case has index zero. The matrix case turned out to be much more involved. In the 1960s, the general index problem was solved by M. Atiyah and I. Singer, with far-reaching consequences for Analysis, Algebraic Topology, Noncommutative Geometry, with further expansions into Theoretical Physics. With his ideas and results in singular integral operators, S.M. became a forerunner of the revolutionary progress in analysis in 60-s, the theory of pseudodifferential operators, enveloping both singular integral, as well as differential operators and their resolvents. R.T. Seeley, one of the pioneers in the field, acknowledges the contribution by Mikhlin to the topic, see [8]: 'It will be clear that the author is indebted to the work of Mikhlin, which introduces many of the concepts and questions considered here.'

The results by S.G. Mikhlin as well as the development of the theory are presented in the books [6, 7].

Vladimir Maz'ya, Grigori Rozenblum

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Sergei Lvovich Sobolev (1908–1989)

Sergei Sobolev started his career in the theoretical department of the Seismological Institute. He invented Sobolev spaces and proved embedding theorems for these spaces. Sobolev was one of Igor Kurchatov's deputies in the atomic bomb project. Last but not least, he initiated the establishment of the Siberian Branch of the Academy of Sciences in Novosibirsk.

Sobolev was born on 6 October 1908. His father Lev and mother Natalia met when they were exiled to Saratov for revolutionary activities. After this exile, his parents returned to St. Petersburg, where his father worked as a lawyer and his mother taught at a grammar school. In 1919 Natalia and her children moved to Kharkov to stay with her relatives because of the lack of jobs in Petrograd[1] and the impending famine. In Kharkov, adults' earnings were not enough to buy food, so the children worked too, and Sergei read textbooks while herding goats. In 1923 they returned to Petrograd.

It was not possible to enter university before the age of 17, and Sergei finished school at 16. So, he enrolled in piano classes at the 1st Leningrad Art Studio, where he continued his studies even after he started university in 1926. It was at this same studio that Sobolev met his future wife in 1927. One day, in the presence of Leonid Kantorovich,[2] Sobolev said:

> Leonid Vitalyevich and I could have started university during the same year, but I was not allowed to start because of my young age, so I had to wait, and wasted the whole year playing piano and chess.

During his university studies, at Professor Günther's lectures on mathematical physics, Sobolev had doubts about Saltykov's theorem. Nikolai Günther suggested that Sobolev read Saltykov's original paper. When Sobolev compre-

[1] The city of St. Petersburg was called Petrograd from 1914–1924, and Leningrad from 1924–1991.

[2] Kantorovich entered university in 1926, being only 14 years old, with special permission from the Ministry of Education, abbreviated in Russian as NarComPros.

hended that paper and presented a counterexample, Günther was extremely surprised and immediately singled him out among the other students.

In 1930, at the First All-Union Congress of Mathematicians in Kharkov, Jacques Hadamard, speaking to Sobolev, who had presented a report on the Cauchy problem for hyperbolic equations, said: "I will be very happy, young colleague, if you will keep me informed about your future work, which is of great interest to me." In 1935, Sergei Sobolev introduced the notion of a generalized function (i.e., distribution).

Sergey Sobolev and N.M. Günther.

Here is a fragment from his article:[3]

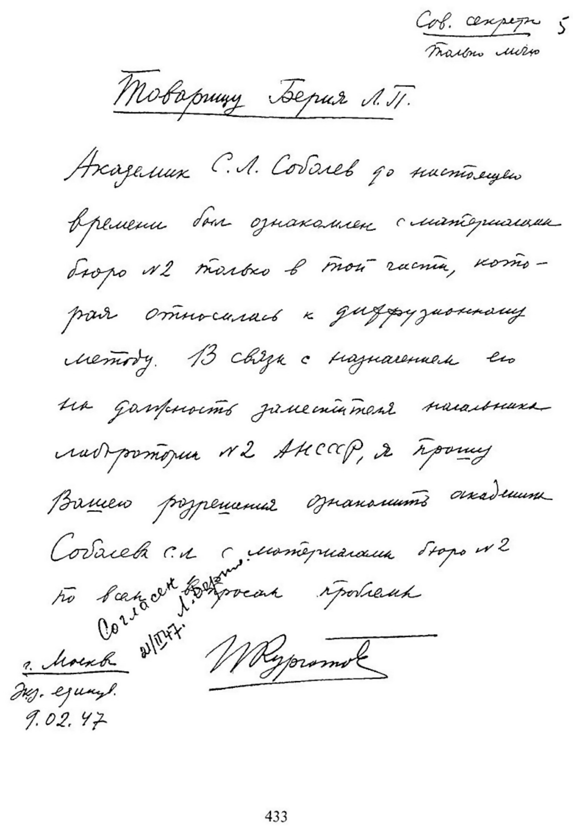
> As we will see further, the studies by Prof. N.M. Günther on the equations of potential theory and theory of heat turn out to be very close to this circle of ideas. In them, Professor Günther shows that it is often useful for these problems of mathematical physics to abandon the differential equation in its classical form and move on to study some integral equalities containing derivatives of orders lower than the original differential equation. To solve the problem of diffraction on logarithmic surfaces, which is the second part of our paper, we will have to use some functions that are solutions to the wave equation in some generalized sense. Not only is it possible that these solutions have no first derivatives, but they could even be unbounded.

The young scientist attracted attention with his strong papers; in 1933, at the age of twenty-five, he was elected as a corresponding member of the USSR

[3] S.L. Sobolev, *General Theory of Diffraction of Waves on Riemann Surfaces*, in: Proc. of the Steklov Mathematical Institute, 9, USSR Academy of Sciences Publishing House, Moscow-Leningrad, 1935, P. 39–105.

Academy of Sciences and as a full member six years later. Sergei Sobolev was the youngest academician in the USSR for a long time. To list some of Sobolev's achievements in mathematical physics: he had developed the theory of seismic wave propagation, had founded the theory of generalized functions as functionals, had introduced the concept of generalized derivatives as locally integrable functions, and had defined the generalized solutions of differential equations. Also, dealing with the Dirichlet problem for a domain whose boundary contains manifolds of different dimensions, he introduced new function spaces in mathematics, now called Sobolev spaces, and proved embedding theorems for them, known as Sobolev embedding theorems.

Sobolev was always interested in applications of science: as a student, he did an internship at the Electrosila plant and calculated the transverse vibrations of asymmetric shafts of electric generators. After graduation from



Сов. секретно 5
Только лично

Товарищу Берия Л.П.

Академик С.Л. Соболев до настоящего времени был ознакомлен с материалами бюро №2 только в той части, которая относилась к диффузионному методу. В связи с назначением его на должность заместителя начальника лаборатории №2 АНСССР, я прошу Вашего разрешения ознакомить академика Соболева С.Л. с материалами бюро №2 по всем вопросам проблемы.

Согласен. Л. Берия 21/II-47.

г. Москва

И. Курчатов

Изг. единств.
9.02.47

433

A confidential Kurchatov's letter to Beria, asking to grant Sobolev access to all parts of the Soviet atomic project.

university, he worked on theoretical issues of seismology together with Vladimir Smirnov and Solomon Mikhlin. In 1943, he started working at the Institute of Atomic Energy, worked with Kikoin,[4] and became Igor Kurchatov's deputy. During his fifteen years of work on the atomic project, Sobolev had to deal with a computationally complex method of uranium enrichment by diffusion of gaseous uranium hexafluoride through semi-permeable membranes. He also coordinated calculations required for membrane design and the stable operation of cascades for diffusion machines, and developed methods for the verification of calculations, etc. During this time, Sobolev took an interest in computational mathematics and cybernetics. Upon completion of the atomic project, he moved to Moscow State University, where he became head of the first chair of computational mathematics and computer center in the country. For Sobolev, as it was for Leonid Kantorovich, estimation of the accuracy and speed of computation was the subject of functional analysis and could be done only after choosing a suitable function space.

S. Sobolev, the construction site of the Mathematical Institute in the Novosibirsk Akademgorodok.

For Sobolev, the aesthetic side of science was also important:

> If a mathematician cannot admire good results, and not only understand what is accurate or inaccurate in them but see what is beautiful and what is ugly, he will never do anything. Aesthetic criteria play a huge role... I think it is one of the most important qualities of a person who has decided to devote himself to mathematics — to be able to see the beauty of a solution, a design, a construction ... One cannot do anything in mathematics without a sense of beauty. I remember how

[4] Isaak Konstantinovich Kikoin (1908–1984), physicist, from 1943, together with Kurchatov, was one of the main organizers and scientific directors of the Soviet atomic project.

> our eyes, the eyes of the young students of Leningrad University, were opened to the gigantic world of motion, to the depth of connections and causes of everything going on around, when for the first time they showed us how to calculate the frequency of the oscillations of a pendulum, knowing its length and the law of resistance of the medium in which it swings. It was a revelation that left an indelible impression for the rest of my life... I would not attempt to explain where the internal laws of mathematics come from and how they work. Aesthetic feeling plays a considerable role here. Any mathematician knows what a 'beautiful result' or an 'elegant proof' is. Sometimes, it is an unexpected thought that suddenly leads to a solution to a problem that has been elusive for a long time. Sometimes, it is a bold generalization, the ability to see in a specific situation the laws that govern phenomena. And sometimes it is the orderliness and wisdom of a complex concept, which, at the end of a long path of calculations and reasoning, produces a bright result... I know for a fact that often during reflection there is a desire to do what is prompted by some inner feeling.

Sobolev was an emotional man. Once, when he was doing calculations on the atomic bomb, someone knocked on his office door, but the lock would not open. So, he opened the door with a kick of his foot. The doctor diagnosed him with some kind of fracture and forbade him to leave the house for six months — and it was during this time that Sobolev wrote his seminal book *Some Applications of Functional Analysis in Mathematical Physics.*

He said in an interview that

> emotions are probably important, but mainly you need to be able to get into the subject, you need to be hardworking and to have intuition. Intuition only starts to work after you get used to the subject, and that takes time. If you are interested and passionate about an issue, you have to make it 'your own' in your head. Only then do fresh ideas begin to emerge. The creative process is preceded by a lot of preparatory work. Inspiration takes a lot of work. It doesn't come by itself. It all has to come together into some kind of system. You have to know the subject extremely well for your intuition to work.

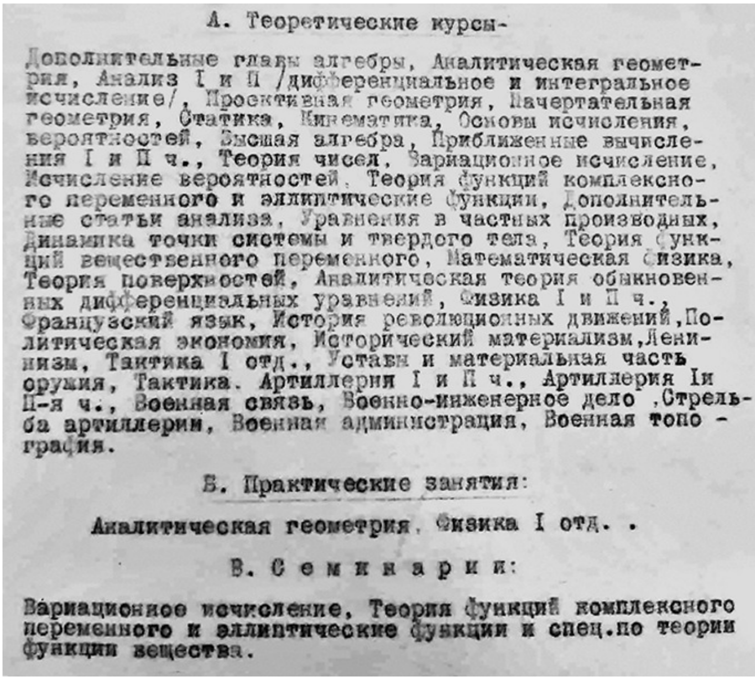
Sobolev was involved in the political life of the country. For example, on the one hand he defended genetics (he was one of the signatories of the "Letter of 300"),[5] cybernetics, and the application of mathematical methods in economics. On the other hand, Sobolev was one of the active participants of the anti-Luzin campaign[6] in 1936, a series of accusations completely destroyed Luzin's career and life, with questionable corroborative evidence. In 1956, together with Academicians Mikhail Lavrentyev and Sergey Khristianovich, he appealed to the Central Committee of the Communist Party with a proposal to create the Siberian Branch of the Academy of Sciences. The proposal was approved.

In 1958, Sobolev moved to Novosibirsk to work as director of the Institute of Mathematics of the Siberian Branch of the Academy of Sciences. Sobolev made sure that all the most important areas of modern mathematical science were represented at the institute. Sobolev himself was never engaged in either cybernetics or mathematical economics, but he did everything to support their development at the Institute. Sobolev was a talented administrator, but sometimes he trusted people too much. At the International Mathematical Congress in Stockholm in 1962, trusting his subordinates, Sobolev made a report on machine decoding of the Maya script (which was in fact made up from beginning to end).[7]

[5] The "Letter of the Three Hundred" was a letter sent to Soviet leadership on October 11, 1955, by a large group of Soviet scientists. It contained criticism of the scientific views and practical activities of then head biologist Trofim Lysenko, and eventually led to the resignation of Lysenko as well as that of some of his followers and protégés in the Academy of Sciences.

[6] Nikolai Luzin was a Russian and Soviet mathematician known for his work in descriptive set theory and aspects of mathematical analysis with strong connections to point-set topology. In 1936, Luzin was heavily criticized for a number of ideological "sins," including publishing his major results in foreign journals and being disloyal to Soviet authorities. The latter accusation was supported by Sobolev. A special meeting of the Commission of the Academy of Sciences of the USSR endorsed all charges of Luzin as an "enemy under the mask of a Soviet citizen." His department at the Steklov Institute was closed, and he lost all his official positions, but he was neither arrested nor expelled from the Academy. This decision was finally reversed in 2012. Aleksei Krylov and Sergei Bernstein defended Luzin, while Pavel Alexandrov, Andrei Kolmogorov, Alexander Khinchin, and Sergei Sobolev were against Luzin. Ivan Vinogradov remained neutral.

[7] The principle of the Maya script was established by Yuri Knorozov in 1955. In 1960, the staff of the computing center decided to automate the translation and a year later reported


А. Теоретические курсы-

Дополнительные главы алгебры, Аналитическая геометрия, Анализ I и II /дифференциальное и интегральное исчисление/, Проективная геометрия, Начертательная геометрия, Статика, Кинематика, Основы исчисления вероятностей, Высшая алгебра, Приближенные вычисления I и II ч., Теория чисел, Вариационное исчисление, Исчисление вероятностей. Теория функций комплексного переменного и эллиптические функции, Дополнительные статьи анализа, Уравнения в частных производных, Динамика точки системы и твердого тела, Теория функций вещественного переменного, Математическая физика, Теория поверхностей, Аналитическая теория обыкновенных дифференциальных уравнений, Физика I и II ч., Французский язык, История революционных движений, Политическая экономия, Исторический материализм, Ленинизм, Тактика I отд., Уставы и материальная часть оружия, Тактика. Артиллерия I и II ч., Артиллерия I и II-я ч., Военная связь, Военно-инженерное дело, Стрельба артиллерии, Военная администрация, Военная топография.

Б. Практические занятия:

Аналитическая геометрия. Физика I отд. .

В. Семинарии:

Вариационное исчисление, Теория функций комплексного переменного и эллиптические функции и спец.по теории функции вещества.

From Sobolev's university diploma.

Sobolev's wish for young scientists seems to be a fitting ending:

> What is the most important thing a scientist should cultivate in himself? One should get rid of excessive ambition. One should not think that only a genius can be happy. One must learn to appreciate even a small achievement, to rejoice in it, and never overestimate oneself. One has to cultivate a love for work. One has to understand and cultivate the joy of learning, which is almost the same as the joy of life. Happiness is when your life's work is needed.

Nikita Kalinin

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a tremendous success, with deciphered phrases about ancient Maya fire-making, the firing of white clay vessels, various priestly rituals, etc.

Sobolev embedding theorems

Sergei Lvovich Sobolev made a fundamental contribution to the development of modern mathematics, to the formation of major scientific schools, to the creation and development of new areas of applied mathematics. The foundations laid by S.L. Sobolev for the application of functional analysis in mathematical physics (the concept of a generalized derivative, a generalized solution of a differential equation, embedding theorems, generalized functions) have been developed in numerous studies in our country and abroad.

In the history of science, the name of Sergei Lvovich will forever be associated with one of the most fundamental mathematical concepts of the 20th century — the theory of generalized functions, which opened up wide opportunities for research in the field of partial differential equations, theoretical physics, and mechanics.

According to Sobolev, generalized functions are functionals on the space of compactly supported functions (that is, functions equal to zero outside a certain bounded domain) of a certain smoothness. For generalized functions, the concepts of linear operations, differentiation, and multiplication by a sufficiently smooth function were introduced. S.L. Sobolev applied the theory of generalized functions to the study of solutions of hyperbolic and elliptic partial differential equations.

On the basis of the theory of generalized functions, S.L. Sobolev introduced the function spaces $W_p^{(l)}$, now called Sobolev spaces, consisting of functions whose generalized derivatives of order l are p-integrable.

Let f and g be locally integrable functions on a domain Ω in $\mathbb{R}^n$. The function g is called the generalized derivative $D^\alpha f = \dfrac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$ of f, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \sum \alpha_i$, if the equality

$$\int_\Omega f(x) D^\alpha \varphi(x) dx = (-1)^{|\alpha|} \int_\Omega g(x) \varphi(x) dx$$

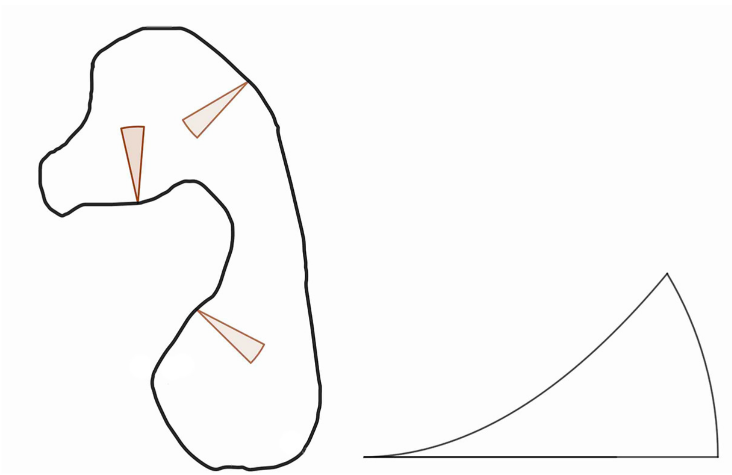
holds for any infinitely differentiable finite function φ on Ω. The set of all functions f defined on Ω, for which all generalized derivatives $D^\alpha f$ of order l, $|\alpha| = l$, exist and are p-integrable, $p \geq 1$, is a Banach space $W_p^{(l)}(\Omega)$ with the norm

$$\|f\|_{W_p^{(l)}(\Omega)} = \|f\|_{L_p(\Omega)} + \sum_{|\alpha|=l} \|D^\alpha f\|_{L_p(\Omega)},$$

where $L_p = L_p(\Omega)$ is the Lebesgue space of p-integrable functions with the following norm: $\|f\|_{L_p(\Omega)} = \{\int_\Omega |f(x)|^p dx\}^{1/p}$.

S.L. Sobolev studied connections of the spaces $W_p^{(l)}(\Omega)$ with Lebesgue spaces and with spaces of continuous functions, he characterized the trace spaces for functions from these spaces on surfaces of different dimensions. These results, now called the Sobolev embedding theorems, laid the foundation and gave rise to the rapid development of a new research direction in the theory of functions and in the theory of partial differential equations. Let us describe the results of Sergei Lvovich on embedding theorems for function spaces.

The main results on the embedding of the spaces $W_p^{(l)}(\Omega)$ refer to domains Ω satisfying the cone condition. This means that each point of Ω can serve as a vertex of a cone of constant height and opening located in Ω. In what follows, we assume that the domain $\Omega \subset \mathbb{R}^n$ satisfies the cone condition.



Areas with a smooth border, as shown on the left, satisfy the cone condition. In regions with external peaks, like the region on the right, the relations between the parameters in the embedding theorems turn out to be different.

Let Ω_m, $1 \le m \le n$, denote the section of Ω by a plane of dimension m, $\Omega_n = \Omega$, and $1 < p < q < \infty$. Let us present Sobolev theorems on embeddings into the space $L_q(\Omega_m)$ of q-integrable functions on Ω_m and into the space $C(\Omega)$ of continuous and bounded functions on Ω with the norms $\|f\|_{L_q(\Omega_m)}$, $\|f\|_{C(\Omega)} = \sup_\Omega |f|$, respectively.

Let $1 < p \le q < \infty$. Then

$$W_p^{(l)}(\Omega) \subset L_q(\Omega_m) \text{ for } l - n/p + m/q \ge 0, \tag{1}$$

$$W_p^{(l)}(\Omega) \subset C(\Omega) \text{ for } l - n/p > 0. \tag{2}$$

ГЛАВА II

СВЯЗЬ МЕЖДУ ПРОСТРАНСТВАМИ $L_p^{(\nu)}$

§ 6. Новое определение производной

Пусть $\varphi(x_1, \ldots, x_n)$ — некоторая функция переменных $x_1, \ldots, x_n$, суммируемая в области D. Пусть D' — какая-нибудь другая область, лежащая внутри D вместе со своей границей.

Рассмотрим какую-нибудь функцию $\psi(x_1, \ldots, x_n)$, имеющую непрерывные производные до порядка α включительно и уничтожающуюся вне D'. Если функция φ сама имеет непрерывные производные до порядка α, то справедливы равенства

$$\overbrace{\int \ldots \int_D}^{n} \left(\varphi \frac{\partial^\alpha \psi}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} + (-1)^{\alpha-1} \psi \frac{\partial^\alpha \varphi}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} \right) dx_1 \ldots dx_n = 0. \qquad (6, 1)$$

Мы будем рассматривать это равенство как определение производной и называть производной от φ,

$$\frac{\partial^\alpha \varphi}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}},$$

такую функцию переменных $x_1, \ldots, x_n$, суммируемую по любой ограниченной части области D, которая удовлетворяет условию (6, 1) при всяких ψ.

An excerpt from Sobolev's work in the journal *Matematicheskiĭ Sbornik* (1938) where he defines the generalized derivative.

It is necessary to clarify that in the theory of Sobolev spaces (as in the theory of Lebesgue spaces), equivalent functions, that is, functions differing on a set of measure zero, are identified. So the embedding (2) means that every function in $W_p^{(l)}(\Omega)$ is equivalent to a continuous one. The embedding (1) for $m<n$ shows that the Ω_m-traces of functions from $W_p^{(l)}(\Omega)$ belong to $L_q(\Omega_m)$. Moreover, for a continuous function f, the Ω_m-trace of f is defined as the restriction of f to Ω; for an arbitrary function $f \in W_p^{(l)}(\Omega)$, the trace is defined as the limit of traces in $L_q(\Omega_m)$ for a sequence of functions continuous on Ω and approximating f in $W_p^{(l)}(\Omega)$.

The embeddings (1) and (2) mean not only the fact that each function from $W_p^{(l)}(\Omega)$ belongs to the spaces $L_q(\Omega_m)$ and $C(\Omega)$, respectively, but also the following inequalities:

$$\|f\|_{L_q(\Omega_m)} \le C\|f\|_{W_p^{(l)}(\Omega)},$$
$$\|f\|_{C(\Omega)} \le C\|f\|_{W_p^{(l)}(\Omega)}$$

for all $f \in W_p^{(l)}(\Omega)$; the constant C in the above inequalities does not depend on f.

S.L. Sobolev established the embedding (1) for $m=n$ and the embedding (2). In the case where $m<n$, the embedding (1) was established for

$p=q=2$. For other values of p and q in this case, the proofs are due to V.I. Kondrashov and V.P. Il'in.

To prove the embeddings (1) and (2), Sergei Lvovich proposed a new research method — the method of integral representations for functions of several variables in terms of derivatives. Such a representation has the following form:

$$f(x) = \sum_{|\alpha|\le l-1} x^\alpha \int_\Omega \zeta_\alpha(y) f(y)\,dy + \int_\Omega \frac{1}{|x-y|^n} \sum_{|\alpha|=l} \omega_\alpha(y,x) D^\alpha f(y)\,dy,$$

where $x^\alpha = (x_1^{\alpha_1} \dots x_n^{\alpha_n})$. The principal part of the above representation is the potential type integrals for derivatives of the function under consideration. With the help of such representations, the problem of proving embedding theorems is reduced to appropriate estimates for integrals of potential type.

After the fundamental work of Sergei Lvovich, the theory of embedding of function spaces has been actively developed in many different directions. One had weakened the conditions on the domain under which the embeddings (1) and (2) hold. One had obtained the embedding theorems depending on the form of the domain Ω, one had studied Sobolev type spaces, the norms of which did not include all derivatives of order l, and one had investigated anisotropic Sobolev spaces, in which the norms were composed of the norms of derivatives of different orders in different variables. Special attention was given to the study of the exact characteristics of the traces of functions from the space $W_p^{(l)}(\Omega)$ on the boundary of Ω. Such characteristics were given in terms of other function spaces of positive smoothness, which stimulated the development of the theory of these spaces. It is difficult to present all the directions of development of the theory of embeddings of function spaces.

The theory of Sobolev embedding theorems and applications of this theory to the solution of differential equations are presented in his book "Applications of functional analysis in mathematical physics" (1950) published by Leningrad State University. The third edition of this book (1988) contains useful additions and remarks, as well as a far from complete list of papers reflecting the development of the theory of generalized functions and function spaces by S.L. Sobolev and many applications of this theory.

Oleg Besov

George G. Lorentz (1910–2006)

Russian-American mathematician George G. Lorentz was born at 12 Pesochny Street,[1] St. Petersburg. His father, Rudolf Fedorovich Lorentz, was a German railway engineer on the private Moscow-Vindava-Rybinsk Railroad. Rudolf Fedorovich was fired from state railways for sympathizing with workers during a strike in 1906. George Lorentz's mother, Milena Nikolaevna, was the daughter of Prince N.V. Chegodaev, who was a Lieutenant Colonel and a teacher of the Nikolai Cadet Corps. In this family, the men were military personnel and/or engineers. Her sister, Yelizaveta, was a physician in St. Petersburg in the 1930s. In 1912, R.F. Lorentz relocated to work for railway companies in the Caucasus. The family lived in Armavir (1913–1918), then in a village near Sochi (1919–1922), then in Tbilisi. Lorentz studied at a Russian school (1923) and later at a German one. In 1926, George Lorentz was accepted to the polytechnical department of Tbilisi State University.[2] His successes in mathematics were so remarkable that his professors, N.I. Muskhelishvili and A.M. Razmadze, advised him to apply to Leningrad State University.

In 1927, Lorentz was accepted as a first-year student at Leningrad State University. He called himself a student of G.M. Fichtenholz, N.S. Koshlyakov, and A.M. Zhuravsky. Lorentz graduated in 1931, and defended his Candidate's thesis on Bernstein polynomials in 1935. From 1936 to 1942, he taught mathematical analysis at the university. He also taught at the Herzen State Pedagogical Institute for some time. He published several papers: On Methods of Linear Summation (1932), Functionals and Operations on Spaces of Sequences (1935), On the Convergence of Stieltjes–Landau Polynomials (1936), About the Theory of Bernstein Polynomials (1937) — the last two of which were topics of his thesis. He edited Ya.S. Besicovitch's

[1] The street is now Professora Popova Street.

[2] Since 1928 — Georgian State Polytechnic Institute, since 1990 — Georgian Technical University.

book on approximations in calculus. At the time, Lorentz lived at Demidov Lane,[3] house No.3 [1].

In 1937, Lorentz's father, who by then was a professor of railway engineering at the Tbilisi Polytechnic Institute, was arrested and wrongfully accused of espionage. He was sentenced to eight years in a labor camp, where he died after one year. This deeply disturbed G. Lorentz. He wrote that despite having support from G.M. Fichtenholz, he never finished his half-written textbook on functional analysis, and barely did any scientific work until 1942. In January of 1942, under Case No. 555 "Union of the Old Intelligentsia", many mathematicians were arrested, including B.I. Izvekov, whose family was close to Lorentz's. The NKVD (People's Commissariat for Internal Affairs) started summoning Lorentz, as well, and danger loomed over him.

When WWII reached the Eastern Front, Lorentz was mobilized as a private in an air defense unit. In April of 1942 he and his wife, Tatyana Pavlovna Belikov (Tanny Belikov), were evacuated to Kislovodsk along with the staff of the Pedagogical Institute. Soon, the Germans occupied the city, and Lorentz was registered as an ethnic German. At the beginning of January 1943, the German troops left Kislovodsk, and Lorentz and his wife were moved to a displaced persons camp in Poland, where his son, Rudolf, was born.

In 1943 he sent two of his papers to Konrad Knopp, and in 1944 he was invited to the University of Tübingen,[4] where he became E. Kamke's assistant.[5] He wrote his thesis, *Einige Fragen der Limitierungstheorie* (Some Questions of Limit Theory), under Knopp's supervision, and got his Doctorate degree in 1944. By the end of WWII, Tübingen was under French control. The French government saw Lorentz as an unwanted foreigner and did not let him become a full professor at the University of Tübingen. In the spring of 1946, Lorentz made his way to the American occupation zone, where he was certified as a stateless person. Lorentz lived 13 years with that document until his naturalization in the USA. After the war, he finished his habilitation[6] in Tübingen, and taught at the Goethe University Frankfurt (1946–1948), then at the University of Tübingen (1948–1949) as an honorary professor,[7] In 1946 Georg Rudolfovich Lorentz changed his name to Georg Gunter Lorentz, a name that (according to him) he made up, and later to George G. Lorentz, which he kept for the rest of his life.

Lorentz remembered the German period of his life as hard and full of deprivations, but scientifically productive. Lorentz wrote about 20 papers

[3] Since 1953, the alley is called Grivtsova Lane.

[4] In his autobiography, Lorentz wrote: "We got lucky, we wanted to be as far from Soviet influence as we could," (3, p.5). He moved to Tübingen with his wife and new-born son, and the Lorentz family had four more daughters there.

[5] Because of his opposition to the Nazism, and because his wife was Jewish, Kamke had to retire, but he was allowed to write books.

[6] This granted him the right to lecture.

[7] Honorarprofessor, so a teacher with a regular salary.

on the theory of differential equations, the Fourier series, and the use of permutations, some of which he wrote with Kamke and Knopp. He gave lectures on Banach spaces to the teacher-professor staff.

Lorentz had an option to seek help of the United Nations Relief and Rehabilitation Administration (UNRRA), but he and his wife wanted to stay as far away from the Soviet occupation zone[8] as possible. They made an independent choice: Canada.

In 1949, Lorentz received a grant from a Canadian charitable foundation[9] and moved to Toronto, where he started as an assistant and later became a teacher at the university. He published his first book Bernstein Polynomials, supervised doctoral students, and gave lectures that included his own results, as well.

Dated March 29, 1931 — Lorentz's class at Leningrad's University. Top row (from the left): Izrail Isaakovich Gordon, Boris Nikolaevich Sokolov, Vasily Nikitich Galich. Bottom row (from the left): Nikolai Nikolaevich Markovets, Maria Danilovna Inpits (?), George Rudolfovich Lorentz. This photo came from I.I. Gordon's private archive, and it's his writing on the back, as well. It was published in the journal Seven Arts in 2011, release 11(24), by E.I. Gordon. I express my appreciation to G.M. Polotovsky, who sent me this photograph.

In 1953, Lorentz was offered a job as a full professor at Wayne State University in Michigan where he started working in approximation theory and

[8] A decree from the Presidium of the Supreme Soviet of the USSR on April 19, 1943: "On the penalties for German-Fascist villains, responsible for the murder and torture of the Soviet population and captured soldiers, for the spies, for the traitors — Soviet citizens who betrayed their homeland, and for their accomplices," the penalties were: death by hanging for the "traitors," and a sentence of 15–20 years of hard labor for the "accomplices."

[9] The Lady Davis Foundation.

taught there until 1959. Then, he was a professor at Syracuse University in the state of New York (1959–1969), where he wrote his famous book, *Approximation of Functions*; then he was a professor at the University of Texas at Austin (1969–1980). In 1973, he spent some time in Stuttgart, with a prize from the German Humboldt Foundation[10] for his scientific achievements. Many of Lorentz's students became successful scientists. In 1980, Lorentz retired, but he continued doing scientific research for 15 more years.

Lorentz's main results belong to mathematical analysis: he has fundamental theorems in approximation theory and functional analysis, he developed theories of interpolation of operators. He also worked in number theory. In his books, he included works of Russian mathematicians, making them available to Western readers. Lorentz spaces, as a generalization of L^p spaces, were introduced by him in two papers: *Some New Function Spaces* (1950) and *On the Theory of Spaces* Λ (1951). Lorentz was called the king of modern approximation theory. For more details, see [3, 4], I especially recommend G. Lorentz's paper *Mathematics and Politics in the Soviet Union from 1928 to 1953* [5].

Let's include Evgeny Izrailevich Gordon's memory in this article:

> I will write how Lorentz and I came into contact. For a long time, my father didn't know what became of Lorentz, other than the fact that he left with the Germans. Sometime in the '50s, Fichtenholz told him with resentment that Lorentz was seen under German occupation basically wearing an SS uniform. My father, who was a very anti-Soviet man, did not believe him. He decided it was a lie deliberately spread by the KGB, that Fichtenholz, who was a very Soviet man, believed. As you know, to look for Lorentz or even try to get information on him in the Soviet Union was very dangerous.
>
> However, in 1966, at the International Congress of Mathematicians in Moscow, my father came up to the topologist Mac Lane, who knew his name from "Gordon rings," and asked him about Lorentz. Mac Lane told him that Lorentz was a professor in Texas, and that he had seven children. My father didn't learn anything else about Lorentz. Sometime in the mid-90s, Vladimir Mikhailovich Tihomirov told me that there was a special release of the Journal of Approximation Theory dedicated to Lorentz's 80th birthday that contained Lorentz's autobiography, which mentioned my father in it. Tihomirov sent me a copy of that article, which I couldn't find now, but I think you can easily find it — it was published in the early '90s. I don't remember whether it was from that or other publications and stories that I learned the details of Lorentz's life.
>
> His father was arrested before the war and he died in a camp when Lorentz was still in Leningrad. He was in the Siege of Leningrad,

[10] This prize gives highly qualified scientists from different countries the opportunity to work on a scientific project of their choice with German scientists.

but sometime around 1942 Lorentz, who was heavily ill, was taken out of Leningrad. Germans were chasing them the whole time they were escaping the city. In the end, they left completely exhausted Lorentz in Tuapse. When the Germans came to that town, they didn't touch Lorentz because he was of German descent and gave him an easy job as a clerk in the office. Aron Grigoryevich Pinsket told his late student, Vladimir Aronovich Geyler, that one night, Lorentz ran to him and warned him that the Germans were planning to kill all the Jews in Tuapse the next day. They warned a few more people, and they all escaped the town that night, saving themselves in the process. Vladimir Geyler, my close friend, told me that himself.

When the Germans were leaving the USSR, they took Lorentz with them. He went to Tübingen, where he took Kamke's (author of a famous textbook on differential equations that was translated to Russian) vacated place as head of the department. A professor from the University of Tübingen, Manfred Wolf, told me that. Wolf and Lorentz became friends when Lorentz visited Tübingen from America.

I found Lorentz's address from the note that Tihomirov sent me, and I sent him a letter telling him of my parents' lives. I added a photograph of their class and asked for the names of the other people on it. He replied quite quickly. Unfortunately, I couldn't find his letter today. I remember one sentence about my dad: 'He introduced us to English literature — Maugham and Joyce.' He told me the names of everyone in that photograph and described them as mathematicians, but said he knew nothing of what became of them. His last phrase in the PS stood out to me: 'I never served in any army — not in the Soviet nor in the German one.' I might add that I didn't mention anything about his departure to Germany in my letter. I thought his response didn't imply a continuation of the conversation, so I replied in short, thanking him for his letter. In the summer of 1999 I moved to America. When I arrived, I didn't call Lorentz, because I was scared that he might think I needed help from him.

My wife had stayed in Russia for a little while longer. Sometime in the beginning of 2002, she told me that a letter came from Lorentz containing some of his papers and asking for my opinion on them. I called him, told him that I was in America and asked him to send those papers to my American address. He sent them to me. After that, we regularly talked on the phone, until he died of a cold in 2006. His son was the one who told me of his death. His last postcard to me was dated April 4, 2005 — from Hawaii! He invited me to his house in Chico, California, where he lived after he retired. In the nursing house, he had a four-room apartment! I was planning on going, but planned for too long...

Well, I have written everything I know about Lorentz. I hope some of it was helpful.[11]

[11] Private message from E.I. Gordon in his letter to G.I. Sinkevich.

Galina Sinkevich

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[5] Lorentz, G.G. (2002) Mathematics and Politics in the Soviet Union from 1928 to 1953, *Journal of Approximation Theory*, 116, pp. 169–223.

[6] http://7iskusstv.com/2011/Nomer11/EGordon1.php

Lorentz Spaces

Much of science and engineering centers on understanding functions that describe physical, chemical, or biological processes. In mathematics, providing such understanding organizes itself into the disciplines of real, complex, and functional analysis. At the coarsest level, functions are typically classified in terms of either their size, or their smoothness, or self-similarities. The earliest measures of size were boundedness and integrability. Smoothness tries to understand how the output of a function depends on small variations in the input and has led to a myriad of notions circling around differentiability. Self-similarities include periodicity and fractal structures.

The best-known quantifications of size are given by Lebesque L_p norms (formulated by M. Riesz). If f is a real-valued measurable function defined on a measure space (Ω, μ), the normed linear space $L_p := L_p(\Omega, \mu)$, $1 \le p \le \infty$, consists of all μ measurable functions f for which the norm

$$\|f\|_{L_p} := \|f\|_{L_p(\Omega,\mu)} := \begin{cases} \left(\int_\Omega |f|^p \, d\mu\right)^{1/p}, & 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{x\in\Omega} |f(x)|, & p = \infty, \end{cases} \tag{1}$$

is finite.

In their work, Hardy and Littlewood stressed the fact that L_p is a *rearrangement invariant space*. To explain what this means, we introduce for any μ measurable function f its *distribution function*

$$\mu_f(y) := \mu\{x \in \Omega : |f(x)| > y\}, \quad y \ge 0. \tag{2}$$

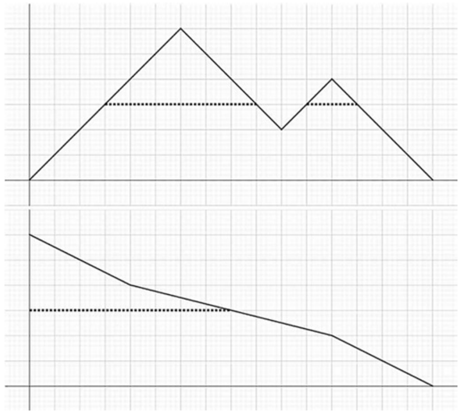
This distribution function is decreasing on $\mathbb{R}^+$. All L_p norms of f are determined from μ_f via the identity

$$\|f\|_{L_p}^p = p \int_0^\infty y^{p-1} \mu_f(y) \, dy, \quad 1 \le p < \infty. \tag{3}$$

Thus, for the purpose of measuring size by L_p norms, two functions f, g which have distribution functions equal almost everywhere have identical L_p norms irrespective of their native measure spaces.

Two measurable functions f and g are said to be equi-measurable if $\mu_f(y) = \mu_g(y)$, a.e. on $\mathbb{R}^+$, with respect to the Lebegue measure. Notice that μ_f is always defined on $\mathbb{R}^+ := [0, \infty)$. One can select a representative among all equi-measurable functions as

$$f^*(t) := \inf\{y \colon \mu_f(y) \le t\}, \quad t \ge 0, \tag{4}$$



A function and its monotonic rearrangement.

which is also a non-increasing function defined on $\mathbb{R}^+$. The function f^* is called the *decreasing rearrangment* of f. It encodes all size properties of f relative to L_p norms. For example,

$$\int_\Omega |f|^p \, d\mu = \int_0^\infty [f^*(t)]^p \, dt, \quad 1 \le p < \infty. \tag{5}$$

In his seminal paper [8], George Lorentz suggested using the rearrangement f^* as a general way to define spaces of functions described by size. The idea is to simply apply a function norm to f^*. He suggested a cadre of possibilities. The most important of these turned out to be the Lorentz spaces $L_{p,q} := L_{p,q}(\Omega, \mu)$, described as all μ measurable functions f defined on Ω for which

$$\|f\|_{L_{p,q}} := \begin{cases} \left\{ \int_0^\infty [t^{1/p} f^*(t)]^q \, \frac{dt}{t} \right\}^{1/q}, & 1 \le p, q < \infty, \\ \sup\limits_{t>0} t^{1/p} f^*(t), & 1 \le p < \infty, \ q = \infty, \end{cases} \tag{6}$$

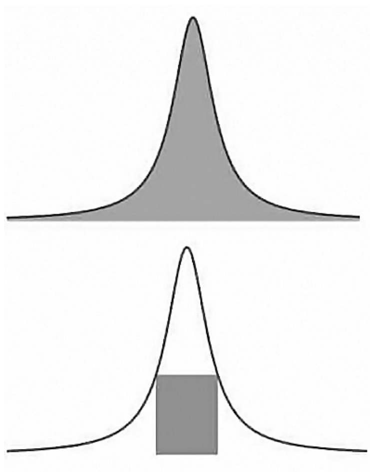
is finite. Notice that these spaces agree with L_p, $1 \le p < \infty$, when we choose $q = p$ but give a fine gradation of spaces near L_p when p is fixed and q varies. The most important of these variants turns out to be the case $q = \infty$, which gives the spaces $L_{p,\infty}$ commonly referred to as weak L_p spaces. The membership condition for weak L_p is that

$$\mu\{x \in \Omega \colon |f(x)| > y\} \le C y^{-1/p}, \quad y > 0, \tag{7}$$

with C an absolute constant.

The subsequent impact of [8] was quite profound in a range of disciplines including harmonic analysis, approximation theory, partial differential equations, and functional analysis. On the one hand, the new Lorentz spaces $L_{p,q}$ were instrumental in explaining the mapping properties of certain fundamental operators of analysis such as the Fourier, Hilbert, and Riesz transforms. On the other hand, they served as the prototype for the development of new function spaces such as Besov classes and approximation classes that remain important to this day in our understanding of numerical methods. Perhaps, the most important consequence was the development of a new branch of analysis called interpolation of operators which helped unify several mathematical disciplines.

Regarding the latter point, the modern theory of interpolation of operators can rightfully be traced to the work of M. Riesz who was interested, among other things, in the mapping properties of the Fourier transform $\mathcal{F}$. The Hausdorff–Young theorem showed that $\mathcal{F}$ mapped L_p into $L_{p'}$, $1 \leq p \leq 2$, where p' is the conjugate index given by $1/p + 1/p' = 1$. Riesz showed that the Hausdorf–Young theorem is actually a consequence of a general principle that any linear operator which is boundedly mapping L_1 to L_∞ and L_2 to itself will necessarily boundedly map L_p to $L_{p'}$ for $1 \leq p \leq 2$. This result was later formally put into interpolation theorems (such as the Riesz–Thorin theorem) and started the study of interpolation of operators. The Lorentz spaces played a significant role in this new development in several important ways.



The Lebesgue L_1 norm corresponds to the area of the subgraph (for positive functions), while the $L_{1,\infty}$ norm corresponds to the area of the maximal inscribed "rectangle."

One of the main chapters of harmonic analysis is the Calderón–Zygmund (CZ) program to understand the mapping properties of the fundamental operators that arise in the theory of differential equations. These include singular integrals and maximal operators such as the well-known Hardy–Littlewood maximal operator. The main vehicle for proving mapping properties of operators is the above theory of interpolation. A prototypical example in the CZ program is the Hilbert transform. M. Riesz showed that this operator mapped L_p into itself when $1 < p < \infty$ but does not map L_1 into itself. This is a typical example where the "endpoint" behavior of the operator is opaque. This endpoint behavior is nicely filled in via the Lorentz spaces. In the case of the Hilbert transform, it turns out that it maps L_1 into weak L_1, i.e., into $L_{1,\infty}$. While this result is interesting in and of itself, even more is true. Namely, any operator that boundedly maps L_1 to weak L_1 and L_2 into itself will automatically boundedly map L_p into itself for $1 < p \le 2$. In other words, to ascertain strong mapping properties of linear operators, it is enough to verify weak mapping properties on pairs of spaces and deduce the strong mapping via interpolation for intermediate spaces.

The subject of interpolation of linear operators began to organize itself into unified studies in the late 1960s (see the treatises [1, 2]). Calderón introduced his complex method of interpolation [4] and subsequently his treatment of weak interpolation via rearrangements [5]. At more or less the same time, the real method of interpolation initiated by Lions and Peetre came to the forefront. The main theme of the real method of interpolation was to show that a linear operator T which boundedly maps Banach spaces $X_i \to Y_i$, $i = 0, 1$, automatically maps $X \to Y$ for a staple of new spaces X, Y. These new spaces are defined via a K-functional which, in the simplest case when X_1 continuously embeds into X_0, is

$$K(f,t) := K(f,t;X_0,X_1) := \inf_{g \in X_1} \|f-g\|_{X_0} + t\|g\|_{X_1}, \quad 0 < t < \infty. \tag{8}$$

Examples of the new intermediate spaces X are the (θ, q)-spaces $X_{\theta,q} = (X_0, X_1)_{\theta,q}$ consisting of all $f \in X_0$ for which

$$\|f\|^q_{X_{\theta,q}} := \int_0^\infty [t^{-\theta} K(f,t)]^q \, dt/t, \quad 0 < \theta < 1;\ 0 < q < \infty, \tag{9}$$

with the obvious modification for $q = \infty$. One can rather easily show that the above operator T boundedly maps $X_{\theta,q}$ into $Y_{\theta,q}$ for each $\theta \in (0,1)$ and $0 < q \le \infty$. While the similarity between these spaces and the Lorentz spaces is apparent, the connections have an even happier ending since when the measure μ is finite,

$$K(f,t;L_1,L_\infty) = \int_0^t f^*(s)\, ds, \quad t > 0, \tag{10}$$

and

$$(L_1, L_\infty)_{\theta,q} = L_{p,q}, \quad \theta = 1 - 1/p. \tag{11}$$

A similar connection occurs when engaging smoothness spaces. In a development quite similar to the Lorentz spaces, O. Besov [3] introduced a fine gradation of smoothness spaces that carry his name. The space $B^s_q(L_p(\Omega))$ consists of functions with smoothness of order s in L_p with q playing the same role of fine gradation as in the case of Lorentz spaces. The connections with Lorentz spaces become even clearer when one realizes that these Besov spaces are interpolation spaces between L_p and the Sobolev space $W^r(L_p(\Omega))$ when $r > s$. Indeed, the K functional for this pair is none other than the modulus of smoothness $\omega_r(f, t)_p$, and so the norm in the Besov space takes the form (9). Besov spaces are now commonly used in the study of solutions to partial differential equations and numerical analysis of these equations. Finally, let us mention that interpolation spaces, which can be viewed as descendants of the Lorentz spaces, are also a staple in approximation theory where they are used to characterize the functions which possess a specified approximation order (see [7]).

Ronald DeVore

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Leonid Vitalyevich Kantorovich (1912–1986)

Leonid Kantorovich contributed both to pure and applied mathematics and always wished for these fields to inform each other. To this day he is the only Russian/Soviet — and, indeed, the only East European — laureate of the Nobel Memorial Prize in economics. His pioneering mathematical contributions include the theory of vector lattices, transport theory, approximation methods (particularly in their relation to functional analysis), and linear programming. As an economist, he is renowned for advocating the optimization methods in planning. He is also recognized as a founder of an entirely new field of study, which some have even labeled a 'mathematical revolution' in Soviet economics [9].

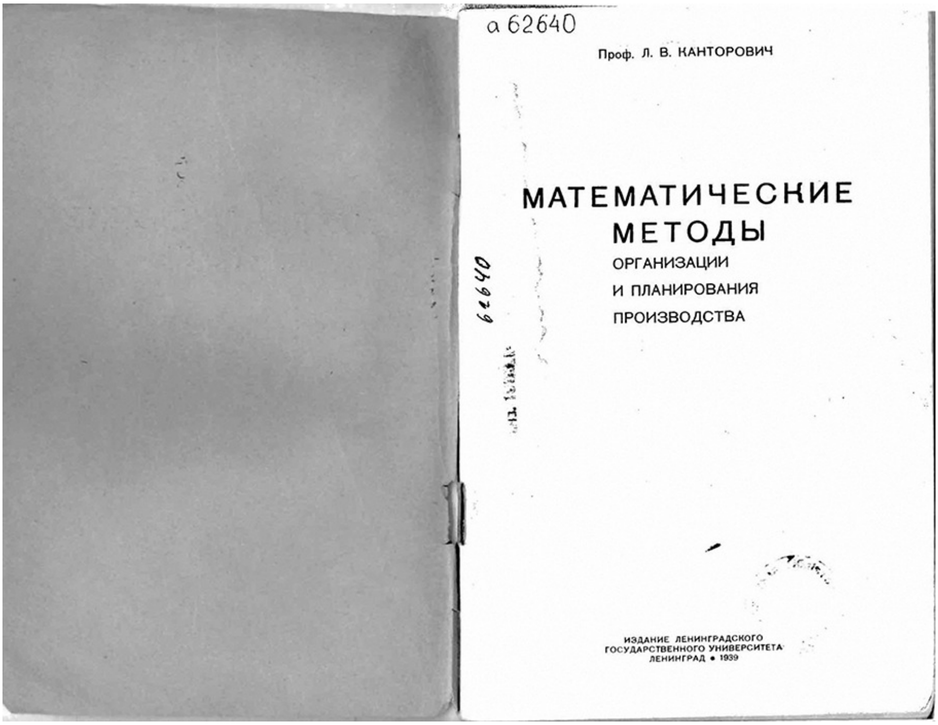
Kantorovich was born in 1912 in Saint Petersburg, to a middle-class Jewish family. A prodigy, he entered university at 14 and graduated at 18, having already published more than ten papers in Russian and French. Nurtured by his teachers Grigory Fichtenholz and Vladimir Smirnov, he very quickly became one of the young leaders of the Leningrad mathematical school, and also established links to the Moscow mathematical community.

Kantorovich started his mathematical work with real analysis. Early on, he also became interested in applied methods, both in numerical analysis (especially in finding solutions to partial differential equations) and in optimization. His key contributions to 'pure' mathematics consist in developing the theory of vector lattices (1935–1937) and in providing an abstract version of — and some important results in — optimal transportation theory (1942). One of the significant postwar results in applied mathematics [1] deals with the convergence of Newton's method.

In 1938, in response to a request from a group of engineers of a plywood trust, Kantorovich formulated and solved several problems in what later came to be called linear programming (linear optimization with linear constraints). His insightful use of convexity theory and real analysis in solving economic problems, as well as, crucially, the interpretation of dual variables as prices, was a pioneering contribution to economics [5]. In 1942, during World War II, he was evacuated from Leningrad with his family. Kantorovich's mother passed away and his 9-month-old son died during the evacuation. As early as in 1942, despite his multiple responsibilities related to the defense efforts, Kantorovich completed a longer manuscript on economics and became a fervent advocate of applying new optimization methods to improve Soviet planning. Specifically, at the end of the 1940s, Kantorovich and his former student Victor Zalgaller were working on optimization techniques for the cutting of materials [7]. Kantorovich's efforts towards the implementation of his ideas, renewed after the 20th Party Congress of 1956 and the wave of de-Stalinization, were met with hostility by the planning authorities and Soviet economists alike. While the former were resistant to change, the latter suspected that Kantorovich's approach to optimal planning was at odds with the official doctrine of Marxist political economy. Because of this, the book he had mostly completed by 1942 was only published in 1959.

Kantorovich wanted to change both the economic organization of the Soviet Union and the culture of its analysis. While he definitely failed in the former task, the latter was a partial success. His influence grew after he received the Stalin Prize in 1949 for his work on functional analysis and applied mathematics and also for his pivotal contribution to the Soviet nuclear project. Calling upon his well-established authority in the Academy of Sciences, his connections to influential Soviet scientists, such as Andrei Kolmogorov, Sergei Sobolev, and Mikhail Lavrentyev, and relying on the support of mathematically minded Soviet economists, such as Vasily Nemchinov, Valentin Novozhilov, and Albert Vainshtein, Kantorovich institutionalized what later became known as 'mathematical methods in economics' in both research and teaching [3]. Thus, Soviet mathematical economics had been born. Across the Soviet Union linear and non-linear optimization, input-output analysis, and even game theory were taught and researched as part of this new field. This became possible thanks to Kantorovich's efforts.

In 1956, Tjalling Koopmans, a Dutch-American mathematical economist, reached out to Kantorovich after reading his short abstract paper on optimal transportation theory. The result of this interaction was the publication of several papers — and a book — in English [8] and the almost universal recognition of Kantorovich's priority in introducing the basic problems and methods of linear optimization [2]. In 1960, Kantorovich moved to Novosibirsk to take part in building the Siberian branch of the Soviet Academy of Sciences. During this time, mathematical methods in economics were officially accepted. Mathematics was now seen as having the potential to improve the Soviet economic system and thus contribute to the success of socialism. In Novosibirsk, Kantorovich and his colleagues (notably Gennady Rubinstein and Valery Makarov) continued developing the ideas behind 'optimal planning', including



The cover page of the book *Mathematical methods in the organization and planning of production*, 1939.

its dynamic versions. Some particular applications of linear optimization were also underway [4].

That period of Kantorovich's career as well as the years that followed are not well-documented. The archival data is still unavailable, both in Novosibirsk and in Moscow — where Kantorovich moved to in 1971, working first at the State Committee for Science and Technology and later, from 1976 onwards, for the newly founded Institute of Systems Research. It might be that in the coming years, we will learn more about Kantorovich's role in Soviet academic and policy debate.

In 1975, Kantorovich shared a Nobel Memorial Prize in economics with Koopmans. Koopmans was frustrated that the Nobel committee did not award the prize to George Dantzig, a mathematician who introduced the simplex method, a powerful technique for solving linear programming problems, and who generally contributed to the creation of linear optimization as a separate field of applied mathematics. Koopmans even suggested jointly declining the prize. But it didn't happen. Other complications were political: this same year, Andrei Sakharov got the Nobel Peace Prize. The press in Stockholm cared less about optimal planning and more about how Kantorovich, at that time a perfectly loyal Soviet citizen and bureaucrat, felt about getting the prize together with Sakharov.

During the last decades of his life, Kantorovich was less active academically but remained quite active as a policy expert and was supporting research in mathematical economics and related fields [6]. His mathematical insights, his extraordinary ability to delve into economic problems and ingeniously solve them, and his sharp vision in both academic and policy contexts, made him a crucial figure in Soviet science and beyond.

Ivan Boldyrev

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Important biographical materials in English are to be found in: Leifman, Lev J. (ed.), Functional Analysis, Optimization, and Mathematical Economics: A Collection of Papers Dedicated to the Memory of Leonid Vital'evich Kantorovich, Oxford: Oxford University Press, 1990.

Kantorovich's influence in optimal transport

Kantorovich, a functional analyst motivated by economics applications, provided a manageable formulation of the transport problem (now called "Kantorovich formulation") and developed the basis of the duality theory in optimal transport, opening the way for impressive developments over the past thirty years. In 1975, Kantorovich was awarded the Nobel Prize in economics, jointly with Tjalling Koopmans, "for their contributions to the theory of optimum allocation of resources."

The origin

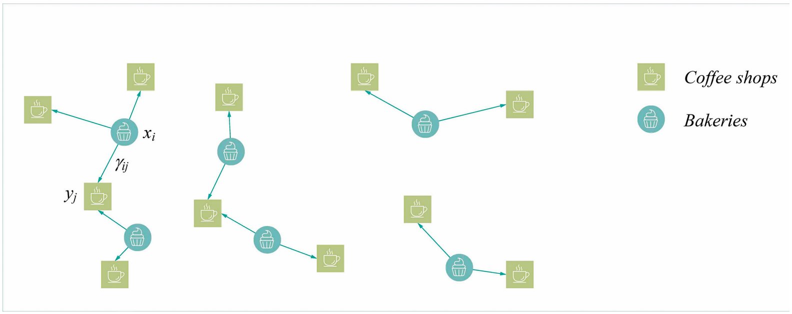
In 1781, the French mathematician Gaspard Monge got interested in the following problem: *Suppose you need to carry some debris from one location to another to build fortifications. How should the transport occur to minimize the average distance?*
In his memoir [10], Monge discussed the two and three-dimensional problems, showing a profound understanding of the problem and its challenging aspects. In particular, Monge had the intuition that, in 3 dimensions, optimal transport paths should be orthogonal to a certain 1-parameter family of surfaces.

More than 150 years later, in 1938, the Soviet mathematician Leonid Kantorovich laid the foundations of linear programming. Then, in 1942, he investigated a variant of the optimal transport problem, that we now describe.

Let X be a compact metric space with the distance function d, let μ_0 and μ_1 be probability measures on X, and let $\Pi(\mu_0, \mu_1)$ denote the space of probability measures on $X \times X$ having μ_0 and μ_1 as marginal distributions. The set $\Pi(\mu_0, \mu_1)$ is called the set of *transport plans*. Then, Kantorovich considered the problem

$$K(\mu_0, \mu_1) := \inf_{\gamma \in \Pi(\mu_0, \mu_1)} \int_{X \times X} d(x, y)\, d\gamma(x, y). \tag{1}$$

To explain the intuition behind this problem, consider the case when $\mu_0 = \sum_{i=1}^{N} \alpha_i \delta_{x_i}$ and $\mu_1 = \sum_{j=1}^{M} \beta_j \delta_{y_j}$. Think of these measures as follows: there are N bakeries located at positions $(x_i)_{i=1,\dots,N}$ and M coffee shops located at $(y_j)_{j=1,\dots,M}$, with the i-th bakery producing an amount $\alpha_i \geq 0$ of bread and the j-th coffee shop needing an amount $\beta_j \geq 0$. The assumption $\sum_i \alpha_i = \sum_j \beta_j = 1$ implies that the demand is equal to the supply.



Then, (1) corresponds to looking at matrices $(\gamma_{ij})_{1\le i\le N,\,1\le j\le M}$ such that:

(i) $\gamma_{ij} \ge 0$ (the amount of bread going from x_i to y_j is a nonnegative quantity);

(ii) $\forall\, i\colon \alpha_i = \sum_{j=1}^{M} \gamma_{ij}$ (the total amount of bread sent to all coffee shops is equal to the production);

(iii) $\forall\, j\colon \beta_j = \sum_{i=1}^{N} \gamma_{ij}$ (the total amount of bread bought from all bakeries is equal to the demand);

(iv) γ_{ij} minimizes the cost $\sum_{i,j} d(x_i, y_j)\gamma_{ij}$ (the total transportation cost is minimized).

It is interesting to observe that the constraint (i) is convex, the constraints (ii) and (iii) are linear, and the objective function in (iv) is also linear (all with respect to γ_{ij}). In other words, Kantorovich's formulation corresponds to minimizing a linear function with convex/linear constraints.

Thanks to the compactness of X, it is not difficult to prove that minimizers in (1) always exist. However, one would like to understand their structure. In [5], Kantorovich established the following fundamental duality theorem.

Theorem 1.

$$K(\mu_0, \mu_1) = \max \int_X \varphi(x)\, d(\mu_0 - \mu_1)(x),$$

where the maximum is taken over all φ, 1-Lipschitz relative to the distance d, namely,

$$|\varphi(x) - \varphi(y)| \le d(x, y) \quad \textit{for all } x, y \in X.$$

These 1-Lipschitz functions are called *potentials* by Kantorovich, and a transport plan γ is called a *potential plan* if there exists a potential function φ such that

$$\varphi(x) - \varphi(y) = d(x, y) \quad \text{for } \gamma\text{-a.e. } (x, y).$$

The above theorem is an infinite-dimensional version of the classical duality theorem of linear programming: the value of the inf in the original problem is the same as the sup in the second "dual" problem. It is important to recall that, at that time, there was no duality theory for infinite-dimensional programming problems.

Thanks to this theorem, Kantorovich deduced that transport plans are optimal if and only if they are potential. Also, in 1947, Kantorovich saw the proceedings of a conference held in Leningrad (St. Petersburg) on the bicentennial of Monge's birth, and realized that the surfaces of Monge are just the level sets of the optimal potentials that he had defined a few years before [6].

Modern formulation

Monge's version of the optimal transport problem is nowadays formulated in the following general form: Given two probability measures μ_0 and μ_1 respectively defined on measurable spaces X_0 and X_1, find a measurable map $T: X_0 \to X_1$ such that $T_{\#}\mu_0 = \mu_1$ (i.e., $\mu_1(A) = \mu_0(T^{-1}(A))$ for every $A \subset X_1$ measurable) and that minimizes the transportation cost. This last condition means

$$\int_{X_0} c(x, T(x))\, d\mu_0(x) = \min_{S_\sharp \mu_0 = \mu_1} \int_{X_0} c(x, S(x))\, d\mu_0(x),$$

where $c: X_0 \times X_1 \to \mathbb{R}$ is a given cost function. When the transport condition $T_\sharp \mu_0 = \mu_1$ is satisfied, we say that T is a *transport map*, and if T also minimizes the cost, we call it an *optimal transport map*.

In this general setting, Kantorovich's formulation of the optimal transport problem becomes

$$\inf_{\gamma \in \Pi(\mu,\nu)} \int_{X \times Y} c(x, y)\, d\gamma(x, y). \tag{2}$$

With these definitions, Monge's problem becomes a particular case of Kantorovich's problem thanks to the following observation:

$$T_{\#}\mu_0 = \mu_1 \quad \Rightarrow \quad \gamma_T := (\mathrm{id} \times T)_{\#}\mu_0 \in \Pi(\mu_0, \mu_1).$$

In other words, every transport map induces a transport plan. Also, one can easily check that T and γ_T have the same transportation cost. Thus, Monge's problem corresponds to a particular case of Kantorovich's problem, where one only considers transport plans induced by maps.

It turns out that a duality result "à la Kantorovich" holds in great generality, and it is at the core of the proof of existence and uniqueness of optimal maps (see, for instance, [4, Chapter 2]). In particular, duality theory is a key tool in the proof of Brenier's Theorem for the quadratic cost in $\mathbb{R}^n$ [1], a milestone in the theory of optimal transport.

Kantorovich/Wasserstein distances

Let X be a compact metric space, and let $P(X)$ denote the space of probability measures on X. One can define the so-called *Wasserstein distances* on

$P(X) \times P(X)$ as follows: for $p \geq 1$,

$$W_p(\mu_0, \mu_1) := \left(\inf_{\gamma \in \Pi(\mu_0, \mu_1)} \int_{X \times X} d(x, y)^p \, d\gamma(x, y) \right)^{1/p}.$$

One can prove that W_p are distances on the space of probability measures that metrize the weak convergence (see, for instance, [4, Chapter 3]). We note that, when $p = 1$, W_1 coincides with the Kantorovich distance K defined in (1). Also,

$$W_p(\delta_x, \delta_y) = d(x, y), \qquad \forall\, x, y \in X.$$

In other words, the embedding

$$(X, d) \ni x \mapsto \delta_x \in (P(X), W_p)$$

is an isometry for any $p \geq 1$.

In [7], Kantorovich and Rubinstein noted that the Kantorovich distance $K = W_1$ can be extended to a norm over the set of signed measures $M(X)$ on X. It is common to call this extension the *Kantorovich–Rubinstein norm*. This norm property is a particular feature of the exponent $p = 1$, and it provides an explicit isometric embedding of an arbitrary metric space (X, d) in a Banach space. Also, as a consequence of Kantorovich's fundamental duality theorem, the Kantorovich–Rubinstein norm on a metric space (X, d) can be characterized as the maximal norm $\| \cdot \|$ on $M(X)$ which is "compatible" with the distance, in the sense that $\|\delta_x - \delta_y\| = d(x, y)$ for all $x, y \in X$.

We refer the interested reader to [15] for a complete historical account of Kantorovich's contributions.

Why Wasserstein? The terminology "Wasserstein distances" is due to Dobrushin [2]: While investigating the existence and uniqueness of random fields with a given system of conditional distributions, it was convenient for him to use the Kantorovich distance K. However, Dobrushin was only aware of the Vasershtein paper [14] (Vasershtein worked in the laboratory he headed) but not of Kantorovich's publications, and therefore he introduced the terminology "Vasershtein distance", which later became (in the transliteration from Cyrillic to the Latin alphabet) Wasserstein. Although unfair towards Kantorovich, the expression "Wasserstein distances" has been very successful, and nearly all recent papers in optimal transport use this convention.

Wasserstein distances and convergence estimates

Wasserstein distances are particularly useful in problems where one wants to quantitatively estimate the distance between two measures, especially when some measures are atomic.

A classic example is provided by a second paper of Dobrushin [3] on kinetic theory, where he investigated the convergence of particle Newtonian dynamics to the so-called Vlasov equation. In that paper, by a fixed point argument involving an appropriate variant of W_1, he proved the existence and uniqueness of solutions to the Vlasov equations with smooth force fields. This result and approach have been a fundamental reference point in the kinetic community.

We mention here a couple of more examples, referring to [4, Chapter 5] for more detail and references.

- **Central limit theorem.** The central limit theorem states that if $\{X_i\}_{i\in\mathbb{N}}$ is a sequence of independent random variables with the same law, mean 0, and variance 1, then the average $\overline{X}_k := \frac{1}{k}\sum_{i=1}^k X_i$ converges in law, as $k \to \infty$, to a standard Gaussian. One may desire to make this statement quantitative: How far can the law of $\overline{X}_k$ be from a Gaussian? Wasserstein distances are a very useful tool to address this question.

- **Random matching.** Let $X_1, \ldots, X_k$ be k independent points uniformly distributed on the cube $[0, 1]^d$, and consider the (random) empirical measure $\mu^k = \frac{1}{k}\sum_{i=1}^k \delta_{X_k}$ associated to the k points. As k grows, one expects μ^k to approximate the uniform measure on $[0, 1]^d$. Then, a very natural question is the following: What is the convergence rate to 0 of the random variable $W_p(\mu^k, dx|_{[0,1]^d})$?

Convexity along Wasserstein geodesics

Let (X, d) be a geodesic space (i.e., a complete separable metric space such that every couple of points can be joined by a minimizing geodesic), and consider the space $P_2(X)$ of probability measures with finite second moment, endowed with the Wasserstein distance W_2. It turns out that $(P_2(X), W_2)$ is a geodesic space too.

Now, given two measures $\mu_0, \mu_1 \in P_2(X)$, let $(\mu_t)_{t\in[0,1]}$ be a constant-speed geodesic joining μ_0 to μ_1. A crucial idea that has found a variety of applications is that the behavior of μ_t captures some information about the geometry of the underlying space.

More precisely, fix a reference measure ν on X, and consider the following functionals on $P_2(X)$: Given a measure $\mu \in P_2(X)$, write it as $\mu = \rho\nu + \mu^s$, where μ^s is the singular part of μ with respect to ν. Then we define

$$H_N(\mu) = -\int_X \rho^{1-1/N}\, d\nu, \qquad N \geq 1,$$

$$H_\infty(\mu) = \begin{cases} \int_X \rho\log(\rho)\, d\nu & \text{if } \mu^s \equiv 0, \\ +\infty & \text{otherwise.} \end{cases}$$

When $X = \mathbb{R}^n$, ν is the Lebesgue measure, and $N \geq n$, McCann proved in [9] that the above functionals are convex along Wasserstein geodesics (in short, *displacement convex*). This fact was the starting point for many applications, which we now describe briefly.

• **Gradient flows.** In [11], Otto understood that many evolution PDEs can be interpreted as gradient flows of some "energy functionals" in the space $P_2(\mathbb{R}^d)$ with respect to W_2. For instance, the gradient flow of H_∞ is the heat equation, while the gradient flow of H_N gives a porous-medium equation. In particular, whenever these energy functionals are displacement convex, such an interpretation turns out to be extremely well-adapted to proving existence, uniqueness, stability, and asymptotic behavior for solutions.

• **Geometric and functional inequalities.** Displacement convexity can be used to prove some geometric and functional inequalities. As an example, given two open bounded sets $A, B \subset \mathbb{R}^n$, consider the probability measures $\mu_0 = \frac{1_A}{|A|}$ and $\mu_1 = \frac{1_B}{|B|}$. As shown in [9], the displacement convexity of H_n allows one to prove the Brunn–Minkowski inequality:

$$|A+B|^{1/n} \geq |A|^{1/n} + |B|^{1/n}.$$

Since such proof relies only on the convexity of H_n, it can be immediately extended to any Riemannian manifold on which H_n is displacement convex.

• **Riemannian manifolds and Ricci curvature bounds.** As a natural extension of McCann's result [9], given a Riemannian manifold (M, g) with reference measure $\nu = \mathrm{vol}_g$, one may wonder when the functionals H_N and H_∞ are displacement convex. The combination of results of many authors can be summarized in the following statement, relating displacement convexity and Ricci curvature.

Theorem 2. *Let $(X, \nu) = (M, \mathrm{vol}_g)$, and let $N \geq \dim M$. Then H_N (resp. H_∞) is displacement convex if and only if $\mathrm{Ric}_g \geq 0$.*

This result gives a robust characterization of the geometric condition $\mathrm{Ric}_g \geq 0$ (there exist also generalizations to characterize $\mathrm{Ric}_g \geq K\,g$, $K \in \mathbb{R}$) in terms of the convexity of suitable functionals, and it has been the starting point for Lott-Villani and Sturm [8, 12, 13] to give a meaning to lower bounds on the Ricci curvature on a metric measured space, a theory that is still very active.

Conclusion

In this short note we have seen the crucial influence of Kantorovich on the development of optimal transport. Also, we have described some examples of applications of Kantorovich/Wasserstein distances to probability, PDEs, and geometry. For an introduction to optimal transport and other applications, we refer to the recent book [4], and the references therein.

Alessio Figalli

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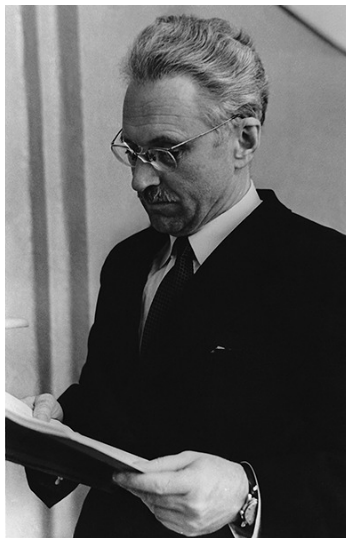
Alexander Danilovich Alexandrov (1912–1999)

A few recollections, speculations, and quotations about A.D. Alexandrov, a prominent mathematician, alpinist, philosopher, and humanist.

The writing below is for the most part based on my recollections of a very superficial acquaintance with A.D. during the last decade of his life, as well as some stories my dad[1] told me. I also make a few general remarks about my own insights regarding some of the many punchlines of A.D.'s life. These remarks are no more than speculations since I am not a historian of mathematics. They come from my own understanding of a few sources, written in Russian, and from speaking to people. There is a very detailed and well-thought-out biographical sketch by O.A. Ladyzhenskaya and some vivid recollections by A.M. Vershik. I use loose translations of small parts of them here. They are in Russian, so I include an abridged retelling, mostly made up of quotations, of these papers in the Russian version of this essay.



In English, a much more detailed and gorgeously written essay about A.D. by Semen Kukateladze can be found on the Internet.[2] I highly recommend it for a smoother read.

A.D. was obviously ingenuous in all areas that attracted him, and everything he touched would bloom or sprout another blossom. Not only in math, where his achievements are surely outstanding but also in service to the scientific community and as a support to remarkable people during those difficult times. In culture, including poetry. Philosophy.

[1] Yuri Burago (born 1936) is a Russian Geometer from Alexandrov's School. Yuri is a creator (with his students Perelman and Petrunin, and M. Gromov) of what is known now as Alexandrov Geometry. With V. Zalgaller, he also authored a classic book "Geometric Inequalities," bringing them to the state of the art. Yuri Burago was Head of the Laboratory of Geometry and Topology at LOMI (St. Petersburg Branch of the Steklov Institute for Mathematics) and Professor at SPbU. He took part in a report for the United States Civilian Research and Development Foundation. He was a recipient of the Steel Prize in 2014.

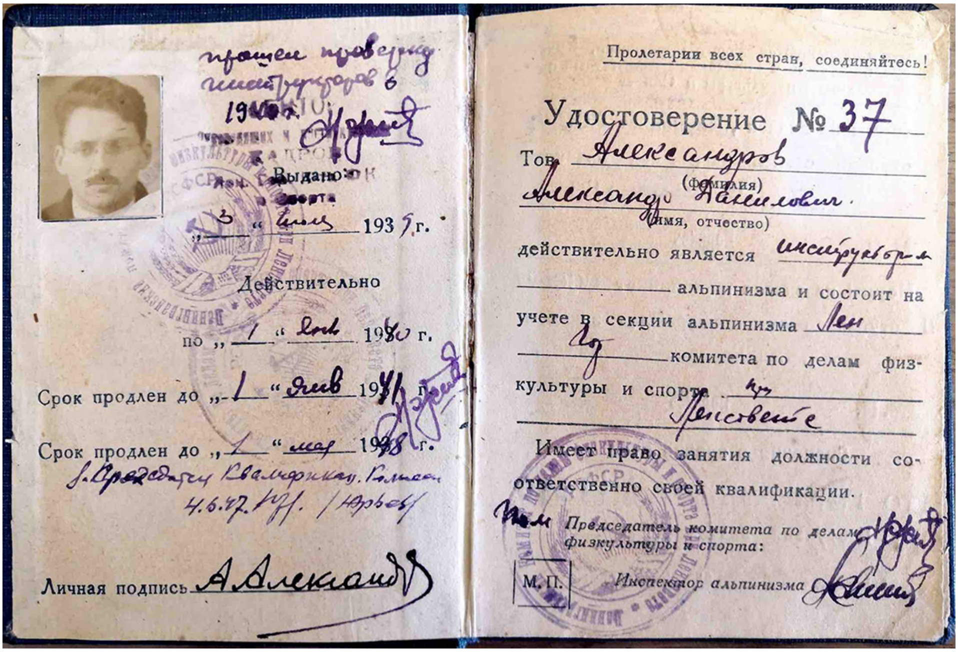
[2] http://www.math.nsc.ru/LBRT/g2/english/ssk/memorye.html

The history of mathematics. In methods of teaching math in schools, including A.D.'s work on textbooks in geometry for school children. In mountaineering. "Alexandrov's School" and Alexandrov's seminar had a huge influence on the mathematical community in the USSR.

It is extremely difficult to write about A.D. The initials A.D. are how Alexander Danilovich Alexandrov's name was abbreviated by most who knew him when talking about him. Too much is written about A.D. already. It would be silly if I spoke a lot about Alexandrov's math: A.D.'s mathematical interests were very broad, and the only area of his work I know reasonably well is the one that gave rise to what is now known as Alexandrov geometry. Actually, there are two Alexandrov geometries, very different from each other, with upper and lower curvature bounds. The starting technology is the same, but then the spirit of the two areas diverges completely. More than a third of a 450+ page book that I wrote with my dad and Sergei Ivanov is devoted to it. Also, my lecture at the ICM-98 is based on applications of the Alexandrov geometry of $k \leq 0$ to dynamics. Well, yet another area worth mentioning is: The theory of surfaces with bounded integral of the positive part of the curvature (the condition can be reformulated without integration, using the excess of angles in triangles). Regardless of the fact that the theory is 2-dimensional, it turned out to be notoriously difficult and highly non-trivial. It is enough to recall Alexandrov's problem, whose formulation is very elementary and looks like an exercise, that is still open, with many mathematicians "breaking their teeth" trying to attack it. S.T. Yau even included this problem in his list of the 50 most important problems in geometry. The problem asks: what is the maximal possible area of a 2-sphere with a length metric of non-negative curvature and an intrinsic diameter ≤ 1? The extremal case is definitely not a round sphere... Note that this metric can be realized as the induced metric on the surface of a convex body in $\mathbb{R}^3$, which may be degenerate. That is, it lies in a plane and the boundary is regarded as the double of the region, like the surface of a coin. By the way, the latter statement is also attributed to A.D. and is known as Alexandrov's theorem (one of many with this name). Anyhow, if I were to be writing about A.D.'s math I would inevitably be repeating myself. There are also results of A.D. that I have only heard about through the interpretations of other people. For instance, below is a free translation of a text by Academician O.A. Ladyzhenskaya:

The years of Rectorship for A.D. were years of personal flourishing. On the one hand, not only does he not leave mathematics, but quite the opposite, he reaches a high summit which is now known as "The Maximum Principle of Alexandrov." Some parts of his result were understood only a quarter of a century later. It is one of the cornerstones for the theory of the existence of solutions for completely nonlinear equations of the elliptic type, and later for the parabolic type. For the theory of viscosity solutions, A.D.'s results, as well as his earlier results on convex functions, were foundational. In 1975, A.D. was elected to the oldest Academy, the National Academy of Sciences of Italy, also known as the "Academy of the Forty."

I can only write based on my recollections, which mostly date back some 33 to 38 years, and on several stories that my dad, A. Vershik and a few others told me. I do not use quotation marks when quoting someone if I am not sure of the precise wording. This is what I remember through the mist of a long time.



Alexandrov's *Mountaineering guide* certificate.

I first met A.D. when I was an undergrad. I attended his topic course, even though I knew this math relatively well already. I enjoyed the show. During the first lecture, A.D. stood behind a desk and just jumped on it pushing himself up with two legs simultaneously. Not such a big surprise: he was a very skilled mountaineer. A.D. was, however, 70 years old at that time. And then he suggested the young men in the class try it. It hurts a lot to hit one's shins against the sharp edge of the desk... And then A.D. said something like this: "If you cannot jump, let me at least teach you some math." A.D. engaged students by surprising them, involving them, and then teaching math in a very clear and definitely not boring way. I think his teaching style influenced many, including me.

He would run down the stairs, leaping over three steps, and would yell at the students: Let me pass, you have plenty of time, and I don't! In his lectures, he used non-standard (at least at that time) terminology and employed a lot of analogies. He called a sequence of points in a metric space "a dotted line." He liked to tell stories, and, having an encyclopedic knowledge, he knew a lot. Later we occasionally met at the chair of geometry, and A.D. slightly distinguished me from the other students. Not because of my math, even though I already had some results. A.D. loved to use quotations from various books, he could quote plenty off the top of his head, and would then ask: Where is this from? I could answer reasonably often. He had very broad interests, not only in math and mountaineering, but also in philosophy, literature, art, and physics.

My dad told me that when A.D. was in love with his future wife Svetlana, he wrote rhymes for her... in English! I recall attending an international conference where A.D. delivered a spectacular talk, in English of course. He was over 80 years old. I remember the beginning. He said:

> At my age, only a fool would be speaking about his own results, but yesterday a friend of mine phoned me from Novosibirsk and told me of two beautiful results he had proven.

And then he spent the rest of the lecture explaining these results.

After he finished, while walking down the stairs from the podium, he said loudly, in Russian: Why don't they let an old man die in peace? Then there were noises echoing throughout the room: Russian-speaking listeners were translating A.D.'s words for their neighbors. After the lecture, someone asked A.D. how he had managed to learn English so well. I am pretty sure A.D. read highly linguistically sophisticated authors like William Somerset Maugham, and obviously a lot of English poetry, but he did not like obvious answers. He said: "I read O'Henry..."

I noticed that A.D. always wore a black glove on one of his hands. I asked my dad why and he told me that A.D. had had encephalitis many years ago. Some of the nerves were damaged and so one hand felt cold even during hot weather. A.D. kept doing high-class mountaineering, however, even after he had almost fully recovered from this oftentimes fatal disease.

A.D. was a Gentleman with a capital G. He always dressed as and behaved like a nobleman with lineage going back ten generations. He was born into a noble and highly educated family of school teachers in a village near Ryazan, where they had a country house. It happens that I visited A.D.'s apartment in Leningrad a couple of times many, many years ago. I was wet behind the ears, having just finished my undergrad studies at Leningrad State University, and A.D. was an Academician and former Rector of the University. He would always take my coat and then help me to put it back on later on. I was impressed and am trying to borrow this courteous habit from him, but for him — this was in his blood.

A.D. hated lying, even though he knew that, during Soviet times, lies were in some cases inevitable. G. Perelman was formally a student of A.D. at LOMI, though Perelman's actual adviser was my dad. Why? Because Perelman is ethnically a Jew, and the only way for him to be able to study at LOMI at those times was this little legerdemain. My dad, however, told me the following story. At Perelman's thesis defense, L.D. Faddeev, who chaired the committee, asked A.D. if he could say a few words on behalf of his advisee. A.D. answered: "I hardly understood a thing from Perelman's results. Do you want me to explain why he is my postgraduate student?" And Faddeev said: "No, thank you."

A.D.'s manner of speech was extremely clear and sharp, at times even harsh, though A.D. was always perfectly polite in the way he expressed himself, although not necessarily in the meaning of what he said... On the other hand, it seems that A.D. even enjoyed it when someone was impertinent and talked back to him. I guess he took this as his interlocutor having individuality and character.

I personally met A.D. when he reached the age of 70. He was an extremely bright, energetic, clear, and original-thinking young old man. I hardly can imagine how fantastic he was as a thirty-year-old!

After A.D. defended his diploma paper "Computation of the energy of divalent atoms using Fock's method", and by obtaining the highest grade for it, he was recommended to continue studies as a postgraduate student. He declined, saying he could not promise he would always be doing proper things. At the time, this most likely meant "the things wanted from me by those in power." He was scolded by the administration but got compliments from advisors. For example, V.A. Fock, a famous theoretical physicist and member of the Academy of Sciences, said: You are too decent a man; and B.N. Delone, a famous mathematician and a corresponding member of the Academy at the time, said: Alexander Danilovich, you really are not enough of a career chaser.

Both the aforementioned teachers of A.D. were right, though at the time they could hardly foresee the complicated consequences brought on by A.D.'s incredible talent and his remarkable personality. This combination of qualities during Soviet times inevitably resulted in a lot of controversies and contradictions.

On the one hand, A.D. had a fantastic career. He was a member of the Communist Party, was elected to the Academy of Sciences, was for some period the Rector of the Leningrad State University, and won lots of the most prestigious prizes and extensive recognition. He helped a lot of people who were in the doghouse, so to speak, and stopped some scoundrels from moving up to leading positions. He supported R.I. Pimenov, who shared an office room with my dad and V. Zalgaller, up until Pimenov was exiled to Syktyvkar for seven years despite the fierce attempts of A.D. to save him. He stopped a dishonest and sham scientist I.I. Prezent, a protégé of Lysenko, and the same breed, from getting an influential position at the Leningrad University, even though N.S. Khrushchev, the leader of the former USSR, supported Prezent in 1955, two years after the death of Stalin when almost everyone was badly scared...

Khrushchev yelled at A.D. He said something like the following: How could you, the Rector, not obey directions! In Stalin's time you would be sentenced to death! And still, A.D. just "...showed his bared teeth." [1] This was A.D.'s "...angry reaction to danger" [1]. This does not mean that A.D. was a man without a sense of fear, however. He just had the capacity to overcome it.

There are many stories like the two described above. One recalls A.D.'s valiant standpoint with regards to the article by N.P. Dubinin "Biological and Social Heredity..." (see [1] for details). A.D. had appraised this composition as an outstanding piece of antiscientific literature, he said: "I am convinced that to read the article by N.P. Dubinin and the relevant controversy is as vital for a young scientist of any specialty as the perusing of the shorthand record of the notorious August Session of the Lenin All-Union Academy of Agricultural Sciences (VASKhNIL in the Russian abbreviation) which took place in 1948." A huge achievement was A.D.'s hiring of V. Rokhlin, a great mathematician and person, who was badly in disgrace, as a faculty member at the Leningrad State University.

Yet A.D. himself was for certain periods of time in disgrace. This is a huge controversy. He was neither a "tame" nor a "loyal" man, this was clear, and yet he was still appointed rector of the second largest university. My understanding, perhaps inaccurate, is that he became a member of the Communist Party partially because this gave him additional leverage for doing the right things. Being a member of the Party was an ambiguous success, as some may now safely say, but times were different then. With A.D.'s unyielding and absolutely honest personality, I can hardly believe that A.D. would join the Communist Party for purely practical reasons, even those as honorable as the ones suggested above. My guess is that the situation was more complicated. From the memoirs written by those who knew A.D., it seems that for some time he sympathized strongly with socialistic ideas a la Marx and Lenin, like many great people of the time. This was no surprise: the injustice and imperfectness of society were evident. Who knows, maybe past loyalty to these ideas also influenced his decisions. I heard from A.M. Vershik that he had very different attitudes individually towards Stalin, Lenin, and Marx. Only lessons learned during the 20th century eventually showed that following these ideas leads to even more unfairness and cruelty, and A.D. surely realized this. A.D. witnessed Perestroika, however, and the first years of its consequences, and did not really appreciate it nor took it too well. From what I know, he was especially concerned with the way that Perestroika affected the scientific community and science education. Perhaps this and what is now known as “criminal Petersburg”, as well as examples similar to it, caused A.D.’s mind to be somewhat obsessed with socialist and communist ideas, oftentimes vaguely referred to as Marxism, during his later years. Maybe A.D.’s approach was more anti-capitalistic. I doubt anyone can responsibly say now what A.D.’s thoughts were. Maybe, looking back, A.D. had a nostalgia for the tempting ideals of the Soviet Times, regardless of the “sloppy” implementation of those. I am only guessing.

Even with A.D.’s espousal of socialistic ideology, he was several times on the verge of being expelled from the Party. He was pretty much forced to leave his position as rector, though formally he quit of his own accord. As funny as it may sound, A.D. himself had been “exiled” to Peterhof along with the University. Apparently, the authorities preferred to keep stubborn people away from Leningrad.

According to S.S. Kutateladze, A.D. had a strong stereotype: Everyone who hates A.D. is a potential, if not complete, scoundrel. As described by his students, friends, and relatives, A.D. was exceptionally kind, even tender, very attentive, and scrupulous.

A man of passion, A.D. always remained self-critical. As expressed by V.I. Smirnov, A.D. controlled the University using the power of moral authority.

According to S.S. Kutateladze, A.D. knew a lot about religion, always contrasting religious belief and scientific search. He was fond of reiterating that he believed in nothing. This statement usually produced the following retort in the audience: “Neither in communism?” This, in turn, always generated an affirmative answer in A.D. It goes without saying that the A.D.’s lectures were often followed by sneaky letters sent to various local party committees.

V. Zalgaller saved in his memory the following lines:

> My heart is full of burning wishes,
> My soul is under spell of thine,
> Kiss me: your kisses are delicious
> More sweet to me than myrrh and wine.
> Oh lean against my heart with mildness,
> And I shall dream in happy silence,
> Till there will come the joyful day
> And gloom of night will fly away.

S.S. Kutateladze wrote:

> Not later than in 1944 A.D. had made this interpretation in the English Language of a celebrated poem was written by A.S. Pushkin in Russian as far back as in 1825 ...

In June of 1993, S.S. Kutateladze received the following lines “in sloppy handwriting”:

> Since legs, nor hands, nor eyes, nor strong creative brain,
> But weakness and decay oversway their power,
> I am compelled forever to refrain
> From everything but waiting for my hour.

I am grateful to D. Belousov, M. Guysinski, and A. Reshetikhina for their help in editing the essay, and to D. Alexandrov, Yu. Burago, N. Kalinin, A. Vershik, and A. Werner for also correcting factual mistakes, and for their recollections, advice, and insights.

Dmitry Burago

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Alexandrov's embedding theorem

Alexandrov's embedding theorem provides a complete description of the intrinsic geometry of surfaces of convex polyhedrons. We give a sketch of its proof.

The intrinsic distance between two points on the surface of a convex polyhedron is defined as the length of a shortest curve on the surface between these points.

Recall that the sum of angles at the tip of a convex polyhedral angle is less than 2π; this statement can be found in a school textbook [6, § 48].

It is easy to see that the surface of a convex polyhedron is homeomorphic to the sphere. Therefore the statements above imply that the surface of a convex polyhedron equipped with its intrinsic metric is an example of a *polyhedral metric on the sphere with the sum of angles around each vertex at most* 2π; a metric is called *polyhedral* if the sphere admits a triangulation such that every triangle is congruent to a plane triangle.

Alexandrov's theorem states that the converse holds if one includes in the consideration *twice covered polygons*. In other words, we assume that a polyhedron can degenerate into a plane polygon; in this case, its surface is defined as two copies of the polygon glued along their boundary.

Further, we assume that a polyhedron can degenerate to a plane polygon.

Theorem. *Alexandrov's theorem*

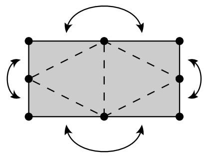
1. *A polyhedral metric on the sphere is isometric to the surface of a convex polyhedron if and only if the sum of angles around each of its vertex is not greater than* 2π.

2. *Moreover, a convex polyhedron is defined up to congruence by the intrinsic metric on its surface.*

A.D. Alexandrov has many remarkable theorems, but in our opinion, this theorem is the most remarkable. At the same time, its proof is elementary; it could be explained to anyone familiar with basic topology.

This theorem has many applications. In particular, it is used in the proof of its generalization [4] that gives a complete description of intrinsic metrics on the sphere that are isometric to convex surfaces in Euclidean space. The latter statement is fundamental in a branch of modern mathematics — the so-called *Alexandrov geometry*.

The first part is central; it is called the *existence theorem*. The second part is called the *uniqueness theorem*; it is a slight variation of Cauchy's theorem about polyhedrons. (There is another uniqueness theorem of Alexandrov that generalizes Minkowski's theorem about polyhedrons.)



According to the theorem, a convex polyhedron is completely defined by the intrinsic metric of its surface. In particular, knowing the metric we could find the position of the edges. However, in practice, it is not easy to do. For example, the surface glued from a rectangle as shown on the diagram defines a tetrahedron. Some of the glued lines appear inside facets of the tetrahedron and some edges (dashed lines) do not follow the sides of the rectangle.

The theorem was proved by A.D. Alexandrov in 1941 [3]; we will present a sketch of his proof. A complete proof is nicely written by A.D. Alexandrov in his book [2]. Yet another proof was found by Yu.A. Volkov in his thesis [8]; it uses a deformation of three-dimensional polyhedral space.

Space of polyhedrons and metrics

Space of polyhedrons. Let us denote by Φ the space of all convex polyhedrons in the Euclidean space, including polyhedrons that degenerate to a plane polygon. Polyhedra in Φ will be considered up to a motion of the space, and the whole space Φ will be considered with the natural topology (an intuitive meaning of the closeness of two polyhedrons should be sufficient).

Further, denote by Φ_n the polyhedrons in Φ with exactly n vertices. Since any polyhedron has at least three vertices, the space Φ admits a subdivision into a countable number of subsets $\Phi_3, \Phi_4, \dots$

Space of polyhedral metrics. The space of polyhedral metrics on the sphere with the sum of angles around each point at most $2 \cdot \pi$ will be denoted by Ψ. The metrics in Ψ will be considered up to an isometry, and the whole space Ψ will be equipped with the natural topology (again, an intuitive meaning of closeness of two metrics is sufficient).

A point on the sphere with the sum of angles strictly less than $2 \cdot \pi$ will be called an *essential vertex*. The subset of Ψ of all metrics with exactly n essential vertices will be denoted by Ψ_n. It is easy to see that any metric in Ψ has at least three essential vertices. Therefore Ψ is subdivided into countably many subsets $\Psi_3, \Psi_4, \dots$

From a polyhedron to its surface. Recall that the surface of a convex polyhedron is a sphere with a polyhedral metric such that the sum of angles around each point is at most $2 \cdot \pi$. Therefore passing from a polyhedron to its surface defines a map

$$\iota \colon \Phi \to \Psi.$$

Note that the number of vertices of a polyhedron is equal to the number of essential vertices of its surface. In other words, $\iota(\Phi_n) \subset \Psi_n$ for any $n \geq 3$.

About the proof

Using the notation introduced in the previous section, we can give the following more exact formulation of Alexandrov's theorem.

Reformulation. *For any integer $n \geq 3$, the map ι is a bijection from Φ_n to Ψ_n.*

We sketch the original proof of A.D. Alexandrov. It is based on the construction of a one-parameter family of polyhedrons that starts at an arbitrary polyhedron and ends at a polyhedron with its surface isometric to the given one. This type of argument is called the *continuity method*; it is often used in the theory of differential equations.

The two parts of the first formulation will be proved separately.

Part 2. Let us show that the map $\iota\colon \Phi_n \to \Psi_n$ is injective; in other words, a convex polyhedron is defined by the intrinsic metric on its surface up to a motion of the space.

The last statement is analogous to the Cauchy theorem about polyhedrons, and the proof goes along the same lines.

The Cauchy theorem states that facets of a polyhedron together with the gluing rule completely describe a convex polyhedron; its proof is given in many classical popular texts [1, 5, 7].

Part 1. Let us prove that $\iota\colon \Phi_n \to \Psi_n$ is surjective. This part of the proof is subdivided into the following lemmas.

Lemma. *For any integer $n \geq 3$, the space Ψ_n is connected.*

The proof of this lemma is not complicated, but it requires ingenuity; it can be done by the direct construction of a one-parameter family of metrics in Ψ_n that connects two given metrics. Such a family can be obtained by a sequential application of the following construction and its inverse.

Let M be a sphere with metric from Ψ_n. Suppose v and w are essential vertices in M. Let us cut M along a shortest line from v to w. Note that the shortest line cannot pass through an essential vertex of M. Further, note that there is a three-parameter family of patches that can be used to patch the cut so that the obtained metric remains in Ψ_n; in particular, the obtained metric has exactly n essential vertices (after the patching, the vertices v and w may become inessential).

Lemma. *The map $\iota\colon \Phi_n \to \Psi_n$ is open, that is, it maps any open set in Φ_n to an open set in Ψ_n.*

In particular, for any $n \geq 3$, the image $\iota(\Phi_n)$ is open in Ψ_n.

This statement is very close to the so-called *invariance of domain theorem*; the latter states that a continuous injective map between manifolds of the same dimension is open.

According to part 2, ι is injective. The proof of the invariance of domain theorem can be adapted to our case since both spaces Φ_n and Ψ_n are $(3n-6)$-dimensional and both look like manifolds, although, formally speaking, they are *not* manifolds. In a more technical language, Φ_n and Ψ_n have the natural structure of $(3n-6)$-dimensional *orbifolds*, and the map ι respects the *orbifold structure*.

We will only show that both spaces Φ_n and Ψ_n are $(3n-6)$-dimensional.

Choose a polyhedron P in Φ_n. Note that P is uniquely determined by the $3\cdot n$ coordinates of its n vertices. We can assume that the first vertex is the origin, the second has two vanishing coordinates and the third has one vanishing coordinate; therefore, all polyhedrons in Φ_n that lie sufficiently close to P can be described by $3\cdot n-6$ parameters. If P has no symmetries then this description can be made one-to-one; in this case, a neighborhood of P in Φ_n is a $(3\cdot n-6)$-dimensional manifold. If P has a nontrivial symmetry group, then this description is not one-to-one but it does not have an impact on the dimension of Φ_n.

The case of polyhedral metrics is analogous. We need to construct a subdivision of the sphere into plane triangles using only essential vertices. By Euler's formula, there are exactly $3\cdot n-6$ edges in this subdivision. Note that the lengths of edges completely describe the metric, and slight changes of these lengths produce a metric with the same property.

Lemma. *The map $\iota\colon \Phi_n\to\Psi_n$ is closed; that is, the image of a closed set in Φ_n is closed in Ψ_n.*

In particular, for any $n\ge 3$, the set $\iota(\Phi_n)$ is closed in Ψ_n.

Choose a closed set Z in Φ_n. Denote by $\bar Z$ the closure of Z in Φ; note that $Z=\Phi_n\cap\bar Z$. Assume $P_1, P_2, \dots\in Z$ is a sequence of polyhedrons that converges to a polyhedron $P_\infty\in\bar Z$. Note that $\iota(P_n)$ converges to $\iota(P_\infty)$ in Ψ. In particular, $\iota(\bar Z)$ is closed in Ψ.

Since $\iota(\Phi_n)\subset\Psi_n$ for any $n\ge 3$, we have $\iota(Z)=\iota(\bar Z)\cap\Psi_n$; that is, $\iota(Z)$ is closed in Ψ_n.

Summarizing, $\iota(\Phi_n)$ is a nonempty closed and open set in Ψ_n, and Ψ_n is connected for any $n\ge 3$. Therefore, $\iota(\Phi_n)=\Psi_n$; that is, $\iota\colon\Phi_n\to\Psi_n$ is surjective. □

Acknowledgments. We want to thank S. Alexander, Yu. Burago, and J. Tsukahara for help.

Nina Lebedeva and Anton Petrunin

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Yuri Vladimirovich Linnik (1915–1972)

Yu. V. Linnik achieved exceptional results in two fields: the number theory and the probability theory. In the number theory, he developed and applied new and original methods, namely the large sieve and the ergodic and variance methods. With the latter, he was able to solve the Hardy–Littlewood conjecture — any natural number is the sum of one prime number and two squares. Linnik derived a new and simple proof to Vinogradov's theorem that any sufficiently large odd number can be written as a sum of three prime numbers (Goldbach's ternary problem), and he proved the uniformity of distribution of integer points on a sphere.

His most important results in probability theory and mathematical statistics are in large deviations theory, decomposition of probabilistic laws, characterization of distributions, and statistical problems with an interfering parameter. In the mid-60s, Linnik and his collaborators solved the famous Behrens–Fisher problem, finding a counterexample to the conjecture believed to be true.

Yuri Vladimirovich Linnik was born in the city of Bila Tserkva (now in Ukraine), to a family of (at that time) teachers. In 1926, Linnik's family moved to Leningrad, where his father (an optical physicist who later became a member of the Academy of Sciences of the Soviet Union) worked at the State Optical Institute; Yuri worked there as a lab assistant after he finished school. He went on to study at Leningrad State University in the Department of Physics and Mathematics, studied there for three years, and then transferred to the newly-established Department of Mathematics and Mechanics (because he "felt an irresistible attraction towards higher arithmetic," according to his autobiography). By the time he graduated from university in 1938, he had already published a paper.

In the winter of 1939, Yu.V. Linnik was drafted into the First Soviet-Finnish War, where he served as commander of an artillery group until his discharge in 1940. When he returned to Leningrad, Linnik wrote a thesis on the arithmetics of quadratic forms, for which he was given a Doctorate degree (at the age of 25!), skipping the preceding Candidate's degree. In July of 1941, Linnik joined the People's Militia and fought in WWII. However, during the autumn of 1941 he was discharged due to muscular dystrophy and evacuated to Kazan to work at the Steklov Institute. This most likely saved his life, as the German Siege of Leningrad lasted 872 days, causing over half a million people to die of starvation. In 1943, Yuri Vladimirovich derived an elementary proof showing that any sufficiently large natural number can be written as a sum of 7 cubes of natural numbers (Waring's problem). In 1944, he returned to Leningrad and combined his work at the Steklov Institute and Leningrad State University, where he created the department of probability theory and mathematical statistics in 1948.

When an administrative order forbade members of the Academy of Sciences to hold side jobs at universities, Yuri Vladimirovich taught for free for two years until this ridiculous decree was removed. Linnik divided his students into two categories: some he gave specific problems to and closely monitored their progress, while others he gathered mainly to talk about what he studied and why. For example, he would send someone a telegram asking them to call him back, then invite them to his house over the phone, and after quickly preparing some food, would talk about his recent results for hours on end.

He was exceptionally hardworking, and his work was a source of happiness for him. Once, after getting another remarkable result, he said: "yes, it's a difficult theorem, it took me three whole days and nights to prove." He often said that when starting a new field of study, "you have to choose a difficult, but well-posed problem; while trying to solve it, new problems will emerge, and they will serve as a testing ground for arising methods. This strategy gradually leads to the creation of a theory and methods of a general nature."

Once, he went to an academic conference in Riga, where he attended lectures during the day and talked with his colleagues during the breaks and evenings. At 11 o'clock one night, Yuri Vladimirovich called B.M. Bredikhin and said: "I have a new idea." Bredikhin came to his hotel room just as someone else from the conference was leaving it. Bredikhin himself left the room sometime between midnight and one in the morning, greeting Linnik's next guest on the way out.

Linnik started with the number theory but later worked in the probability theory, as well. One of the reasons he went into the probability theory was a piece of advice he received from A.Ya. Khinchin: "You always have to work in at least two fields, so that when something isn't working out in one field, you can always switch to the other. For you, this extra field could be the probability theory." Both in the number theory and the probability theory, Yu. V. Linnik was interested in hard analytical problems, which he solved with great success. A quote from A. Weil's book *Basic Number Theory* is appropriate here: "It has been clear for a long time (since the fundamental works of Khinchin and

1943 МАТЕМАТИЧЕСКИЙ СБОРНИК Т. 12 (54), N. 2
RECUEIL MATHÉMATIQUE

On the representation of large numbers as sums of seven cubes

U. V. Linnik (Leningrad)

§ 1

In the present paper we prove that $G(3) \leqslant 7$, *i. e.* every sufficiently large number is the sum of 7 non-negative cubes. This is an improvement of a result of E. Landau [1]: $G(3) \leqslant 8$. Our proof is based on some elementary identities as well as on our previous results on the theory of positive ternary quadratic forms. In particular, we use the following

Theorem [2]. *Every positive ternary quadratic form $F(x, y, z)$ with odd invariants $[\Omega, 1]$ such that there exists a prime ω/Ω with $\left(\frac{F}{\omega}\right) = \left(\frac{-1}{\omega}\right)$ represents all sufficiently large numbers that are prime to 2Ω and satisfy the genus conditions of F**.

In particular, such forms that $\left(\frac{F}{\omega}\right) = \left(\frac{-1}{\omega}\right)$ for any ω/Ω are called «convenient» [2]; they represent also every even number $m \not\equiv 0 \pmod 4$ satisfying the conditions: $(m, \Omega) = 1$, $\left(\frac{m}{\omega}\right) = \left(\frac{-1}{\omega}\right)$, $m > c_0(\Omega)$.

As one can prove by the methods of [2], the reciprocal to convenient forms $f(x, y, z)$ have genus conditions modulus 8 only and represent, in particular, every number $m \not\equiv 0 \pmod 4$, for which the congruence $m \equiv f(\xi, \eta, \zeta) \pmod 8$ is solvable and $m \equiv 1 \pmod 5$, when $m > c_0(\Omega)$ and $\Omega \equiv 1 \pmod 5$. Such forms will be used here**. Such are, in particular, the forms

$$f(x, y, z) = A_1^{\tau_1}x^2 + A_2^{\tau_2}y^2 + A_3^{\tau_3}z^2,$$

where A_1, A_2, A_3 are primes satisfying the conditions:

$$\left(\frac{A_1A_2}{A_3}\right) = \left(\frac{-1}{A_3}\right); \quad \left(\frac{A_1A_3}{A_2}\right) = \left(\frac{-1}{A_2}\right); \quad \left(\frac{A_2A_3}{A_1}\right) = \left(\frac{-1}{A_1}\right), \qquad (1)$$

and τ_1, τ_2, τ_3 are odd; $A_1 \equiv A_2 \equiv A_3 \equiv 1 \pmod 5$.

* As for G. Pall remarks (Math. Rev., November 1941), see [4].

** To prove it by the methods of [2] the following quaternion equations should be considered: $b + L = Q \cdot P \cdot X$ where $L^2 = -\Omega m$; $\text{Norm}\, P = \Omega$; $\text{Norm}\, Q = 5^{\beta}$; $b \equiv \pm 1 \pmod 5$; $b \equiv 0 \pmod \Omega$.

The first page of Linnik's article about Waring's problem.

Kolmogorov), that the probability theory is applied analysis — in other words, it is a calculus that can be applied to certain types of problems. The same can be said about analytic number theory."

Linnik was interested in military history, and collected wartime memoirs and historical literature. His knowledge of the subject was quite broad, and because of his exceptionally good memory, he remembered the exact dates of a great number of events. He was also a deputy in the Leningrad City Council, although he was not a member of the Communist Party. He liked to travel and met with many foreign colleagues.

Yu.V. Linnik and his collaborators wrote a series of papers on practical applications of statistics and probability in the study of power systems, gridding quality, and the gyroscope theory.

Yuri Vladimirovich was fluent in English, French (although still worse than Russian, according to him), and German; he spoke Ukrainian, Polish, Serbian, and Italian, loved reading books in foreign languages, and even wrote witty poems in Russian, German and French. Linnik wrote the "axiomatic theory of the Communist Party", let us look at the first lines of it.

Axiom: From each according to his ability, to each according to his needs.

Existence theorem: Let us examine, for example, a graveyard. Both parts of the axiom are clearly met.

In 1953 Linnik was elected corresponding member, and in 1964 became a full member of the Academy of Sciences of the Soviet Union. He died of a heart attack in June of 1972.

Nikita Kalinin

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Vladimir Abramovich Rokhlin (1919–1984)

Rokhlin's main results belong to low-dimensional topology, the topology of real algebraic varieties, and measure theory. In measure theory, Rokhlin defined Lebesgue–Rokhlin probability spaces and developed a theory of their measurable partitions. He also obtained significant results on entropy.

Rokhlin introduced cobordism groups and computed them in dimensions 3 and 4, he discovered the divisibility by 16 of the signature of closed spinor four-dimensional manifolds, proved the combinatorial invariance of Pontryagin classes in a joint paper with Albert Schwarz, and established the additivity of signature in a joint paper with Sergei Novikov.

Rokhlin applied the technique of algebraic topology to the geometry of real algebraic varieties, which allowed him to prove a congruence relation for the Euler characteristic of maximal real algebraic varieties.

Vladimir Abramovich Rokhlin was born in Baku. His mother, Henrietta Emmanuilovna Levinson, came from a wealthy Jewish family, received a European medical education, and worked as head of Baku's sanitary and epidemiological station. In 1923 she was murdered there during the mass riots caused by an epidemic. Her half-brother was the famous children's writer Korney Chukovsky.

Rokhlin's father, Abram Veniaminovich Rokhlin, was an economist and member of the RSDLP.[1] He publicly opposed the Bolsheviks during the revolution. In 1918 he was a food commissioner for the Revolutionary Socialist–Menshevik government of the "Centrocaspian Dictatorship"[2] in Baku.

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[1] The Russian Social Democratic Labor Party, was founded in 1898; in 1903 it split into Bolsheviks and so-called Revolutionary Mensheviks factions, with the former eventually becoming the Communist Party of the Soviet Union.

[2] A revolutionary socialist anti-Soviet administration that existed in Baku in July-September 1918.

He was later a *glasny*[3] at the Duma for the Democratic Party of the Turkish Federalists, Musavat. He joined the All-Union Communist Party in 1920 but was expelled during the purge in 1921. In 1930 Abram Rokhlin was exiled to Kazakhstan, then returned to Baku, arrested in 1939, and executed in 1941.

In 1934, the fifteen-year-old Vladimir, skipping several years, graduated with distinction from school in Alma-Ata, and in 1935, he was admitted to the Department of Mechanics and Mathematics of Moscow State University. He studied under the most distinguished mathematicians: Pavel Alexandrov, Andrei Kolmogorov, Israel Gelfand, Lazar Lyusternik, Lev Pontryagin, and Abraham Plessner.

All of them recommended Rokhlin for postgraduate studies. While still a student, he published two scientific papers. In addition, in 1940, Rokhlin wrote an extensive survey on homotopy groups and prepared detailed notes of Plessner's lectures on the spectral theory of operators in the Hilbert space. These two reviews (the second in co-authorship with Plessner) were published in Uspekhi Matematicheskikh Nauk, one of the leading mathematical journals in the USSR, only in 1946, when Rokhlin was in a gulag camp, that is, he did not take part in the preparation of the publication. Several generations of mathematicians learned these subjects from these two papers.

Rokhlin's thesis was about measurable partitions and became the beginning of the series of his famous papers on the foundations of measure theory.

At the beginning of the war, in 1941, Vladimir Rokhlin was in his first year of postgraduate study. He was sent to the militia, which immediately fell into the so-called Vyazma pocket: a large group of troops (over half a million men) was encircled and defeated near Vyazma. Wounded in both legs, Rokhlin was left in the village in the care of the locals. His wounds were not healing, and he was moved to a local hospital. When the territory was occupied by the Germans, Rokhlin was arrested and sent to a P.O.W. (Prisoners of War) camp. There, he contracted typhus, recovered, tried to escape several times, and was moved to another camp in Belarus and then to Poland.

Rokhlin managed to conceal that he was a Jew. He said that one day a specialist came to the camp who could identify Jews by the shape of their skulls. The specialist told Rokhlin, with a sigh, that the Russians had such a variety of skulls that he could not tell anything about anyone. Rokhlin had spoken fluent German since childhood, and the guards chatted with him eagerly. The retreating Germans dragged the POWs with them, but one day, the POWs learned that the next day, the Germans would withdraw and continue to retreat while the POWs would be shot. Their barrack was guarded by one soldier; the POWs killed him and left to meet the Red Army.

The war had already moved to Germany, and the Soviet army needed interpreters who knew German, which is where Rokhlin's German came in

[3] In the Russian Empire, a member of a meeting, e.g., local or regional assembly, with a casting vote.

handy. On the last day of the war, Rokhlin and other former prisoners of war were loaded into *teplushkas*[4] and taken under escort to Komi for clearance. They were interrogated in order to find out whether they had collaborated with the Germans during their captivity. The logic of the Soviet authorities was to send them to a lumber camp: if they worked hard and fulfilled their quotas, they were "our people," but if they did not do their job (or just died), then they were traitors and should be punished. The convoy was told that their wards were *Vlasovtsy*[5] and treated them accordingly.

> To the People's Commissar of Internal Affairs of the USSR
> Comrade Kruglov
> from laureates of the Stalin Prize
> Academician A.N. Kolmogorov
> and L.S. Pontryagin, Corr. Member of the USSR Academy of Sciences
>
> We hereby request your attention to the fate of Vladimir Abramovich Rokhlin, who returned from a German prison camp and is now living in Komi. [...] we believe that in the interests of the development of Soviet mathematics it would be highly desirable to give V. A. Rokhlin the opportunity to get back to his postgraduate studies in the near future to continue his scientific work under our supervision.
>
> 13 February 1946.

> To the Director of the Institute of Mathematics
> Professor V.V. Stepanov
>
> In accordance with your letter of 5/09 1946, regarding the return of Vladimir Abramovich Rokhlin to the Moscow University the GULAG of the Ministry of Internal Affairs of the USSR has instructed me to dismiss V.A. Rokhlin from the guard and to transfer him to your custody.
>
> Head of the Security and Regime Department of the Gulag
> of the USSR Ministry of Internal Affairs
> Colonel I. Smirnov.
>
> December 4, 1946.

Rokhlin later recounted that during the clearance process he claimed that he did not remember anyone from the German camps because he had been seriously wounded. It guaranteed that his testimony would not contradict anyone else's, and he would not incriminate anyone. At the time, most Moscow mathematicians were convinced that Rokhlin had died during the war, but he

[4] *Teplushka* is a heated goods wagon, widely used for transporting troops and prisoners at the time.

[5] Common name for the soldiers of the Russian Liberation Army that was led by general Andrey Vlasov, fought under German command during World War II, and was primarily composed of Russians.

managed to send a word of himself. Andrei Kolmogorov and Lev Pontryagin wrote a letter to the head of the NKVD asking him to release Rokhlin. By the time this letter reached the addressee Rokhlin had already been cleared of charges and had been assigned to serve in the guard as was often the case.

Through the war and the camps, Rokhlin carried the notebook in which he wrote down his ideas and plans, and within a year and a half of passing the clearance he had defended both his dissertations (Ph.D. and Higher Doctorate). Lev Pontryagin was blind, and he used this to argue for employing Rokhlin at the Steklov Mathematical Institute in Moscow as his assistant and secretary. This is where Rokhlin worked until he completed his doctoral thesis.

While editing Pontryagin's papers on topology, Rokhlin took an interest in the computation of homotopy groups of spheres by the geometrical method (the invariant-free method, as Pontryagin called it) and computed the third stable homotopy group of spheres. This was followed by other profound results in algebraic topology, which are described in [1] and the accompanying mathematical paper.

During this period (1940–50), Andrei Kolmogorov, Lev Pontryagin, Israel Gelfand, and Abraham Plessner wrote rave reviews of Rokhlin's dissertation, his articles, and his teaching activities. However, the beginning of the 1950s in the Soviet Union was marked by an anti-Semitic campaign, and after defending his doctoral thesis Rokhlin could not stay at the Steklov Institute (he was told that they could not keep him there even if he had computed all the homotopy groups of all spheres).

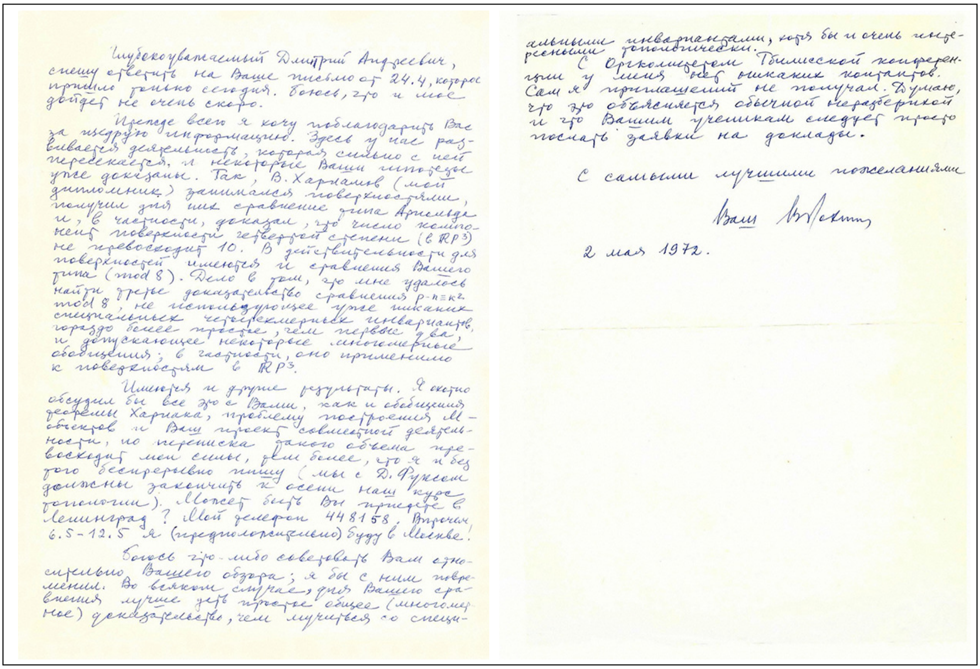
So, Vladimir Rokhlin had to leave Moscow with his family. They moved to Arkhangelsk, then to Ivanovo, then to Kolomna (which is not far from Moscow, so it became possible to participate in seminars occasionally), where Rokhlin was a professor of mathematics at a pedagogical and technical institute with a significant teaching load.

Rokhlin greatly contributed to the notion of Kolmogorov entropy, first by showing that Kolmogorov's definition is not complete in general case, and then, together with Yakov Sinai, by developing the theory of K-automorphisms.

Rokhlin was a bright figure in the mathematical community then, and his influence on many young mathematicians (Sergei Novikov, Yakov Sinai, Vladimir Arnold, etc.) was great. But it was not until 1960 that Rokhlin got a job befitting his standing when the rector of Leningrad State University, the geometer A.D. Alexandrov, invited him to Leningrad.

From 1960 to 1980 Rokhlin was a professor at Leningrad University. He immediately became one of the informal leaders of Leningrad mathematicians. In particular, as soon as he arrived, he organized topological and ergodic seminars attracting the best students, many of whom later became distinguished scientists.

He was also involved in modernizing the curriculum of the Faculty of Mathematics and Mechanics and introduced a compulsory course in topology, the



Глубокоуважаемый Дмитрий Андреевич,
спешу ответить на Ваше письмо от 24.4, которое пришло только сегодня. Боюсь, что и мое дойдет не очень скоро.

Прежде всего я хочу поблагодарить Вас за щедрую информацию. Здесь у нас развивается деятельность, которая сильно с ней пересекается, и некоторые Ваши гипотезы уже доказаны. Так, В. Харламов (мой дипломник) занимался поверхностями, получил для них сравнение типа Арнольда и, в частности, доказал, что число компонент поверхности четвертой степени (в $\mathbb{R}P^3$) не превосходит 10. В действительности для поверхностей имеются и сравнения Вашего типа (mod 8). Дело в том, что мне удалось найти третье доказательство сравнения $p-n\equiv k^2$ mod 8, не использующее уже никаких специальных четырехмерных инвариантов, гораздо более простое, чем первые два, и допускающее некоторые многомерные обобщения; в частности, оно применимо к поверхностям в $\mathbb{R}P^3$.

Имеются и другие результаты. Я охотно обсудил бы все это с Вами, как и обобщения теоремы Харнака, проблему построения М-объектов и Ваш проект совместной деятельности, но переписка такого объема превосходит мои силы, тем более, что я и без того беспрерывно пишу (мы с Д. Фуксом должны закончить к осени наш курс топологии). Может быть Вы приедете в Ленинград? Мой телефон 448158. Впрочем, 6.5–12.5 я (предположительно) буду в Москве!

Боюсь что-либо советовать Вам относительно Вашего обзора; я бы с ним повременил. Во всяком случае, для Вашего сравнения лучше дать простое общее (многомерное) доказательство, чем мучиться со специальными инвариантами, хотя бы и очень интересными топологически.

С Оргкомитетом Тбилисской конференции у меня нет никаких контактов. Сам я приглашений не получал. Думаю, что это объясняется обычной неразберихой и что Вашим ученикам следует просто послать заявки на доклады.

С самыми лучшими пожеланиями

Ваш В. Рохлин

2 мая 1972.

A letter from Vladimir Rokhlin to Dmitry Gudkov, 2 May 1972, [5]. Rokhlin writes about diploma theses of his student V. Kharlamov, new Gudkov-type conjugacy results, and mentions his new proof of $p-n\equiv k^2$ (mod 8) not using special facts about 4-manifolds. Rokhlin and Fuchs plan to finish their topology textbook before autumn.

first in the country and probably in the world. Generation after generation of his postgraduate students led students' self-study clubs. Rokhlin's topological seminar was, for a long time, one of the leading mathematical seminars in the USSR, on par with those led by Arnold, Gelfand, and others.

He was an excellent lecturer and generously shared new ideas and whole scientific areas with his students. He was always elegant, amiable and polite, punctual, ironic, and witty.

He had a complicated and tragic life. His friends and students remember him for his strong moral principles, profound theorems, and for the fact that a whole generation of strong mathematicians, the Leningrad school of topology, was tutored by him.

Nikita Kalinin

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Rokhlin's signature theorems

The scope of Rokhlin's mathematics. Rokhlin's mathematical heritage includes a number of outstanding contributions to various fields: measure and ergodic theories, topology, and real algebraic geometry. He used to change the focus of his research interests in average once in five years. In each of the periods, he managed to obtain fundamental results and then turned to a new field, being attracted by a challenging problem.

Rokhlin started in the measure and ergodic theories. The results of his PhD were presented in his paper *On the fundamental concepts of measure theory.* The second (doctoral) dissertation defended soon after his PhD was called *On the most important metric classes of dynamical systems.* The first ergodic theory period finished in 1950. In the late fifties, Rokhlin came back to ergodic theory in his research related to entropy of dynamical systems. In the first half of the fifties, the period of topology followed.

A good challenge. When Rokhlin turned to topology, his initial challenge was to find the third stable homotopy group of spheres, that is $\pi_{n+3}(S^n)$ with $n \geq 5$. Prior to that Pontryagin found the groups $\pi_{n+2}(S^n)$ with $n \geq 2$ and $\pi_{n+1}(S^n)$ with $n \geq 3$ by his method, which bridges the homotopy theory and differential topology. Technically, Pontryagin's calculation was a study of curves and surfaces embedded in Euclidean space with trivialized normal bundles. It relied on a comparatively simple topology of curves and surfaces. A topology of 3- and 4-dimensional smooth manifolds that was needed for a similar calculation of $\pi_{n+3}(S^n)$ had not yet been developed.

Groundbreaking results. In a striking tour de force, Rokhlin developed a technique, which allowed him to prove that $\pi_{n+3}(S^n) = \mathbb{Z}/_{24}$ for $n \geq 5$, and found himself in a new research area, where he could make groundbreaking discoveries. Below we overview the results published by Rokhlin in the notes of 1951–52, where he calculated $\pi_{n+3}(S^n)$. The results grew out of calculations of cobordism groups Ω_3 and Ω_4. (Cobordisms and the cobordism groups Ω_n and $\mathfrak{N}_n$ were introduced by Rokhlin in the same notes.)

• *Any oriented closed smooth 3-manifold bounds a compact oriented 4-manifold.* (In other words, $\Omega_3 = 0$.)

Rokhlin introduced the signature $\sigma(M)$ of an oriented closed $4n$-manifold M. The signature $\sigma(M)$ is the difference between the numbers of positive and negative coefficients in a diagonalization of the intersection form $H_{2n}(M;\mathbb{R})\times H_{2n}(M;\mathbb{R})\to\mathbb{R}$ of M. By discovering the properties of signature stated below Rokhlin turned it into one of the central invariants in topology of manifolds.

• *The signature of an oriented smooth closed 4n-manifold vanishes if the manifold bounds an oriented smooth compact* $(4n+1)$*-manifold.* (In other words, the signature defines a homomorphism $\Omega_{4n}\to\mathbb{Z}$.)

• *An oriented smooth closed 4-manifold M bounds a compact oriented 5-manifold if and only if* $\sigma(M)=0$*. In general, an oriented smooth closed M is cobordant to the disjoint sum of* $\sigma(M)$ *copies of the complex projective plane.* (In other words, $\Omega_4=\mathbb{Z}$.)

• *The signature of an oriented smooth closed 4-manifold M equals one third of the Pontryagin number* $p_1(M)[M]$.

• *The signature of an oriented smooth closed 4-manifold M with* $w_2(M)=0$ *is divisible by 16.*

Most of these achievements are often attributed to other authors, who discovered these things later. However, the last item in this list stands apart. It is commonly referred to as *the Rokhlin Theorem* and considered Rokhlin's most famous result. Many times it played a substantial rôle in the subsequent development of topology. Below we concentrate on a few of them that seem the most important.

Reformulations of the Rokhlin Theorem. To a non-specialist, the Rokhlin theorem sounds a bit cryptic and technical. Let us take a closer look. The theorem establishes a relation between two basic characteristics of an oriented smooth closed 4-manifold M: its second Stiefel–Whitney class $w_2(M)$ and the signature $\sigma(M)$. It claims that if $w_2(M)=0$, then $\sigma(M)\equiv 0 \mod 16$.

What does the assumption $w_2(M)=0$ mean? The Stiefel–Whitney classes $w_k(M)\in H^k(M;\mathbb{Z}/_2)$ measure complexity of the tangent bundle TM. Orientability of M means $w_1(M)=0$. For an orientable smooth closed 4-manifold M, $w_2(M)$ is the only obstruction for M being *almost parallelizable*, that is admitting 4 tangent vector fields linear independent at all points but one. With this in mind we can reformulate the Rokhlin Theorem as follows:

• *The signature of an almost parallelizable smooth closed 4-manifold is divisible by 16.*

A smooth closed manifold M admits a Spin-structure iff $w_1(M)=0$ and $w_2(M)=0$. Thus we can reformulate the Rokhlin Theorem as follows:

• *The signature of a smooth closed Spin 4-manifold is divisible by 16.*

For an orientable smooth closed 4-manifold M the assumption $w_2(M)=0$ holds true iff the $\mathbb{Z}/_2$ intersection form $H_2(M;\mathbb{Z}/_2)\times H_2(M;\mathbb{Z}/_2)\to\mathbb{Z}/_2$ is even, that is the self-intersection number of any closed surface in M is even. This interpretation does not depend on the smooth structure or tangent bundle of M. Thus we get one more reformulation:

• *If an oriented smooth closed 4-manifold M has even $\mathbb{Z}/_2$ intersection form, then* $\sigma(M) \equiv 0 \mod 16$.

Causes for divisibility: Topology versus Algebra. Divisibility of some integer by 16 looks quite mysterious. Sixteen is a large non-prime number. As we will see, a half of divisibility of the signature by 16, that is divisibility by 8, can be extracted by purely algebraic arguments from simple topological facts, which are not specific for smooth 4-manifolds. On the contrary, the last factor 2 in 16 manifests a crucial obstruction for a topological manifold to admit a smooth structure.

Due to Poincaré duality, the intersection form $H_2(M) \times H_2(M) \to \mathbb{Z}$ of a closed oriented 4-manifold M is a symmetric integral unimodular form. If $w_2(M) = 0$, then this form is even, because its reduction modulo 2 is a part of the $\mathbb{Z}/_2$ intersection form, which is even as $w_2(M) = 0$. The signature of an even symmetric integral unimodular form is divisible by 8. This is a purely algebraic fact.

The Rokhlin Theorem claims that $\sigma(M)$ is divisible not only by 8, as the algebra ensures, but by 16. So, it imposes a restriction on a symmetric even unimodular form realizable as the intersection form of an oriented smooth closed 4-manifold M with $w_2(M) = 0$. Namely, the signature of a realizable form cannot be congruent to $8 \mod 16$. For example, the form E_8 (a famous even unimodular integral form of rank and signature 8) is prohibited.

Obstructions to Diff or PL on a 4-manifold. In Rokhlin's theorem, the assumption that M is *smooth* is necessary: as Freedman proved in 1982, there exists an oriented closed topological simply-connected 4-manifold with any unimodular intersection form. If the form is even and its signature is not divisible by 16 (say, if this is E_8) then by the Rokhlin Theorem the manifold does not admit a smooth structure. In lower dimensions this does not happen: any n-manifold with $n < 4$ admits a smooth structure.

In dimension four, smoothability is equivalent to existence of a piecewise linear structure (PL-structure). Thus the Rokhlin theorem provides an obstruction to existence of a PL-structure.

The key to PL in high dimensions. In the eighties Donaldson discovered other numerous obstructions to existence of PL or smooth structure on 4-manifolds. However, they are less robust. The Rokhlin Theorem is formulated in terms invariant under cobordisms and gives rise to *high-dimensional* results, while the other obstructions do not.

For example, if M is a simply-connected closed 4-manifold with signature non-divisible by 16 (and does not admit a PL-structure by the Rokhlin

Theorem), then $M \times (S^1)^n$ does not admit a PL-structure for any $n > 0$. This can be deduced from the Rokhlin Theorem.

Complete obstructions to existence or equivalence of PL-structures on manifolds of dimension ≥ 5 (discovered by Kirby and Siebenmann in the seventies) rely on the Rokhlin Theorem.

More about the Rokhlin Theorem. Due to the space restrictions, our story is very incomplete. We said nothing about numerous generalizations of the Rokhlin Theorem, their applications, proofs, misconceptions, etc. You can find all of this in a note by S. Finashin and V. Kharlamov, *A glimpse into Rokhlin's Signature Divisibility Theorem.* arXiv:2012.06389.

See also a book by L. Guillou and A. Marin *A la Recherche de la Topologie Perdue. I Du côté de chez Rohlin. II Le côté de Casson.*

Sergey Finashin, Viatcheslav Kharlamov, Oleg Viro

Victor Abramovich Zalgaller (1920–2020)

Victor Abramovich Zalgaller was a person with a dramatic but, in the end, happy life. A front-line signalman who traveled throughout the entire war from Leningrad to Rügen, a bright mathematician of Alexandrov's school, an author of not only theoretical but also applied papers, an excellent teacher, one of the founders of a specialized physical-mathematical school № 239 in Leningrad, Zalgaller was a character of exceptional morality and integrity. His life is closely tied to the Department of Mathematics and Mechanics of Leningrad State University (now St. Petersburg State University).

In geometry, the main focuses of Zalgaller's research were the theory of convex bodies, the theory of convex surfaces, the intrinsic geometry of irregular surfaces, polyhedra, and optimization problems. Equally important was his research on applied problems, primarily the method behind the optimal cutting of materials.

Victor Abramovich was born on December 25, 1920, in the vicinity of Novgorod. His father, A.L. Zalgaller, was an engineer and his mother, Tatyana Markovna Shabad–Zalgaller, was a lawyer. Tatyana Markovna's oratory skills played a notable role — she taught her son how to speak eloquently, a skill that Victor Abramovich later used to become a legendary lecturer at Leningrad's math department.

Once in 1982, he was asked to give a lecture on teaching methods in mathematics (just a regular lecture on his class schedule). The large auditorium in the Petergof building of the math department was filled with undergraduate and postgraduate students as well as professors. The diverse audience benefitted Victor Abramovich, for he was simultaneously saying and showing "how to do it properly." "*To make the audience lighten up a little,*" he said,

> tell a funny joke or anecdote in the middle of the lecture. The bigger the audience is, the wittier the joke should be. In a small auditorium, a sarcastic smile is enough.

The room exploded with loud applause after the lecture, and Victor Abramovich bowed happily.

Victor Abramovich's life was not easy — he belonged to a generation that almost perished during the war. Shortly after the start of the German invasion of the USSR he joined the volunteer corps and was almost immediately sent to the front lines. He was a signalman throughout most of the war. His journey as an active duty soldier was also difficult: the Leningrad defense, the Oranienbaum Bridgehead, an injury during the lifting of the Leningrad blockade, the storming of Vyborg, battles in the Baltic states, the storming of Danzig, and the arrival to the Elbe, among others. Victor Abramovich was awarded the Order of the Red Star, a medal "for courage," three medals "for battle merit," a medal "for the defense of Leningrad," and more. He ended the war as a Senior Sergeant. His older brother, Lieutenant Leonid Zalgaller, died in 1942 when beseiged near Myasnoy Bor. Victor Abramovich left a remarkable wartime memoir called Life in War, from which the following quote takes the reader back to those days:

> ...we're carrying Boris on a stretcher. One can see his heart beating through his wound. We cover it with some cotton wadding. We reached an open field. Not a soul to be seen. It's hot. Two people carry the stretcher, one rests, then we switch. A German airplane passed us three times, trying to shoot the long shadow that the stretcher drops on the sunny road. "It's hard for you to carry me, I'll sing." And so he sings: "Arms, like two large and warm birds, how you flew, how you lit up everything around you..." His voice disappears when he loses consciousness, then reappears again. "Arms, how easily you could wrap yourself around me..."

After the war, Victor Abramovich returned to his university studies, which he finished with honors in 1948. That same year, he applied to the Steklov Institute, where he worked for over 50 years.

Victor Abramovich was an outstanding teacher with an unusual ability to clearly demonstrate the most difficult ideas. He also stood out because of his friendliness towards students and young people in general. He started teaching for a mathematical circle that met in the Pioneers' Palace.

"I had some good kids," remembered Victor Abramovich, "Yura Reshetnyak, Garald Natanson, Misha Solomyak — they were all in my circle." (All three later became well-known mathematicians.) Victor Abramovich continued to enjoy working with school and university students. He was one of the organizers and teachers at the specialized mathematical school School No. 239, where he developed the school's first curriculum.

In his monograph, *Convex Polyhedra with Regular Faces*, Zalgaller thanks twelve students from the 239th School who helped him with calculations. It is rare for high school students to be mentioned in serious scientific work.

The story of this problem is as follows: Any mathematician who knows about the five platonic solids has thought of the following question: what (convex) polyhedra are there with all regular faces, but not necessarily the same ones? This question is a lot harder than it appears at first glance, and its solution is accredited to Victor Abramovich: there are exactly 28 simple (i.e., not composed of two other ones) regular-faced polyhedra, except for prisms or antiprisms. Victor Abramovich decided to engage the students in solving this problem.

Boris Belinsky, one of the participants in this project, remembered those days:

> It was like listening to a fairy tale, that's how clearly he explained everything. I remember one of his examples about how to tie a dog so that it protects exactly half of a circle. Later I learned that this was part of a serious geometric problem. And then Victor Abramovich started a project describing regular-faced polyhedra [a.k.a. Johnson solids], which I was more than happy to participate in. That's when we realized that it's one thing to get a good grade for something your teacher taught you, and quite a different thing entirely to try to do something that has never been done before. First a paper was published, and then a book, where our names were written in parentheses after certain lemmas. Everyone here knows that feeling when you see your name printed for the first time.

Victor Abramovich's first papers were written under the influence of his teacher, Alexander Danilovich Alexandrov. Zalgaller came to Alexandrov's school during its golden age. We cannot grasp the full scope of Victor Abramovich's scientific legacy with one story, nor can we describe all of his main achievements. Therefore, we will focus on two of his works, which are so accessible that even a non-specialist would understand their formulation.

One of Zalgaller's (and Yu.D. Burago's) results, inspired by J. Nash's theorems, is as follows: every 2-manifold with a polyhedral metric admits piecewise linear isometric immersion into 3-dimensional Euclidean space. To simplify further, one can say that if you have a collection of paper triangles (non-stretchable and non-compressible) that you can glue together to create the surface of a sphere with handles and cross-caps, then these glued triangles can be realized in a 3-dimensional space, if you allow the triangles to be fanfolded, and allow self-intersections of the obtained surface.

To prove this, Zalgaller needed the following nontrivial result: every polygon M can be cut into acute triangles that are adjacent to each other by whole sides. (A good exercise for the reader: try to cut an obtuse triangle into acute ones.) The idea of the construction is that at first, the faces of a 2-manifold were divided into acute triangles, but possibly not adjacent to triangles on another face along a whole side. Then, the divisions were subdivided (refined) even more, and the correct triangulation appeared from slight shifts of vertices of the finer subdivision along the edges of the original division. Surprisingly, he made use of the famous theorem about the simultaneous good approximation of a number set by a collection of rational numbers with a common (arbitrarily large) denominator.

Zalgaller with his grandson, 1979.

Let us mention another one of V.A. Zalgaller's great achievements, which he obtained when he was in Israel at the age of 80. Together with A.Yu. Solynin, he proved an old conjecture of Pólya and Szegö: the logarithmic capacity of a flat n-sided polygon with a given area attains its minimum at a regular polygon. Their proof is beautiful and not simple. It consists of two parts: a geometric one and an analytical one, the first of which which belongs to Victor Abramovich. A special triangulation of an n-sided polygon was needed here: a polygon is covered by triangles with non-overlapping neighboring sectors such that the ratio of α_k to the area of T_k does not depend on k, where α_k is the angle of the triangle T_k opposite to the side A_{k+1} of the polygon. As Victor Abramovich told his colleagues, he was constantly thinking about this problem over the course of a whole year, and in the end the solution came to him in a dream. The proof of this conjecture became one of the best gifts he gave himself for his 80th birthday. In 2004, this result was published in one of the most prestigious journals, Annals of Mathematics.

V.A. Zalgaller was the author of a few books that ended up in the hall of fame of mathematical literature. Some of his most famous books that are loved by geometers are the following:

• Yu.D. Burago, V.A. Zalgaller, *Geometric Inequalities* — a real encyclopedia, dedicated to various inequalities for subsets of Euclidean and Riemannian spaces, a handbook for geometry lecturers at mathematics departments.

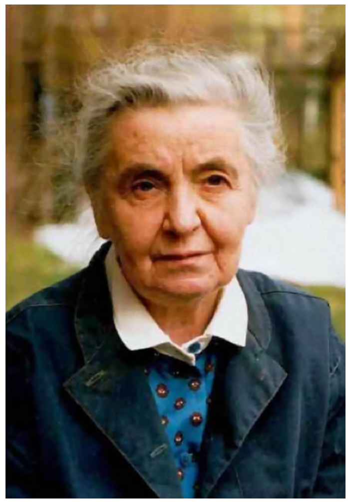
• Yu.D. Burago, V.A. Zalgaller, *Introduction to Riemannian Geometry* — the only textbook on Riemannian geometry in Russian where the subject is presented from the point of view of 'Riemannian geometry at large' while still acting as a good introduction to the subject.

Colleagues, students, friends, and people close to him remember Victor Abramovich Zalgaller as a wonderful and multifaceted scientist with a complicated biography, and at the same time as an extremely kind person with a soft and charming smile.

Gaiane Panina

Olga Alexandrovna Ladyzhenskaya (1922–2004)

O.A. Ladyzhenskaya was born on March 7, 1922, in the town of Kologriv (now Kostroma region). Her skills in mathematics were noticeable from a young age. In 1937, her father, a math teacher and former military officer, was executed as an "enemy of the people." Because of this, Olga Alexandrovna was not accepted to Leningrad State University in 1939, despite her brilliant exam results. She could only attend the Pokrovsky Leningrad State Pedagogical Institute. When the Great Patriotic War[1] started, Olga Alexandrovna returned to Kologriv where she taught math at a school.



In 1943, Ladyzhenskaya enrolled at Moscow State University as a second-year student. After graduating with honors in 1947, with I.G. Petrovsky as her advisor, she got married and moved to Leningrad, where she became a postgraduate student at Leningrad State University (LSU), with S.L. Sobolev acting as her advisor. In 1949, she got her Candidate's degree,[2] and in 1953 — her Doctorate degree.[3] In 1950 she became an Assistant Professor at the Physics Faculty of LSU (and in 1955 — a Full Professor). Since 1954, she was a Fellow of the Leningrad Department of the Steklov Institute (LOMI). In 1961 O.A. organized the Laboratory of Mathematical Physics at LOMI; she headed this Laboratory until 1998. From 1999 onwards, she was a Principal Researcher at the Steklov Institute. She became a Corresponding Member of the Academy of Sciences of the Soviet Union in 1981, and a Full Member in 1990.

O.A. Ladyzhenskaya wrote over 250 works, including 7 monographs and 1 textbook. Her achievements were recognized with the bestowal of numerous prestigious awards, including the highest award of the Russian Academy of

[1] The Great Patriotic War (1941–1945) is a term used in Russia and former Soviet Republics to describe the war between the Soviet Union and Nazi Germany during World War II.

[2] The Candidate's degree is roughly equivalent to a PhD.

[3] The Doctorate degree is roughly equivalent to Habilitation.

Science — the Lomonosov Gold Medal. She was elected as a foreign member of the German National Academy of Sciences Leopoldina (1985), the Academia Nazionale dei Lincei (1989), and the American Academy of Arts and Sciences (2001), and as an honorary doctor at the University of Bonn (2002). In 2020, the National Committee of Mathematicians of Russia, St. Petersburg University, and the Organizing Committee of the International Congress of Mathematicians established a special medal in honor of O.A. Ladyzhenskaya.

Associated with her name are results in spectral theory of elliptic operators, diffraction theory, and a justification of convergence of the Fourier method for hyperbolic equations. During the '60s, Olga Alexandrovna and her student, N.N. Uraltseva, wrote a series of papers about the regularity of solutions to quasilinear elliptic and parabolic partial differential equations. In fact, these works contained the solutions to the 19th and 20th Hilbert problems.

In the second half of the 20th century, Ladyzhenskaya was a "trendsetter" in the theory of partial differential equations, a real mathematical strategist. It is important to note that she was more interested in creating new problems than solving existing ones. It is thanks to Ladyzhenskaya that we have concepts such as generalized settings and generalized solutions.

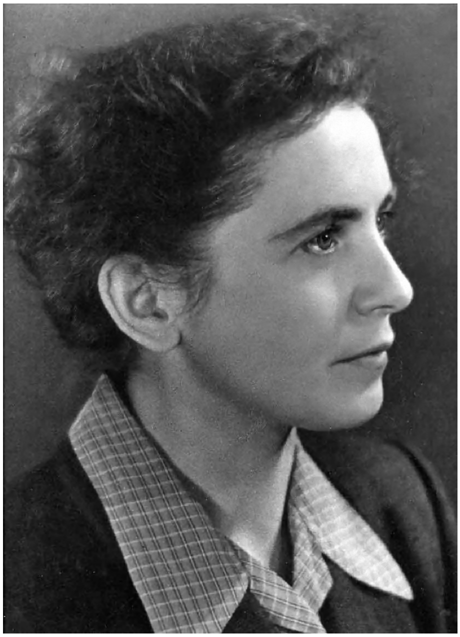
Ladyzhenskaya's contribution to mathematical fluid dynamics is fundamental. She was the first to prove uniqueness in the flow problem for the 2-dimensional Navier–Stokes system. And together with A.A. Kiselev, she established several theorems on the solvability of the 3-dimensional system. Olga Alexandrovna introduced the concept of the attractor for 2-dimensional Navier–Stokes systems, and proved its existence. This result became the basis of the general theory of global stability for evolutionary partial differential equations.

For many years, Olga Alexandrovna taught advanced courses for students in two faculties (Mathematics and Mechanics, and Physics) simultaneously. Her charm, her ability to recognize capable students, and her readiness to help beginners out allowed her to foster brilliant scientists, such as L.D. Faddeev, N.N. Uraltseva, V.A. Solonnikov, V.S. Buslaev and others, whose names make up the glory of the St. Petersburg school of partial differential equations and mathematical physics.

Ever since V.I. Smirnov organized the City Seminar of Mathematical Physics in 1947, Olga Alexandrovna actively participated in it, and later became its leader for many years. Almost all experts in partial differential equations and their applications who studied in Leningrad (now St. Petersburg) participated in that seminar. The chance to give a talk in front of Olga Alexandrovna was considered an honor for mathematicians across the Soviet Union. Distinguished foreign mathematicians gave lectures at her seminar as well, including R. Courant, J. Leray, P. Lax, and others.

Since the revival of the St. Petersburg Mathematical Society in 1959, O.A. Ladyzhenskaya became one of its most active members. For over 40 years she served as a board member and vice president, and from 1990 to 1998 she was the President of the Society. In 1998 she was elected as an Honorary Member of the St. Petersburg Mathematical Society.

Despite her high status, Olga Alexandrovna was always very respectful to everyone and was interested in what they said. You could argue and disagree with her, but she would not force her opinions on you and was not afraid to overlook her own point of view.



This portrait of Olga Alexandrovna stood on A.D. Alexandrov's desk.

Olga Alexandrovna was deeply religious, but never let it show. Her behavior around people was not defined by the phrase "for God's sake" but "for people's sake." She was loyal to the slogan "Who, if not me?", and was always willing to help those around her, reaching out before they did. Her help appeared in many forms: money, clothes, a place to live, organizing shifts for someone who needed 24-hour care, taking care of administrative hassles, etc. She consistently stood up for students who were subjected to discrimination for political reasons when applying to postgraduate school. During the '90s, a terrible decade of economic hardship in Russia, Olga Alexandrovna walked around with her pockets full of change and gave it out to the poor.

Olga Alexandrovna was a unique and many-sided person. She was interested in almost everything in the world: she was well-versed in literature, art, and music. Famous poets, authors and musicians valued their interactions with her; among them were J. Brodsky, A. Solzhenitsyn, and B. Tishchenko.

Solzhenitsyn added Ladyzhenskaya to the list of 257 "witnesses of The Gulag Archipelago."[4]

O.A. Ladyzhenskaya and A.A. Akhmatova, early 1960s.

Ladyzhenskaya's friendship with Anna Akhmatova, the great Russian poet, should be considered separately. Despite their big age difference, they were very close: Olga Alexandrovna was one of the 11 people whom Akhmatova entrusted to read her manuscripts "Poem without a Hero" and "Requiem", which were not allowed for publication at that time. Moreover, Olga Alexandrovna convinced Anna Andreevna to make a tape recording of "Requiem" (Ladyzhenskaya kept it hidden for 20 years). It is important to note that if the KGB (the Soviet political police) had found that recording in Olga Alexandrovna's possession, it would have put her professional career in serious jeopardy. Today, thanks to Ladyzhenskaya, we can listen to the author performing the timeless words of Requiem. A well-known poem In Vyborg by Akhmatova is dedicated to Olga Alexandrovna.

Everyone who knew Olga Alexandrovna remembered her relentlessness — in mathematics as well as during excursions and tourist trips. It was only natural for her to spend a couple of hours questioning a lecturer on the details of a proof. With the same level of ease, she could visit 4 different art galleries in one day (and this was at an advanced age too!) during a trip overseas. There were even legends about her tendency to "get lost" in the mountains.

[4] The Gulag Archipelago is a three-volume non-fiction text by Alexander Solzhenitsyn. It was first published in 1973, and translated into English and French the following year. It covers life in what is often known as the Gulag, the Soviet forced labor camp system. The "Witnesses of the Archipelago" is the list of 257 names of those whose stories, letters, memoirs, and corrections were used in the making of the book.

Finally, it is impossible not to mention that Olga Alexandrovna was a very beautiful woman. In a congratulatory letter composed for her 60th birthday, Alexander Danilovich Alexandrov wrote: “To have so much beauty and talent in one person would seem impossible, if it wasn’t for Olga Alexandrovna.” After visiting Leningrad, the French mathematician Jean Leray said that he “saw the Hermitage Museum, the Peterhof Palace, and Ladyzhenskaya.”

Darya Apushkinskaya and Alexander Nazarov

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O.A. Ladyzhenskaya and the problem of the global unique solvability of the Navier–Stokes equations

One of the most important models in mathematical hydrodynamics is the following Navier–Stokes system of equations in $\mathbb{R}^3 \times (0, +\infty)$:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\,\Delta u + (u\cdot\nabla)u + \nabla p \;=\; 0; \\ \qquad \operatorname{div} u \;=\; 0, \end{cases} \tag{1}$$

which describes the flow of a viscous incompressible fluid. The variables in system (1) are the vector field u and the scalar function p, they play the role of the fluid velocities field and pressure, respectively. One of the central questions regarding this system is whether the given model provides a deterministic description of fluid dynamics, in other words, if the definition of the initial data

$$u|_{t=0} \;=\; u_0 \tag{2}$$

uniquely determines the solution of system (1) for all t.

To answer these and similar questions in the 20th century, the theory of partial differential equations began to develop new approaches based on the methods of functional analysis. Among the scientists who contributed to the development of new ideas, one can name N.M. Günther, S.L. Sobolev, J. Leray, R. Courant, K. Friedrichs, and many others. In particular, using the concept of weak derivatives introduced by S.L. Sobolev, J. Leray [14] proved the global existence of weak solutions, later called the Leray–Hopf solutions, and established the existence of strong solutions on a finite time interval. By now, the uniqueness of the Leray–Hopf solutions is an open problem. On the other hand, the strong solutions are unique in the class of Leray–Hopf solutions, but their global existence is unknown. This was the situation in the theory of the Navier–Stokes equations by the time when O. A. Ladyzhenskaya joined the research in this area.

In 1957, the paper *On the existence and uniqueness of the solution of the nonstationary problem for a viscous, incompressible fluid* was published; see [3]. It can be said that already in this work Olga Alexandrovna's position was determined, which she adhered to in her studies of the Navier–Stokes equations throughout her life. Olga Alexandrovna always considered the primary question of the uniqueness of solutions (and not of their regularity) for nonstationary problems, and considered the formulation of a problem to be "correct" if the question was to find functional classes in which it would be possible to prove simultaneously both the global existence of solutions and their uniqueness. While for the Navier–Stokes equations the uniqueness problem for solutions is closely related to the question of their regularity (for the Navier–Stokes system, the following "weak-strong" uniqueness theorem holds: the existence of a smooth solution implies that any weak solution, the Leray–Hopf solution, must coincide with it), Olga Alexandrovna admits that the solutions of equations describing the dynamics of viscous fluids in principle can form singularities over time, but despite this, should be described mathematically by models that give a deterministic description of such flows.

In [3], different variants of functional classes are presented, in which the theorem of uniqueness of solutions to initial-boundary value problems for equations (1) is valid; the existence of solutions in these classes on a finite time interval is proved, with a lower bound for the lengths of the corresponding intervals. Olga Alexandrovna adhered to the philosophy of this work all her life — to look for functional classes in which the global unique solvability holds. She believed that the class of Leray–Hopf solutions is unacceptably large and there is no uniqueness in it. This problem is still open, but the paper [2] partly confirms the conjecture.

In 1958, O.A. Ladyzhenskaya [4, 5] proved the global unique solvability of the initial-boundary value problem for the following two-dimensional Navier–Stokes equations in $Q_T := \Omega \times (0, T)$:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\, \Delta u + (u \cdot \nabla) u + \nabla p \;=\; 0; \\ \qquad \operatorname{div} u \;=\; 0; \\ \qquad u|_{t=0} \;=\; u_0, \quad u|_{\partial\Omega\times(0,T)} \;=\; 0, \end{cases}$$

where Ω is an arbitrary domain in $\mathbb{R}^2$. This generalizes the results of J. Leray [15] for the two-dimensional Cauchy problem. In contrast to J. Leray, she uses her concept of choosing the correct functional class. Her proof is based on the following interpolation inequality:

$$\|u\|^4_{L_4(\mathbb{R}^2)} \;\le\; 2\, \|u\|^2_{L_2(\mathbb{R}^2)} \|\nabla u\|^2_{L_2(\mathbb{R}^2)},$$

which is currently called the Ladyzhenskaya inequality.

Later, Olga Alexandrovna continued to search for cases such that the global unique solvability of the Navier–Stokes equations takes place. In 1961, the first edition of her book *The Dynamics of a Viscous Incompressible Fluid* was published (see [6]). This book, translated into English in 1963, became the basic textbook on the mathematical theory of the Navier–Stokes equations for many generations of mathematicians around the world for many years. In 1968, the paper [9] was published, in which the global unique solvability of the Cauchy problem for the Navier–Stokes equations for axisymmetric initial data without the angular velocity component was proved (similar results were obtained in [18], also published in 1968). In [8], O.A. proves that the uniqueness theorem for the three-dimensional Navier–Stokes equations holds in the class of weak solutions with the following norm:

$$\int_0^T \|u(\cdot,t)\|^l_{L_s(\Omega)} \quad dt < +\infty, \qquad \frac{3}{s} + \frac{2}{l} \leq 1, \quad s \in (3,+\infty], \quad l \in [2,+\infty]. \tag{3}$$

Observe that the class (3) was earlier considered in [16] and [17]. In the same paper, O.A. also proves the smoothness of the solutions satisfying condition (3). For the axisymmetric Navier–Stokes equations without the angular velocity component, O.A. [10] constructs an example of non-uniqueness of weak solutions with a finite energy norm. In the constructed counterexample, an initial-boundary value problem is considered in the axisymmetric domain Q_T defined in cylindrical coordinates (r,φ,z) by

$$Q_T := \{\ t \in (0,T),\ r,z \in (a\sqrt{t}, b\sqrt{t})\ \},\ \ 0 < a < b.$$

The non-unique solutions constructed by O.A. belong to the Leray–Hopf class and, moreover, satisfy condition (3) for all $s>3$ and $l \geq 2$ such that

$$\frac{3}{s} + \frac{2}{l} > 1.$$

O.A. indicates that the constructed example of non-uniqueness gives reason to believe that in the Cauchy problem for the Navier–Stokes equations, the class of Leray–Hopf solutions is possibly too large for uniqueness, and in the energy class, the initial-boundary value problem for the Navier–Stokes equations may be incorrect. This was a very bold hypothesis at that time.

On the other hand, when studying the three-dimensional Navier–Stokes equations, O.A. also considered it illegal to deliberately narrow the functional class of "physically correct" solutions to the class of infinitely smooth functions. She always emphasizes (see, for example, [11]) that the primary question concerning the Navier–Stokes equations is that of global unique solvability, in fact, the question of finding a functional class in which one can establish both the global existence of solutions and their uniqueness. She believed that the formulation of the "Sixth problem of the millennium" proposed by Ch. Fefferman (see [1]) and replacement of the problem of a deterministic description of fluid dynamics by the question of studying the global existence of smooth solutions, to some extent, transferred the problem from a philosophical plane into the category of purely sporting achievements.

O.A. presented her views on the "Sixth problem of the millennium" in [11], as well as in her talk at a seminar on May 3, 2001, at Princeton University.

In this regard, in parallel with the study of the Navier–Stokes equations, Olga Alexandrovna was also looking for other nonlinear hydrodynamic models that, on the one hand, would allow the existence of nonsmooth solutions. On the other hand, it was expected that global unique solvability would hold in the energy class. Such models were announced in the report [7] at the Mathematical Congress in Moscow in 1966 and are currently called "Ladyzhenskaya models." Later it was found that this class includes many models well known in fluid mechanics and turbulence theory, in particular, generalized Newtonian fluids and the Smagorinsky model.

The proof of the global unique solvability of the "modified Navier–Stokes equations" (as O.A. called them) was published in [12, 13]. Later, in the 90s of the 20th century, these equations became a favorite topic of O.A. and she devoted numerous papers to their study.

Grigorii Seregin and Timofei Shilkin

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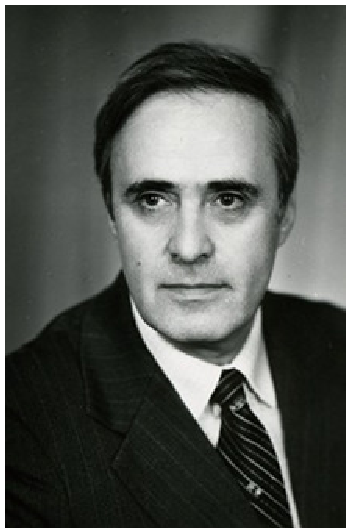
Mikhail Shlemovich Birman (1928–2009)

It is hard to overestimate M.S. Birman's contributions to the spectral theory of operators and to mathematical physics more generally. Especially famous is the Birman–Schwinger principle in the theory of discrete spectra, which serves as a starting point for many problems in quantum mechanics. To a large extent, Birman's work on scattering theory sets the course for the development of this field. In particular, he discovered the invariance principle for wave operators. Another well-known result is the Birman–Krein formula, linking the scattering matrix to the spectral shift function. M.S. Birman and M.Z. Solomyak created the theory of double operator integrals and developed the method of piecewise-polynomial approximations of Sobolev classes. Using it they got precise estimates and asymptotics for the spectra of differential and integral operators. Jointly with his students and colleagues, he extended the spectral theory of the Maxwell operator to the non-smooth case. He also studied discrete spectra that appear in gaps of self-adjoint Hermitian operators under perturbations of various classes. Together with T.A. Suslina, he contributed a lot to the spectral theory of periodic differential operators by solving the problem about the absolute continuity of the spectrum and by developing an operator-theoretical approach to the homogenization theory.



Mikhail Shlemovich (Solomonovich) Birman was born on January 17, 1928, in Leningrad. His father was a specialist in theoretical mechanics. He was a professor at the Leningrad Institute of Refrigeration. His mother was a schoolteacher.

During WWII, Birman's family was evacuated to Sverdlovsk (now Ekaterinburg), where Mikhail graduated from high school. When the war ended, and the family moved back to Leningrad, he enrolled at the Leningrad Electrotechnical University (LEU). The math professors there noticed Mikhail's extraordinary mathematical abilities and advised him to transfer to the Faculty of Mathematics and Mechanics of Leningrad State University (LSU). Mikhail followed their advice.

During his studies at the math department, he specialized in numerical analysis. Mikhail Solomonovich considered his teachers to be Mark Konstantinovich Gavurin, who supervised his thesis, and Leonid Vitaliyevich Kantorovich. When he was still a student, Mikhail Solomonovich worked at the Steklov Institute in Kantorovich's laboratory. Leonid Vitaliyevich recognized his young colaborator's strong intellect and independent thinking and started giving him tasks that greatly surpassed the level of standard technical work. In 1950, Mikhail Solomonovich graduated from the University. Although he was one of the best students in his graduating class, he was not accepted to the postgraduate school because of the tacit anti-Semitic policies during that time.

In 1947, Mikhail Solomonovich married his classmate, Tatyana Petrovna Il'ina. In 1948 they had a son, Zhenya. He and Tatyana Petrovna lived happily together throughout their whole lives; she died just two years before he did. Thanks to her love, loyalty, patience, and care, he could fully dedicate himself to mathematics.

After graduating from the university, Mikhail Solomonovich worked as a teaching assistant at the Leningrad Mining University, in the department of mathematics. Despite his heavy teaching load, he was very active in scientific research. In 1954 he got his Candidate's degree.[1]

In 1956, when state policy eased up, Mikhail Solomonovich got a position at the chair of mathematical physics of LSU, at the initiative and with the serious support of V.I. Smirnov and O.A. Ladyzhenskaya. It was said that Smirnov declared an ultimatum to the Rector of the University: "Either you take Birman, or I leave LSU."

Smirnov's demand was granted, and in time, Birman became one of the best lecturers in the Physics department. In 1962, he got the Doctor of Sciences[2] degree for his thesis *On the Spectra of Singular Boundary Value Problems*. Mikhail Solomonovich worked at the chair of mathematical physics for the rest of his life — over 50 years.

With the help of his colleague, Mikhail Zakharovich Solomyak, Birman created a strong scientific school in the spectral operator theory rocognised worldwide. Many of his students became famous scientists and are now working at the best universities in Russia and the West.

Mikhail Solomonovich Birman is the author of over 160 scientific papers and 2 books. He was a member of the editorial boards for the journals *St. Petersburg Mathematical Journal* and *Functional Analysis and its Applications.*

[1] A Candidate's degree in the Soviet Union was comparable to a Ph.D. in the American university system.

[2] A Doctor of Sciences degree is a higher doctoral degree that can be obtained after the Candidate's degree; similar to Habilitation in Germany.

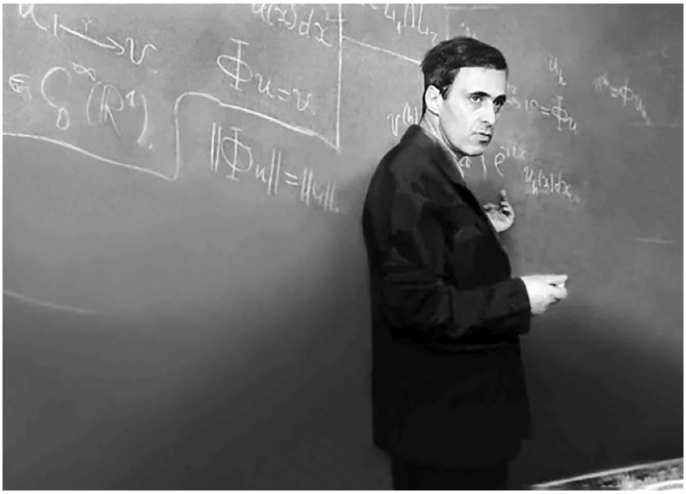
An important role for Mikhail Solomonovich was his active participation in the Leningrad Seminar of Mathematical Physics, which was founded by Vladimir Ivanovich Smirnov in 1947. Now the seminar carries Smirnov's name. For many years, Mikhail Solomonovich was head of the V.I. Smirnov Seminar, along with Olga Alexandrovna Ladyzhenskaya. This photo was taken in 1968. From left to right: M.Z. Solomyak, N.N. Uraltseva, N.F. Morozov, M.S. Birman, M.M. Smirnov.

Some of Birman's awards include the title of Honored Scientist of the Russian Federation, Honored Professor at St. Petersburg State University, and the Chebyshev Award granted to him by the St. Petersburg government. Birman's works have obtained international recognition and are often quoted in the literature. On multiple occasions, he was a plenary speaker at international conferences. He was personally invited to some of the best universities and scientific centers in the world.

Let us describe Mikhail Solomonovich's scientific style a bit. He rarely thought in terms of separate problems, no matter how interesting they seemed to be by themselves. His typical approach was as follows. First, he looked for a general pattern that included the problem, and then that pattern was analyzed from every angle. Any theories emerging from this process could be applied to a wide range of similar problems. Finally, he figured out how parts of the theory worked in the case of the original problem. Usually, this approach led to an exhaustive analysis not only of the original problem but also of the whole class of similar problems.

His approach to developing theories to solve many related problems did not mean, however, that Mikhail Solomovich was worse at solving specific problems. His papers are filled with technical findings that are still widely used.

He was always pushing forward, although he never left a topic unfinished. Mikhail Solomonovich liked to repeat the phrase: "All my life, I am writing



Mikhail Solomonovich was a brilliant lecturer. His lectures were not only informative and well-thought-out, but inspiring as well. He was one of the department's leaders and had an exceptionally high reputation. He always had very high professional standards, which, above all, he applied to himself. Mikhail Solomonovich was very attentive to the people around him. Many of his colleagues regarded him as an excellent professor and a wise person in general.

the same paper," even though he contributed a significant amount of work to different fields of mathematical physics. When he moved from one topic to another, however, there was always a string connecting them.

M.S. Birman and O.A. Ladyzhenskaya hiking in Azau, Caucasus, 1972.

His mindset was not purely mathematical. When choosing a new topic of research, he was often guided by direct applications to natural physics problems. He had many talents, so mathematics should consider itself lucky that he preferred it to other fields.

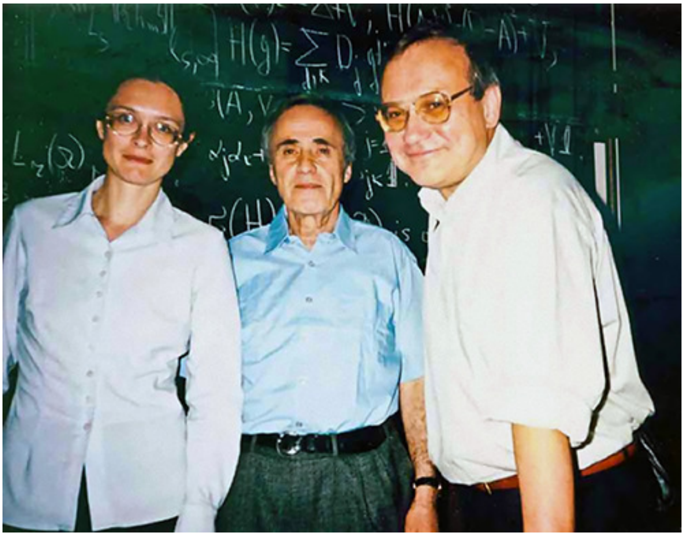


Photo with the authors of this text — T.A. Suslina (left) and D.R. Yafaev (right).

Photo with his students, St. Petersburg, 1999. Bottom row (from the left): V.A. Slousch, A.B. Pushnitsky, R.G. Shterenberg. Top row: N.D. Filonov, E.L. Korotyayev, O.L. Safronov, M.S. Birman, T.A. Suslina, A.A. Laptev.

Mikhail Solomovich had a sharp mind and a well-rounded education. It was interesting and edifying to listen to him. Sometimes this perspective was overly romantic. He wrote poems. He especially loved nature. He was known to go on walks or bike rides outside the city and enjoyed hiking. He knew St. Petersburg and its outskirts very well.

Despite his bad health, he actively worked on mathematics until his last days. Mikhail Solomonovich Birman died on July 2, 2009, after a severe and long illness.

Tatiana Suslina and Dmitri Yafaev

M.Sh. Birman: Spectral and Scattering Theories

Perturbation theory plays an important role in the spectral theory of self-adjoint operators. It draws conclusions about a self-adjoint operator B given an information regarding a simpler operator A close to B in some sense. In particular, perturbation theory on the absolutely continuous (a.c.) spectrum is known as the scattering theory.

Originally, M.Sh. was not an expert in scattering theory. In the fifties he wrote a row of seminal papers on essential and discrete spectra of self-adjoint differential operators. In particular, he proved the stability of the essential spectra of elliptic operators under a wide class of perturbations of their coefficients and of the associated boundary conditions. For the discrete spectrum we only mention the famous estimate on the total number N of negative eigenvalues of the Schrödinger operator $-\Delta + V(x)$ in $L^2(\mathbb{R}^3)$:

$$N \leq \frac{1}{16\pi^2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{V_-(x)V_-(y)}{|x-y|^2} dxdy, \quad V_- = \min(V, 0).$$

This was the first quantitative estimate for the negative spectrum in the multi-dimensional case. It was independently found by J. Schwinger and is usually called the Birman–Schwinger estimate. Its proof relied on a general result of operator theory, which is now known as the Birman–Schwinger principle.

M.Sh. turned his attention to the a.c. spectrum after the famous theorem by T. Kato and M. Rosenblum appeared in 1957. This theorem concerns a pair of self-adjoint operators A and B acting in a separable complex Hilbert space. It states that if the difference $B - A$ belongs to the trace class, then the wave operators defined as strong limits

$$\lim_{t\to\pm\infty} e^{iBt}e^{-iAt}P_{ac}(A) =: W_\pm(B, A)$$

exist; here $P_{ac}(A)$ is the orthogonal projection on the a.c. subspace of A. In particular, this result implies that the a.c. parts of the operators A and B are unitarily equivalent.

Wave operators appear naturally in quantum mechanics. Indeed, consider two systems described by vectors f_0 and f at the time $t = 0$ and governed by the "free" A and "perturbed" B Hamiltonians. Their time-dependent evolutions are described by the unitary groups $e^{-iAt}f_0$ and $e^{-iBt}f$. It turns out that these evolutions are asymptotically close as $t \to \pm\infty$, that is, $e^{-iAt} f_0 \sim e^{-iBt} f$ if $f = W_\pm(B, A) f_0$.

If the wave operators exist, then the scattering operator

$$\mathbf{S} := W_+^*(B, A) W_-(B, A)$$

commutes with A and thus acts as multiplication by an operator-valued function $S(\lambda)$ in the diagonal representation for A. The scattering operator $\mathbf{S}$ and the scattering matrix $S(\lambda)$ are usually of great interest in problems of mathematical physics because they connect the initial (for $t \to -\infty$) and the final (for $t \to +\infty$) characteristics of the process directly, bypassing its consideration for finite times. This also explains the term "scattering theory" borrowed from physics.

Although beautiful and very sharp in the general framework of operator theory, the Kato–Rosenblum theorem cannot be directly applied to differential operators where $B - A$ is a multiplication operator. Naturally, the problem of applications of this general result attracted the attention of T. Kato himself, S.T. Kuroda, and many other mathematicians. The contribution of M.Sh.'s contribution to this highly competitive domain was crucial.

The study of the a.c. spectrum was for M.Sh. a natural continuation of his analysis of the essential spectrum. The connecting point is his paper of 1962 where the invariance of the a.c. spectrum was verified for perturbations of the boundary and of the boundary condition for elliptic operators in unbounded domains. The initial, and as it turned out later very fruitful, idea of M.Sh. was to consider suitable functions φ (for example, inverse powers) of the operators A, B and to apply the Kato–Rosenblum theorem to the pair $\varphi(A)$, $\varphi(B)$.

The invariance of the absolutely continuous spectrum allowed M.Sh. to approach the conjecture that for the trace class difference $\varphi(B) - \varphi(A)$ not only the a.c. spectrum is preserved, but also the wave operators $W_\pm(\varphi(B), \varphi(A))$ exist and

$$W_\pm(\varphi(B), \varphi(A)) = W_\pm(B, A).$$

This result, proven by M.Sh. in 1963 for a wide class of functions φ, was later called *the invariance principle.* This is an important generalization of the Kato–Rosenblum theorem which can be directly applied to differential operators of the Schrödinger type.

At the same period, jointly with M.G. Kreĭn, M.Sh. found a link between the scattering theory and the theory of the spectral shift function. The concept of the spectral shift function $\xi(\lambda)$ appeared in the early fifties in the physics literature in the papers of I.M. Lifshitz in connection with the trace formula

$$\mathrm{Tr}\,\big(\varphi(B) - \varphi(A)\big) = \int_{-\infty}^{\infty} \varphi'(\lambda)\xi(\lambda)d\lambda.$$

M.Sh. and M.G. Kreĭn showed that the scattering matrix $S(\lambda)$ differs from the identity by a trace class operator (so that the determinant of $S(\lambda)$ is well defined) and found a remarkable formula

$$\det S(\lambda) = \exp(-2\pi i\xi(\lambda))$$

valid for almost all λ in the a.c. spectrum. This elegant relation (known as the Birman–Kreĭn formula) is often used as the definition of the spectral shift function on the a.c. spectrum.

Motivated by applications, M.Sh. developed the scattering theory in various directions. He carried over (jointly with M.G. Kreĭn) the Kato–Rosenblum theorem to unitary operators, introduced local wave operators related to some interval of the spectral axis, and constructed the scattering theory for self-adjoint operators A, B acting in different Hilbert spaces.

M.Sh. was a brilliant lecturer. He had an exceptional ability to present difficult things in a particularly transparent way and to find non-trivial connections between apparently different facts. It is noteworthy that M.Sh. always had a firm hand over the audience, which was very beneficial for everybody.

Relations between M.Sh. and his numerous students were very tight and went far beyond purely mathematical subjects. In particular, the authors of this text benefited a lot from his human personality.

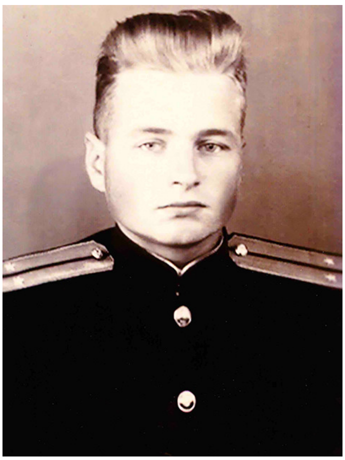
M.Sh. was a very wise person. He permanently thought about different events happening in the world and, as he put it himself, always tried to create the correct world picture. Sometimes this picture was overly romantic.

This very cursory presentation can be supplemented by a more detailed paper by M.Z. Solomyak, T.A. Suslina, and D.R. Yafaev *On the mathematical works of M.Sh. Birman*, Saint-Petersburg Math. J., v. 23, n.1, 5–60, 2011.

Tatiana Suslina and Dmitri Yafaev

Askold Ivanovich Vinogradov (1929–2006)

Askold Ivanovich Vinogradov made major contributions to number theory. His most famous result is the celebrated Bombieri–Vinogradov theorem on the distribution of primes in arithmetic progressions, averaged over a range of moduli. In many problems of analytic number theory this result replaces the generalized Riemann hypothesis. Vinogradov proved this theorem in *On the density hypothesis for Dirichlet L-series*,[1] and independently of him, Enrico Bombieri proved it[2] in *On the large sieve*;[3] Bombieri's proof is based on the development of Linnik's large sieve method.



Askold Ivanovich Vinogradov was born on October 1, 1929, in the Vsevolozhskiy district of the Leningrad region, in the settlement near Shlisselburg Fortress.[4] His parents, Maria Alexandrovna Sorikhina (Vinogradova) and Ivan Georgievich Vinogradov, were the descendants of peasants from the Tver governance. During WWII, the settlement, where young Askold lived with his mother and younger sister Diana, was near the front line: the ring around the Siege of Leningrad had closed on September 8, 1941, with the right bank of the Neva and the Schlisselburg fortress controlled by Soviet troops and the left bank by the Germans. In February 1942, Askold ran[5] away from home across Lake Ladoga, was detained in Vologda, and sent to an orphanage. In the summer of the same year, he

[1] See Izvestia AS USSR. Mathematical Series. 29 (1965), P. 903–934.

[2] For this and other works, Enrico Bombieri was awarded the Fields Medal in 1974, see https://www.mathunion.org/fileadmin/IMU/Prizes/Fields/1974/index.html

[3] See Mathematika. 12 (1965). P. 201–225.

[4] The fortress, also called Oreshek, served as a prison until 1919. Alexander Ulyanov, Lenin's elder brother, was executed there for an assassination attempt on Alexander II. After the revolution, the nearby village on the right side of the Neva was converted into a workers' settlement named after the member of "Narodnaya Volya" N.A. Morozov. There was a gunpowder factory there, where Ivan Georgievich Vinogradov probably worked.

[5] That is according to his 1952 autobiography. However, in his 1991 autobiography, Vinogradov wrote that he was evacuated from Leningrad via the "Road of Life" in March 1942.

was sent to another children's home in the Vologda Region, where he finished 7th grade in 1945. His mother and sister were evacuated from Leningrad in March of 1942. His mother found him in 1944; in June 1945, he left the children's home to join his mother, and in 1947 his mother and Diana returned to Leningrad.

Let us follow with a direct quotation from the 1991 autobiography (that is kept in the LOMI[6] archives):

> In 1945, I entered the Baku Naval Preparatory School. Two years later, our school relocated to Kaliningrad (former Königsberg), where I graduated from the Preparatory School in 1948 and was then transferred to the 1st year of the 2nd Baltic Higher Naval School,[7] which was based on and meant as the continuation of our Preparatory School. I graduated from Higher Naval School in 1952 with a torpedoman specialty and received an officer-torpedoman certificate, №743413. I was transferred to the reserve during our allocation and, by special order of the Minister of the Navy, was sent to Moscow, to the postgraduate school of the V.A. Steklov Mathematical Institute of the USSR Academy of Sciences to Academician Ivan Matveyevich Vinogradov. Although the whole thing was done at the special request of the then President of the USSR Academy of Sciences S.I. Vavilov, I.M. Vinogradov and Y.V. Linnik were in fact behind it. These two men followed my fate throughout the years that I studied at the Naval School.

So, after seven years of military experience, on November 15, 1952, young Askold Vinogradov entered postgraduate school at the Steklov Institute of Mathematics with I.M. Vinogradov as his supervisor. Askold learned the famous I.M. Vinogradov method in number theory and, according to his friends, told his senior namesake, "Now I will solve problems using your method."

I.M. Vinogradov said in response,

> were there any interesting ones [problems], I would have solved them myself. I will send you to postgraduate school in Leningrad to study other things under Y.V. Linnik's supervision.

On February 15, 1953, Vinogradov was transferred to LOMI, and Linnik became his scientific supervisor.

On December 24, 1953, A.I. Vinogradov passed the postgraduate examination with the grade "excellent," answering three questions in front of a commission consisting of Y.V. Linnik, L.V. Kantorovich, and A.A. Markov. The questions were the following: the Cauchy integral, the Fourier transform, and the notion of the Jordan curve. In the report for 1953, there is a list of books that Vinogradov read during his first year of postgraduate study at LOMI:

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[6] LOMI (now POMI or PDMI) is the Russian abbreviation for the Leningrad branch of the V.A. Steklov Mathematical Institute of the USSR Academy of Sciences.

[7] 42nd Baltic Higher Naval School of Surface Navigation (unit 78347)

Fichtenholz' *Differential and Integral Calculus*, in three volumes; Privalov's *Introduction to Complex Analysis*; Vinogradov's *Method of Trigonometric Sums in Number Theory*; Ingham's *The Distribution of Prime Numbers*; and Titchmarsh's *The Riemann Zeta Function.*

When the 1954 annual report of LOMI was being discussed, Linnik characterized Vinogradov in the following way: "I think Askold Ivanovich will do well with his subject: he is an assertive man."

Having defended his Ph.D. thesis, *Additive Problems with Two Prime Numbers and Additional Terms* in 1955, Vinogradov was retained at LOMI as a junior researcher. At that time, Linnik was interested in algebraic geometry and conducted a seminar on André Weil's *Foundations of Algebraic Geometry.*

The thing is, that from the Riemann hypothesis for zeta-functions of curves over finite fields, proved by André Weil in 1948, follows the best possible estimate for special trigonometric sums, the Kloosterman sums. Weil's estimate is inaccessible for the usual methods of the theory of trigonometric sums, including the poweful I.M. Vinogradov's method, so Linnik wanted to understand and generalize Weil's result. Vinogradov recalled that the keynote speaker of the seminar was Linnik's student Boris Skubenko, later a remarkable expert in the geometry of numbers, and each talk began with Linnik's request, "Boris Faddeyevich, remind us, please, the definition of sheaves."[8]

On January 4th, 1963, Askold Ivanovich defended his habilitation[9] thesis, *A Study of Properties of Euler Products for Zeta Functions of Various Algebraic Number Fields and Their Application to Problems of Analytic and Algebraic Number Theory*; from 1964 onwards, he worked as a senior researcher at LOMI (and as a leading researcher from 1986). As mentioned above, in 1965 he proved the density conjecture for the Dirichlet L-series, averaged over the moduli. This remarkable result was not properly appreciated at the Steklov Institute at the time; only a quarter of a century later, A.I. Vinogradov was awarded the I.M. Vinogradov Prize for this work.

From the mid-1960s onwards, Vinogradov's research interests broadened considerably. For instance, in 1967, he tried (unsuccessfully) to prove the well-known Kummer's conjecture on the distribution of the arguments of cubic Gauss sums and proposed an interesting approach to the analytic continuation of the Artin L-function and its connection with the reciprocity laws. The year 1973 was a turning point in his scientific philosophy when he attended L.D. Faddeev's lectures on the Selberg trace formula at a mathematical school in Vilnius. Vinogradov was one of the first to realize that the spectral theory of automorphic functions can provide a powerful new method in analytic number

[8] The story is possibly apocryphal: Ludwig Faddeev said that when he was giving lectures at LSU at the student seminar on quantum field theory, Olga Ladyzhenskaya began each talk with "Ludwig, please remind us the definition of creation and annihilation operators."

[9] The Russian "doctor nauk" degree has no academic equivalent in North America; it is a higher doctoral degree and is roughly comparable to the German *Habilitation*, the French HDR, and British higher doctorates.

theory, and he devoted himself wholly to that subject.[10] A complete list of A.I. Vinogradov's papers can be found on his page at Math-Net.Ru [1].

At this point, a friendship began between Askold Vinogradov, Ludwig Faddeev, and Ludwig's young students, Aleksei Venkov and Leon Takhtajan. Vinogradov participated actively in life at LOMI and organized a weekly seminar on modern number theory with Aleksei Venkov, Nikolai Proskurin, Maxim Skriganov, and Leon Takhtajan. Boris Venkov and Boris Skubenko gave talks at the seminar; Sergei Stepanov and Andrei Tyurin came from Moscow to participate, along with many others. Askold Vinogradov actively supported Nikolai Kuznetsov during the difficult moments of his life. New ideas led N.V. Kuznetsov to the "Kuznetsov trace formula," one of the basic elements of modern analytic number theory. Young Viktor Bykovskii, a graduate of Moscow State University and a student of N.M. Korobov, also came from Moscow. He was at a crossroads, as I.M. Vinogradov had advised him to "improve" the zero-free region of the Riemann zeta-function. Vinogradov introduced Bykovskii to Kuznetsov's trace formula, and from that moment on, Viktor's successful work in number theory has started: in 1982, Ludwig Faddeev submitted Bykovskii's paper to the *Doklady of the USSR Academy of Sciences*; now, V.A. Bykovskii is a well-known Russian mathematician, a corresponding member of the Russian Academy of Sciences.

Vinogradov at work.

The joint paper, *Zeta function of the additive divisor problem and the spectral decomposition of the automorphic Laplacian*, by Vinogradov and

[10] Starting with the pioneering work by Nikolai Kuznetsov on Kuznetsov's trace formula, the method has been successfully applied to various problems in number theory. We refer the reader to the papers by the Polish-American mathematician Henryk Iwaniec and the South African-born American mathematician Peter Sarnak and their co-authors working in the USA.

Takhtajan, was recognized as the best mathematical work of the year 1984 in the Mathematics Department of the Academy of Sciences of USSR.

A.I. Vinogradov was actively involved in the scientific and public life at LOMI and participated in all the informal activities of the Laboratory of Mathematical Problems in Physics, led by L.D. Faddeev. The geography of the offices on the 5th floor of LOMI facilitated interactions: Ludwig Faddeev in office 506, Aleksei Venkov and Leon Takhtajan in 507, Askold Vinogradov in 508. Askold Ivanovich came on "presence days," [11] Mondays and Thursdays, and his office became the center of interactions on scientific, literary, and socio-political subjects. Vinogradov traditionally hosted tea parties, of which Boris Venkov and Boris Skubenko were frequent participants, gathering friends both from neighboring offices and from other floors of LOMI.

While Ivan Matveyevich Vinogradov was alive, Askold Ivanovich had good relations with the people at MIAN (the Steklov Mathematical Institute at Moscow), but they took a turn for the worse because of the conflicts at MIAN after I.M. Vinogradov death in 1983. So, in 1987, Vinogradov moved to Khabarovsk to join Kuznetsov and Bykovskii; on February 15, 1987, he became a chief researcher at the Institute for Applied Mathematics of the Far East Branch of the USSR Academy of Sciences. In December of 1991, he returned to LOMI. During these years, Vinogradov worked only on the major problems of number theory, remaining confident in the possibilities of the spectral theory of automorphic functions till the end of his days.

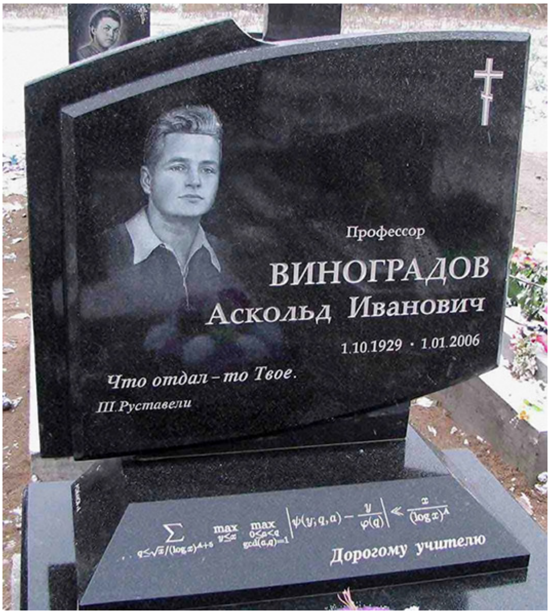
Askold Ivanovich Vinogradov was a unique person: a naval officer and a true gentleman, in the 19th-century sense of the word, a famous mathematician, a pleasant companion for conversation and discussion who never raised his voice, a fan of ballet and rhythmic gymnastics (in his younger years) and a sambo[12] athlete who could deal with any bully. While studying at the Naval School, he was a member of the Komsomol.[13] In 1956, he went to Kazakhstan to help with the harvest on tselina[14] land, and in 1957–1961 he was an instructor of air defense at LOMI. He did not join the Communist Party, as he was critical of the many aspects surrounding its policies and ideology. At the same time, he was a patriot of his country and had a deep respect for the Great Patriotic War and the Supreme Commander of the Soviet Army. Like Boris Faddeyevich Skubenko, he was very keen on war memoirs and avidly read the memoirs of both Soviet commanders and German generals.

[11] In academic institutions, there was usually no need to come to work every day: one or two days a week were compulsory and called "presence days," and on the other days, called "library days," academics could go to the library or work at home.

[12] Sambo is a martial art that originated in the Soviet Union in the 1920s.

[13] Komsomol was a political youth organization in the Soviet Union, de facto the youth division of the Communist Party, although officially independent.

[14] Tselina or virgin lands is an umbrella term for underdeveloped, sparsely populated, highly fertile lands often covered with chernozem soil, which were mostly located in the steppes of the Volga region, Northern Kazakhstan, and Southern Siberia.



Vinogradov's tomb.

His lifestyle was very modest, and he was consistent in all his habits. For example, during Soviet times he used to go to Novy Svet in Crimea every summer, tried to walk as much as possible, regularly came to Komarovo, and during his walks with Takhtajan liked to discuss new approaches to various problems in number theory. As for drinks, he could allow himself a little champagne, being at the same time indulgent towards the tastes and predilections of his colleagues. He was very generous and always helped old friends who were in trouble. The Khabarovsk period of Vinogradov's life is reflected upon in a review by V.A. Bykovskii [2].

Askold Ivanovich would often marvel at descriptions in the tabloids of "contact with aliens" and people allegedly "abducted for experiments." He used to say, "If I had met them, I would immediately ask how to prove the Riemann hypothesis!" Perhaps now he knows...

Leon Takhtajan

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Victor Petrovich Havin (1933–2015)

Victor Havin, an outstanding mathematician working in Complex and Harmonic Analysis, was one of the leaders of the Saint Petersburg analysis community during a 50-year period starting from the mid-1960s. He has founded and shaped the modern St. Petersburg analysis school, which still continues to bring forth new generations of bright scientists. His main mathematical achievements are in the field of complex approximation, spaces of analytic functions, potential theory, and various manifestations of the Uncertainty Principle in Harmonic Analysis.

Havin organized and led the joint University/Steklov Institute Analysis seminar for several decades. A great number of young mathematicians, many of whom later achieved world renown, started their work within that seminar.

Victor Havin was born on the 7th of March, 1933, in Saint Petersburg (at that time Leningrad). His father (Petr Yakovlevich Havin) was a philologist, slavist, and one of the founders of the Journalism division (which later became a department) at Leningrad University; during WWII he served at the Red Army propaganda department and received several medals. The rest of the family succeeded in avoiding the apocalypse of the Siege of Leningrad, having been evacuated to Tashkent (now, Uzbekistan). Havin's mother (Dina Yakovlevna Havina) was a musician, a violinist at the Leningrad Philharmonic Orchestra and the Mikhailovsky (Maly) Opera Theater Ensemble.

After returning to St. Petersburg, Havin completed his education at one of the best high schools, the former "First St. Petersburg Gymnasium," in 1950. According to Havin, during this period he was more inclined to pursue a scholarly career in the humanities. He spoke at least three foreign languages fluently (German, French, and English), and the Foreign Languages Department of the University would have been a natural choice for him.

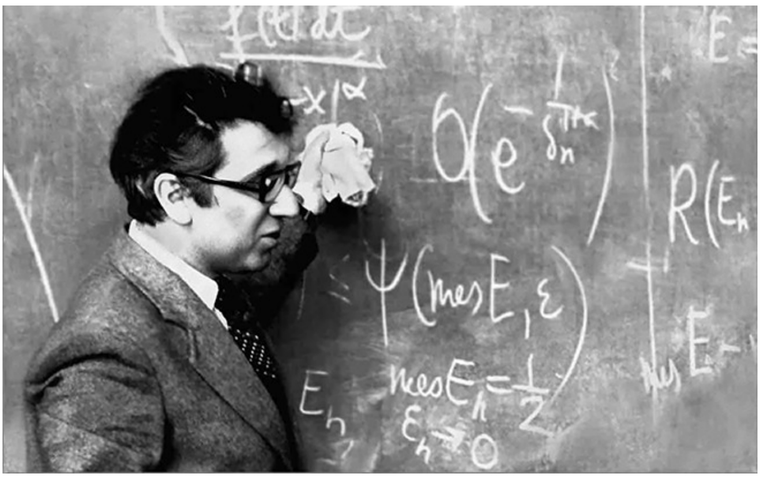
However, his father strictly forbade him to even think about linguistics and ordered his son to make a choice between physics and mathematics for his future studies. The reason for this was the publication of Josef Stalin's "scientific" opus, "Marxism and Problems of Linguistics," and the well-founded fear of Havin's father that linguistics would become an area of ideological purging as had happened with biology at the time. Luckily for the St. Petersburg mathematical community, Havin chose mathematics, and in 1950 he entered the Department of Mathematics and Mechanics (known simply as Mat-Mekh).

Among Havin's teachers at Leningrad University were G.M. Fichtenholz, the author of a famous analysis course, L.V. Kantorovich, future Nobel Prize laureate in economics, and V.I. Smirnov. Havin regularly participated in the research seminar of the Division of Analysis. In 1953 he gave a series of talks on duality theory at this seminar, which was novel and influential at this time, followed by a survey on "analytic functionals" by P. Lévy; later on, Havin was invited to give these talks again at the famous Gelfand seminar in Moscow.

In 1955 Victor Havin started his graduate studies under L.V. Kantorovich and, informally, V.I. Smirnov. Havin always considered their influence on his mathematical education to be very important. However, neither of them was a supervisor in the traditional sense of the word: neither directed the young PhD student in his choice of future research themes. For instance, Leonid Kantorovich said at the very beginning that he was busy at the time with economics applications, and Havin was free to choose his research direction himself. In this way, Havin turned out to be, in a sense, an "autodidact" in mathematics.

Havin defended his Candidate of Science (PhD) thesis in 1959. By this time, he already had 6 papers published in peer-reviewed journals and was the author of several remarkable results. One of them was the solution to a problem on the generalized Laurent representation of an arbitrary function analytic in the complement of a simple rectifiable arc, posed by a prominent analyst, V.V. Golubev, in the 1920s. Another important result from this period is related to the "separation of singularities" — a recurring topic among Havin's interests. Here the problem is to represent a function analytic in the intersection of two domains by a sum of functions that are analytic in the respective domains. Using the "soft" duality arguments, Havin gave a short proof of the results by Aronszajn.

In spite of his brilliant Ph.D. defense, Havin passed through a difficult period of his life right afterward. Due to the (of course, unofficial) anti-Semitic restrictions, widespread during this time in the Soviet Union, he was refused several teaching positions at various St. Petersburg technical universities. During this desperate situation, Prof. Dmitry Faddeev, mobilizing all his authority, greatly helped by insisting in front of the university administration that a position be opened for Havin.



A lecture in Ufa, 1980.

The entire life of V.P. Havin was connected to the Department of Mathematics and Mechanics of St. Petersburg (Leningrad) University — first (from 1959 to 1962) as an assistant and then (1962–1970) as an associate professor (a "docent" in Russian); in 1971 Havin was promoted to the rank of full professor, a position he occupied till his death in 2015. From 1997 to 2004 Havin served as the Head of the Analysis Division. In the 1960s, Havin played a crucial role in the modernization of the teaching of analysis at Leningrad University. Together with G.P. Akilov (one of his mentors and friends), he modernized the course of analysis; specifically, they included the Lebesgue integral and the integration of differential forms in $\mathbb{R}^n$ to the standard curriculum. At that time, this was a revolutionary step: these changes in Havin's analysis course were made earlier than at the Sorbonne, Moscow University, and other first-class universities around the world. Since then, he delivered this course 17 times and published a textbook on the material of his lectures.

He also delivered many specialized lectures as courses in Leningrad (Saint Petersburg) and, as an invited professor, in many other cities of the former Soviet Union and, later, abroad (for instance, he taught at McGill University in Montreal for several semesters from 1995 to 2002). The profound and state-of-the-art content of Havin's lectures, as well as their extremely vivid and clear presentation, always gathered full lecture halls and attracted the most brilliant students.

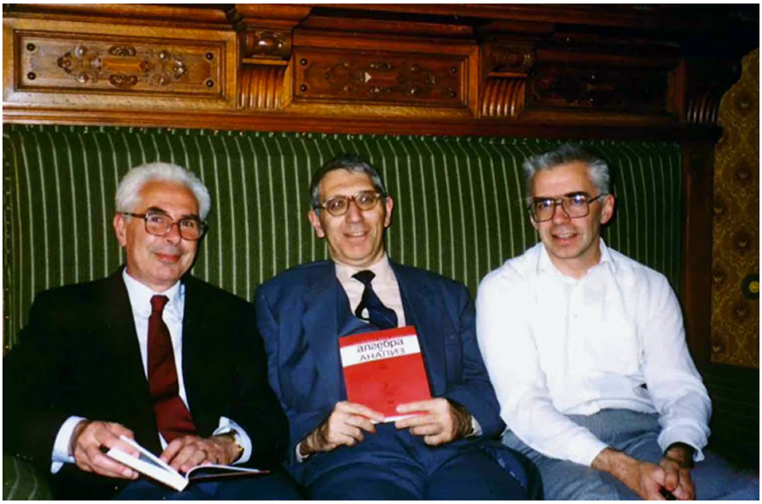
What follows is a brief account of the mathematical interests and main contributions of V.P. Havin to Complex and Harmonic Analysis. One of his early achievements was finding integral analogs of the Vitushkin theorem on uniform rational approximation. It turns out that in the mean square case the role of analytic capacity is played by the classical logarithmic capacity. Havin returned to the subject of the separation of singularities with estimates (e.g., in the class of bounded functions) several times; in particular, on joint papers with A. Nersessyan and J. Ortega–Cerdà. His last paper on this topic appeared in 2007.

In the 1970s Havin discovered and studied the phenomenon of the twofold decrease of smoothness of an analytic function compared with the smoothness of its absolute value on the boundary (this property is said to have been noticed for the first time by L. Carleson and S. Jacobs who, however, did not publish the result). A separate series of papers by Havin dealt with properties of holomorphic functions representable by the Cauchy integrals of complex measures.

In a long-time collaboration with V.G. Maz'ya, Havin founded the so-called nonlinear potential theory, which was applied to problems of uniqueness and approximation for analytic and harmonic functions and to the study of the Cauchy problem for the Laplace equation.

In a series of joint papers with E. Malinnikova and S. Smirnov, approximation properties of harmonic vector fields and differential forms were studied; multidimensional analogs of the Runge theorem and Hartogs–Rosenthal theorem were proven, and it was shown that the analog of Bishop's localization principle is not valid in dimensions higher than two.

One of Havin's favorite themes was the so-called Uncertainty Principle in Harmonic Analysis — a heuristic statement saying that a function and its Fourier transform cannot be small simultaneously. As a classic example one should mention the Heisenberg Uncertainty Principle. However, the "smallness" may be understood in a number of completely different ways



J.-P. Kahane, V.P. Havin (with the first volume of "Algebra and analysis" journal), and N.K. Nikolski after Kahane's talk at PDMI, 1989.

(smallness of support, fast decay at infinity, etc.), many of them leading to deep and important problems.

In a series of papers (written jointly, in part, with his student Burglind Jöricke) some "uncertainty principles" were proven for the convolution integral operators. Another profound result that Havin achieved deals with the uncertainty principle for M. Riesz potentials on a line and the (im)possibility of its extension to higher dimensions.

The work of V.P. Havin in the area of the Uncertainty Principle culminated in his seminal book *The Uncertainty Principle in Harmonic Analysis* written jointly with B. Jöricke and published in 1994 by Springer-Verlag. This book is a true encyclopedia on the subject that covers not only classical results of UP (often with new and more conceptual proofs) but also numerous original results obtained by members of Havin's seminar.

A list of Havin's publications (as complete as possible) can be found in [1].

However, probably the most important mathematical result of V.P. Havin, his "Best Theorem", was the mathematical tradition and community that he created. This community was formed around the joint University/Steklov Institute Analysis seminar. The seminar was started by Havin in around 1963 with only four of his young students, and has since grown to become the important cultural event for the entire city's mathematical community, a "must-attend" for all analysts in Saint Petersburg. Many prominent analysts started their mathematical life in this seminar.

Havin spent a considerable amount of his time and effort nurturing young talents. He supervised 31 PhD theses, and, according to the Mathematics

V.P. Havin and J.E. Brennan, Euler institute, 2008.

Genealogy Project, he currently has 184 descendants. Four of his former students (A. Alexandrov, F. Nazarov, S. Smirnov, A. Logunov) won the prestigious Salem Prize in Harmonic analysis; 6 more Salem prizes can be found among his "grandchildren" (A. Volberg, S. Treil, N. Makarov, S. Petermichl, Dapeng Zhan, D. Chelkak). Last but not least, Stanislav Smirnov was awarded the Fields Medal in 2010.

Undoubtedly, Havin's success in creating this special environment, a true hotbed for talent, owed much to his charismatic personality and an inexhaustible emanation of mathematical ideas and enthusiasm. Invariably, he attracted students and his colleagues with his amicability, openness, and intellectual generosity. He was a remarkably kind and decent person; a very rare pure not materialistic type, who never understood, nor wanted to understand, anything about money.

V.P. Havin's scientific and teaching activity got well-deserved (though, in the authors' opinions, quite delayed and insufficient) recognition. He got several national and international awards, including the degree of Doctor Honoris Causa from Linköping University (Sweden) in 1993. V.P. Havin was also elected the Spencer Lecturer at Kansas State University (USA, 1996) and the Onsager Professor at Trondheim University (Norway, 2000). In 2004, his exquisite results on admissible majorants for model subspaces were awarded the Robinson Prize by the Canadian Mathematical Society. Havin was also awarded the Chebyshev Prize by the St. Petersburg Government (2011) and an Honorary Professorship at St. Petersburg University. He became an Honored

Scientist of the Russian Federation in 2003 and was awarded the Order of Friendship in 2011.

Classical music and literature played a very important role in Havin's life. Among his favorite authors were Lev (Leo) Tolstoy, in particular Tolstoy's *War and Peace* (which V.P. studied like the Bible), Pushkin, Tyutchev, Bulgakov, and Vassily Grossman. With his mastery of languages, Havin preferred to read Western writers in the original, not as translations, especially in German (Goethe, Heine, T. Mann) and French (e.g., J.-P. Sartre's novels and essays). Havin often quoted famous literary maxims and formulae in everyday speech when they corresponded to his feelings at that moment. One of his last mathematical masterpieces, a purely real and, probably, most direct proof of the famous Beurling–Malliavin Multiplier Theorem, appeared in 2005 in a paper entitled *Beurling–Malliavin Multiplier Theorem: the Seventh Proof.* The title emphasizes the fact that several (about six) different proofs of this extremely profound theorem were known previously and at the same time it made an allusion to M.A. Bulgakov's famous novel *The Master and Margarita.* As an epigraph Victor Petrovich used the following quotation:

> ...На это существует седьмое доказательство, и уж самое надежное! И вам оно сейчас будет предъявлено. ("...Yet the seventh proof of this exists, which is reliable beyond a doubt! And it will be shown to you in a while," recall that Bulgakov/Woland speaks about the proof of the Devil's existence.)

Havin also had a deep interest in philosophy. Unfortunately, he did not write on these subjects himself, with one notable exception: his inaugural lecture for the Doctor Honoris Causa from Linköping University, "Mathematics as a source of certainty and uncertainty" [2], is a remarkable short essay on the worldview and philosophical values of the mathematical vision.

A detailed mathematical biography of Victor Havin can be found in the memorial volume, *50 Years with Hardy Spaces: A Tribute to Victor Havin* [1], while [4, 5] contain Havin's vivid recollections of his years as a young man at Mat-Mekh.

Anton Baranov, Nikolai Nikolski

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V.P. Havin and the Uncertainty Principle in Harmonic Analysis (VP and UP)

One of the main research areas of Victor Petrovich (VP) Havin and many of his outstanding students is called the Uncertainty Principle (UP) in Harmonic Analysis. This vast collection of mathematical problems and deep results originates in the ideas of Norbert Wiener from the 1920s. The initial postulate of the principle belongs to Fourier analysis and can be formulated as follows:

A function (measure, distribution) f and its Fourier transform $\hat{f}$ cannot be simultaneously small.

The name of the principle clearly refers to physics as Heisenberg's inequality presents one of the early examples of UP, showing that second moments related to a function and its Fourier transform cannot both be small:

$$\int_{\mathbb{R}} x^2 |f(x)|^2 dx \int_{\mathbb{R}} \zeta^2 |\hat{f}(\zeta)|^2 d\zeta \geq \frac{||f||^4_{L^2(\mathbb{R})}}{16\pi^2}. \tag{1}$$

As it turns out, "smallness" in Wiener's broad statement can be understood in a number of different ways (smallness of support, fast decay at infinity, largeness of the zero set, etc.), many of them leading to deep and important problems. At present, the area of UP grew far beyond its original borders and now, in addition to problems of classical Fourier analysis, includes studies of bases and frames in Banach spaces, harmonic analysis on groups, spectral problems for differential operators and numerous other problems, see for instance [4] or [10].

Like in several other areas of analysis, the contribution of V.P. Havin in the area of UP is broad and significant. It includes not only original results but also promoting and teaching the subject of UP to his many colleagues and friends in the St. Petersburg analysis group. Due to his efforts, many classical problems of UP were studied and solved by the members of the said group and a number of outstanding talks were delivered at the Havin–Nikolski seminar.

As an advisor, VP had a rare gift for selecting the right problem for each of his pupils. The problem had to satisfy two necessary conditions: first, VP must not have known how to solve it. Second, VP must have known, through his tremendous powers of intuition, that its solution lied somewhere within the abilities of the particular student. Many advances in the area of UP appeared this way.

VP and UP: f and $\hat{f}$.

One of the "systemic" contributions of VP to UP is the assertion, quite surprising at the time it was formulated, that UP holds not only for the Fourier transform but for many convolution operators, which set up the natural problem of description of kernels with this property. This idea was developed to the comprehensive theories of "epsilon-local" and "anti-local" operators with numerous natural examples (see Part II, Chapter 5 of [7] for details).

As an example of a "non-Fourier" variation on the theme of the UP, let us consider a well-known theorem by M. Riesz, which says that if a measure and its Riesz potential both vanish on the same non-empty open set in $\mathbb{R}^n$ then the measure is zero identically. In his paper [5] VP made a significant improvement of this theorem in the one-dimensional case proving that if the measure is absolutely continuous with Hölder density of certain precise order, then the open set in Riesz' statement can be replaced with an arbitrary Borel set of positive measure. The problem of extending this result to $\mathbb{R}^n$ and improving the Hölder condition to continuous densities stood open for many years and was studied by VP together with several of his outstanding students. The $\mathbb{R}^n$ question was finally answered in the negative by Havin and D. Beliaev (VP's "mathematical grandson") in [1]. Their sophisticated construction used some of the techniques of J. Bourgain and T. Wolff.

One of the deepest parts of UP is the famous Beurling–Malliavin (BM) theory, which produced the celebrated theorem on completeness of families of exponential functions $\{e^{i\lambda_n z}\}$ in $L^2(0, 1)$. The original proofs which appeared in the 1960s [2, 3], continue to fascinate harmonic analysts to this day.

A significant part of VP's work in UP concerned the study and promotion of BM theory to his students and colleagues. Over the years he made a number of essential improvements and extensions in various parts of the theory, which opened possibilities for further studies, still pursued by many mathematicians. One of the gems in this part of VP's work is the paper "Seventh proof..." joint with his former students J. Mashreghi and F. Nazarov. The title of the paper refers to the famous line from M. Bulgakov's novel "Master and Margarita" and its text contains, among other things, the only "real" proof of the Beurling–Malliavin multiplier theorem by F. Nazarov. The paper raised natural questions on so-called admissible majorants later answered by his students A. Baranov (independently) and Yu. Belov (jointly with VP).

The work of V.P. Havin in the area of UP culminated with the seminal book *The Uncertainty Principle in Harmonic Analysis* [7] written jointly with his student B. Jöricke (a shorter survey, presenting a digest of the book is contained in [6]). The book covers the classical results of UP and gives some of them new updated proofs, many of which were improved by the authors themselves, including a full treatment of the BM theory. Some of the theorems are given up to three different proofs. The results are presented in two groups: those that can be proved with real methods and those that require complex analysis.

The book, written in the 1980s, was a result of a large research project completed by the authors in a tight collaboration with a number of people (mostly, VP's pupils) in the Havin–Nikolski seminar, including F. Nazarov, S. Hruschev, A. Borichev, A. Volberg, and many others. Collaborators' deep original results entered the book as sections and chapters during long hours of mathematical work spent at VP's three-piece apartment on Golodai Island. The relentless search for perfection resulted in a text destined to be a reference source for experts for many years to come. Together with the book by VP's friend and colleague at McGill University P. Koosis, *The Logarithmic Integral* [9], the book by Havin and Jöricke is among the most significant texts in the field of UP.

Alex Poltoratski, Fedor Nazarov

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Ludwig Dmitrievich Faddeev (1934–2017)

Ludwig Faddeev (Ludwig Dmitrievich Faddeev) was one of the giants of mathematical physics. His fundamental contributions include the solution to the quantum three-body problem, developing quantization of gauge theories, the discovery of the Hamiltonian complete integrability of the Korteweg–De Vries equation, and laying out the foundations for integrable quantum field theories and quantum groups. He created one of the most influential research groups in mathematical physics, for many years he was the director of the Leningrad branch of the Steklov Mathematical Institute of the Academy of Sciences of the USSR (and later of Russia). He was the president of IMU from 1987–1990.

Ludwig Faddeev was born in Leningrad (now St. Petersburg), USSR (now the Russian Federation) on March 23, 1934, into the family of mathematicians Dmitry Konstantinovich Faddeev and Vera Nikolaevna Faddeeva. His father was also a talented pianist, so Ludwig grew up surrounded by mathematics and classical music. During the time he was graduating from high school, he seriously considered the career path of a professional musician, but in the end, he decided to go to University. At the time, his father was the Chair of the Mathematics Department, so Ludwig enrolled in the Physics Department of Leningrad University. When he was in his 3rd year of studies, a unit within the department specializing in Mathematical Physics ('kafedra Matematicheskoi Fiziki') was established. L.D. Faddeev and N.N. Uraltseva were among the first five students to graduate with this specialization.

While he was an undergrad, Faddeev took a reading seminar with O.A. Ladyzhenskaya on K.O. Friedrichs' book *Mathematical aspects of the quantum theory of fields*. After this class Ladyzhenskaya became his undergraduate adviser. Her first suggestion for Ludwig was to study the works of N. Levinson on inverse scattering theory, and to present his investigations at the seminar. This was a very profound moment: the inverse scattering problems influenced Ludwig's research for a very long time. Regarding these times, he wrote the following about O.A. Ladyzhenskaya: "I am forever grateful for the direction she gave me..." He kept Ladyzhenskaya's portrait on his desk till his last day.

He graduated from Leningrad University with an equivalent of a master's degree in 1956, and continued on to postgraduate school. O. A. Ladyzhenskaya took him as her Ph.D. student. For his Ph.D. thesis, Faddeev solved the inverse scattering problem for a one-dimensional Schrödinger operator on a line with rapidly decaying potential. He defended his PhD thesis in 1959. This was the beginning of Faddeev's long, productive life in mathematical physics.

After completing his Ph.D. thesis he continued working on scattering problems. He found the complete solution to three-particle scattering, solved the inverse problem for the 3-dimensional Schrödineger operator and had other important results, such as the study of modular functions using methods of scattering theory. Then he moved on to a completely different subject: to quantization of gauge fields. Together with V. Popov he developed perturbation theory for the quantum Yang–Mills theory. These results, known as Faddeev–Popov gauge fixing, were a revolutionary discovery. They are still used in theoretical high-energy physics. After this, Faddeev moved on to a different subject again. He focused on soliton equations and, together with V.E. Zakharov, he presented the Korteweg–De Vries equation as a completely integrable infinite dimensional Hamiltonian flow. This was the

A. Sedrikyan, V. Garzadyan, A. Polyakov, A.B. Migdal, R.E. Kallosh, A.A. Migdal, and L.D. Faddeev on a conference hike in Tsaghkadzor, Armenia, 1983.

first known example of an infinite dimensional Hamiltonian integrable system. Together with L.A. Takhtajan and V.E. Zakharov, he established the complete integrability of two-dimensional relativistic field theory, known as the Sine–Gordon equation.

For Faddeev, the research on integrable non-linear classical field theory was a stepping stone to achieving the goal of finding a non-perturbative construction of quantum field theory. So he started to work on the quantization of classical integrable field theories and made instrumental discoveries such as the algebraic Bethe ansatz; he coined the term "Yang–Baxter equation," emphasizing its importance. This series of works leads to the discovery of quantum groups. Many of his works during this period were done in collaboration with his younger colleagues from the Laboratory of Mathematical Methods in Theoretical Physics at LOMI. More on this and for a more detailed description of Faddeev's research contributions, see [1].

Faddeev liked sports. In his young years he was a stroke of the rowing eight team and a member of the cross-country skiing team. He kept his love for skiing and hiking throughout his life. He loved literature, history, and music. Among his favorites were V. Nabokov's Ada and K. Hamsun's Pan. He had an excellent collection of music and was very proud of it. Isadore Singer recalled that during Ludwig's first visit to Boston, they spent a night visiting jazz joints.

Mathematical physics became an established field around the 1950s–1960s. There are many views on what mathematical physics means, and a variety of opinions on how much mathematical rigor should be present in mathematical physics, and how close it should be to physics. According to Faddeev, the goal

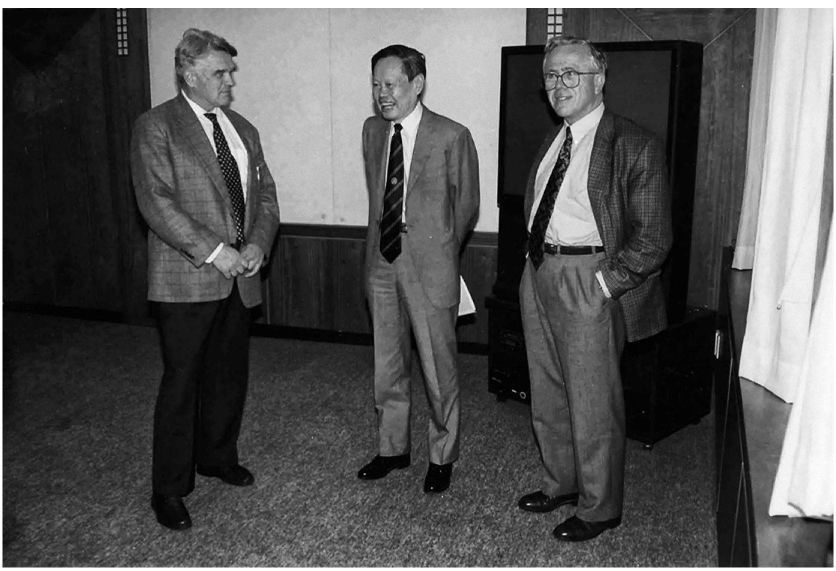
Mathematical Physics group at the Physics Department. Second row, left to right: V.L. Oleynik, B.S. Pavlov, V.B. Matveev, I.A. Molotkov, V.F. Lozutkin, S.Yu. Slavyanov, A.R. Its, A.N. Popov. First row: V.S. Buslaev, L.D. Faddeev, M.S. Birman, V.S. Buldyrev, N.V. Smirnov, 1984.

of a mathematical physicist is not to make rigorous what is already understood, to the extent of being true beyond reasonable doubt by physicists, but to go above and beyond, discovering new phenomena and structures using physical and mathematical intuition, on the basis of mathematical knowledge. In an article intended for the general audience, he wrote: “If someone asked me who among the twentieth-century physicists impressed me the most, I would answer: P.A.M. Dirac, H. Weyl, and V.A. Fock.” It is an interesting illustration of his thesis on mathematical physics: Faddeev regarded H. Weyl as a mathematical physicist. To questions regarding the aim of mathematical and theoretical physics, Faddeev always answered, “There is only one direction and goal: the understanding of the structure of matter and space-time.”

On the physics side he thought of himself as continuing the tradition of V.A. Fock. On the mathematical side, he was influenced by V.I. Smirnov’s famous mathematical physics seminar in Leningrad. In many ways, Faddeev is a descendent of the St. Petersburg mathematical tradition, which goes back to L. Euler and includes names such as M.V. Ostrogradsky, P. L. Chebyshev, A.M. Lyapunov, A. A. Markov, V.I. Smirnov, and many others.

Ludwig had many friends and colleagues around the world. P. Lax, L. Nirenberg, I. Singer, J. Moser were among the mathematicians whose friendship Ludwig particularly valued.

In 1972, he became the head of the Laboratory of Mathematical Methods in Theoretical Physics at the Leningrad Branch (LOMI) of the Mathematics Institute of the Academy of Sciences in Moscow. In 1976 he was elected full



L.D. Faddev with C.N. Yang and R. Baxter, Seoul, 1997.

member (academician) of the Soviet Academy of Sciences and became the director of LOMI.

Faddeev initiated a remarkable series of conferences called "Quantum solitons," which would take place every three years during the late 1970s and early 1980s.

During the cataclysmic dismantling of the Soviet Union, state-funded science became the first victim of privatization. With no alternative to state funding, science was in free fall, along with the whole country in general. Faddeev was one of the very few people in academia of high standing who did not leave the country for long periods of time. During this period he was offered the directorship of the Institute for Theoretical Physics at Stony Brook. The director of ITP at the time was C.N. Yang, who was about to retire. Though Ludwig was very pleased with the offer, he declined it.

Ludwig always valued the international nature of science. In this respect it is only natural that for the period of 1987 to 1990, he was president of the International Mathematical Union.

From the Faddeev's certificate of Pomeranchuk prize.

L.D. Faddeev was elected to leading academies, including the Royal Swedish Academy of Sciences (1989), the National Academy of Sciences[1] (1990), the French Academy of Sciences (2002), and the Royal Society[2] (2010). Among many awards that he received are the Dirac Medal (1990), the Max Plank Medal (1996), the Euler Medal[3] (2002), the Henri Poincaré Prize (2006), the Shaw Prize (2008, jointly with V. Arnold), and the Lomonosov Medal (2013).

More details on the life and work of L.D. Faddeev can be found on the website: faddeev.com. Acknowledgements: The author is grateful to L. Takhtajan for his comments, suggestions, and for sharing the draft of his article on L. Faddeev written for the Royal Society. He is also grateful to F. Smirnov for his multiple helpful comments.

Nikolai Reshetikhin

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1 NAS, USA.
2 Royal Society of London for Improving Natural Knowledge.
3 A medal of the Russian Academy of Sciences.

Scattering and solitons. Three selected papers

The reader may find a comprehensive review of Faddeev's work in [1]. Here we give a short summary of three of his papers with the common theme being spectral theory and quantum scattering theory. The first one is based on his PhD thesis. The second one is his fundamental work on solution to the three-body problem in quantum scattering. The last one is his paper with V.E. Zakharov where they show that the KdV equation is an infinite dimensional integrable Hamiltonian system. The results of the first paper became instrumental here.

We did not even try to give here an overview of Faddeev's, perhaps, most fundamental discovery, the Faddeev–Popov gauge fixing [12]. This revolutionary paper is at the heart of all computations physicists do with the standard model. Nobel Prize laureate (1957 Prize) C.N. Yang in the foreword to selected works of Ludwig Faddeev wrote: "Many people, including myself, felt that Faddeev should have shared the Nobel Prize of 1999 with t'Hooft and Veltman." See [1] for details about this and other works, which are not mentioned here.

1. One dimensional Schrödinger equation. In his Ph.D. thesis *Properties of the S-matrix for scattering by a local potential*, published in [4], L. Faddeev gave a complete study of the direct and inverse spectral map for one-dimensional Schrödinger operator:[1]

$$L = -\frac{d^2}{dx^2} + u(x). \tag{1}$$

If the potential is sufficiently smooth and rapidly decaying at infinity L has two-folded absolutely continuous spectrum and a finitely many isolated eigenvalues. He proved that the potential $u(x)$ can be uniquely recovered from so-called *scattering data*. To define it consider Jost solutions to the differential equation $Lf = k^2 f$:

$$f_1(x,k) = e^{ikx} + o(1), \quad x \to \infty, \qquad f_2(x,k) = e^{-ikx} + o(1), \quad x \to \infty.$$

These solutions can be analytically continued to the upper half-plane. Transition coefficients are defined by the relation

$$f_2(x,k) = a(k) f_1(x,-k) + b(k) f_1(x,k),$$

[1] We use nonstandard notation L for the Schrödinger operator having in mind applications to KdV equation, where it is known as the Lax operator.

which holds when $k \in \mathbb{R}$, where $|a(k)|^2 = 1 + |b(k)|^2$. The coefficient $a(k)$ has analytical continuation to the upper half plane and the following *dispersion relation* holds:

$$a(k) = \exp\left(\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\ln(1-|r(k)|^2)}{k-p}dp\right)\prod_{l=1}^{N}\frac{k-i\kappa_l}{k+i\kappa_l} \tag{2}$$

where $r(k) = \dfrac{b(k)}{a(k)}$ is called *reflection coefficient*, $a(k)$ on the left side is understood as the limit from the upper half plane, and zeros $i\kappa_l$ of $a(k)$ define the discrete spectrum of L with eigenvalues $-\kappa_l^2$. The scattering data is a triple $(r(k), \kappa_l, m_l)$ where $r(k)$ and κ_l are as above and m_l is the normalization constant defined as $(m_l)^{-1} = ia'(i\kappa_l)/c_l$, where c_l is such that $f_1(x, i\kappa_l) = c_l f_2(x, \kappa_l)$.

In his PhD Faddeev proved that potentials $u(x)$ satisfying the condition

$$\int_{-\infty}^{\infty}(1+|x|)|u(x)|dx < \infty$$

are in bijection with the scattering data, i.e. with the triples $(r(k), \kappa_l, m_l)$.

This is an elegant result, but its true significance emerged only with the study of soliton equations (see the review of Faddeev's paper on the integrability of the KdV equation).

2. Three body problem in quantum mechanics. One of the most known works of L.D. Faddeev is the solution to the scattering problem for three particles in quantum mechanics [5], [6], [7], [8].

Consider the differential operator, quantum Hamiltonian, describing the system of three interacting non-relativistic quantum particles with positions $\mathbf{x}_1$,$\mathbf{x}_2$, and $\mathbf{x}_3 \in \mathbb{R}^3$:

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{V}_{12} + \mathbf{V}_{23} + \mathbf{V}_{13}. \tag{3}$$

It acts in the Hilbert space $L^2(\mathbb{R}^9)$. Here $\mathbf{V}_{ij} = v_{ij}(\mathbf{x}_i - \mathbf{x}_j)$ are pairwise interaction potentials and

$$\mathbf{H}_0 = -\frac{1}{2m_1}\Delta_1 - \frac{1}{2m_2}\Delta_2 - \frac{1}{2m_3}\Delta_3$$

is the Hamiltonian describing non-interacting particles. It is assumed that the potentials $v_{ij}(\mathbf{x})$ are smooth and rapidly decaying.

The problem of three-particle scattering consists of describing the spectrum of the operator $\mathbf{H}$ and constructing the scattering operator, which is a unitary operator mapping incoming scattering states to outgoing scattering states. This unitary operator, known as the $\mathbf{S}$-matrix is the central object in physical applications.

In simple cases, in particular, in two-body scattering, the continuum spectrum of $\mathbf{H}$ coincides with the continuum spectrum of $\mathbf{H}_0$. As a result, the resolvent $\mathbf{R}(z) = (\mathbf{H} - z)^{-1}$ is determined by an integral equation with compact kernel. This means iterations rapidly converge and numerical methods may be applied. In this case the **S**-matrix is defined as

$$\mathbf{S} = (\mathbf{U}^+)^*\mathbf{U}^-$$

where $\mathbf{U}^\pm = \lim_{t\to\pm\infty} e^{it\mathbf{H}}e^{-it\mathbf{H}_0}$ are the *wave operators.*

The difficulty of the three-particle scattering problem is that the continuum spectrum of $\mathbf{H}$ differs from the one for $\mathbf{H}_0$. This happens because there are directions in $(\mathbb{R}^3)^{\times 3}$ where the potential $\mathbf{V}_{12} + \mathbf{V}_{23} + \mathbf{V}_{13}$ is not decaying at infinity.

After separating coordinates and momenta of the center of mass, Faddeev passed to the coordinates $(\mathbf{k}_{12}, \mathbf{p}_3)$ that are Fourier conjugate to $(\mathbf{x}_1 - \mathbf{x}_2, \mathbf{x}_3)$. Another natural choice of coordinates on the same space is $(\mathbf{k}_{23}, \mathbf{p}_1)$, and $(\mathbf{k}_{31}, \mathbf{p}_2)$. Denote them by all by $(\mathbf{k}_\alpha, \mathbf{p}_\alpha)$. Assuming that the total momentum is zero, we have $\mathbf{k}_{12} = \mathbf{p}_1 - \mathbf{p}_2$ etc. In the center of mass, after the Fourier transform, the Hamiltonian $\mathbf{H}$ becomes an integral operator acting in $L_2(\mathbb{R}^6)$ (with coordinates in $\mathbb{R}^6$ being any pair (k_α, p_α)) with $\mathbf{H}_0$ being the operator of multiplication by

$$\widehat{\mathbf{H}}_0 = -\frac{1}{2m_1}\mathbf{p}_1^2 - \frac{1}{2m_2}\mathbf{p}_2^2 - \frac{1}{2m_3}\mathbf{p}_3^2 = \frac{\mathbf{k}_\alpha^2}{2m_\alpha} + \frac{\mathbf{p}_\alpha^2}{n_\alpha}.$$

Here the right side of the equality does not depend on which α to choose, and m_α, n_α can be easily computed in terms of m_1, m_2, m_3. The operator of multiplication by $\mathbf{V}_\alpha(\mathbf{x}_\alpha)$ becomes the integral operator with the kernel

$$\mathbf{V}_\alpha(\mathbf{k}_\alpha, \mathbf{p}_\alpha; \mathbf{k}'_\alpha, \mathbf{p}'_\alpha) = v_\alpha(\mathbf{k}_\alpha - \mathbf{k}'_\alpha)\delta(\mathbf{p}_\alpha - \mathbf{p}'_\alpha).$$

Assume now that the spectrum of each of the two-particle Hamiltonians $-\Delta_i/m_i - \Delta_j/m_j + V_{ij}$ have only one point of discrete spectrum (one eigenvalue) and no virtual levels at the lower end of the continuous spectrum (the first assumption is not essential, see [8] for details).

Faddeev provided a tour de force study of the resolvent $\mathbf{R}(z) = (\mathbf{H} - z)^{-1}$ of the Hamiltonian (3). By subtracting the first three singular iterations, and by proving that the resulting equation has compact kernel, he derived integral equations for the resolvent (for the **T**-operator defining resolvent) which can be solved by iterations (since the kernel is compact). These equations are known as *Faddeev equations* in three particle scattering. Using these equations he proved the following.

1) The projection of $\mathbf{H}$ to the subspace of absolutely continuous spectrum in $L^2(\mathbb{R}^6)$ is unitary equivalent to the operator

$$\widehat{\mathbf{H}} = \widehat{\mathbf{H}}_0 \oplus \widehat{\mathbf{H}}_{12} \oplus \widehat{\mathbf{H}}_{31} \oplus \widehat{\mathbf{H}}_{23}$$

where $\widehat{\mathbf{H}}_0$ is as above and

$$\widehat{\mathbf{H}}_\alpha = \frac{\mathbf{p}_\alpha^2}{2m_\alpha} - \kappa_\alpha^2.$$

Here $-\kappa_\alpha^2$ is the eigenvalue of the corresponding two-particle Hamiltonian and the operators $\mathbf{H}_0, \mathbf{H}_\alpha$ act as multiplication operators in

$$\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_{12} \oplus \mathfrak{H}_{31} \oplus \mathfrak{H}_{23}$$

where $\mathfrak{H}_0 = L^2(\mathbb{R}^6)$ are functions of $(\mathbf{k}_\alpha, p_\alpha)$ (different choices of α are related by simple coordinate transformations) and $\mathfrak{H}_\alpha = L^2(\mathbb{R}^3)$ are functions of $\mathbf{R}_\alpha$.

2) He also proved the existence of building blocks $\mathbf{U}_0^\pm : \mathfrak{H}_0 \to L^2(\mathbb{R}^6)$ and $\mathbf{U}_\alpha^\pm : \mathfrak{H}_\alpha \to L^2(\mathbb{R}^6)$ for wave operators. Here $L^2(\mathbb{R}^6)$ is the three-particle Hilbert space with separated center of mass and $\mathfrak{H}_0, \mathfrak{H}_\alpha$ are as above. The operators $\mathbf{U}_0^\pm, \mathbf{U}_\alpha^\pm$ are defined as

$$\mathbf{U}_0^\pm = \lim_{t\to\pm\infty} e^{it\mathbf{H}} \mathbf{J}_0 e^{-it\widehat{\mathbf{H}}_0}, \quad \mathbf{U}_\alpha^\pm = \lim_{t\to\pm\infty} e^{it\mathbf{H}} \mathbf{J}_\alpha e^{-it\widehat{\mathbf{H}}_\alpha},$$

Here $\mathbf{J}_0$ identifies $\mathfrak{H}_0$ with the original $L^2(\mathbb{R}^6)$ and $\mathbf{J}_\alpha$ embed $\mathfrak{H}_\alpha$ isometrically in $L^2(\mathbb{R}^6)$:

$$(\mathbf{J}_\alpha f)(\mathbf{k}_\alpha, \mathbf{R}_\alpha) = \phi_\alpha(\mathbf{k}_\alpha) f(\mathbf{R}_\alpha).$$

Here $\phi_\alpha(\mathbf{k}_\alpha)$ is the Fourier transform of the eigenfunction of the corresponding two-particle Hamiltonian.

Faddeev defined the wave operators $\mathbf{U}^\pm : \mathfrak{H} \to L^(\mathbb{R}^6)$ as

$$\mathbf{U}^\pm = \mathbf{U}_0^\pm \oplus \mathbf{U}_{12}^\pm \oplus \mathbf{U}_{31}^\pm \oplus \mathbf{U}_{23}^\pm$$

and he proved that they satisfy the right identities

$$(\mathbf{U}^\pm)^* \mathbf{U}^\pm = \mathbf{I}, \quad \mathbf{U}^\pm (\mathbf{U}^\pm)^* = \mathbf{I} - \mathbf{P}, \quad \mathbf{H}\mathbf{U}^\pm = \mathbf{U}^\pm \widehat{\mathbf{H}}$$

where $\mathbf{P}$ is the projection to the discrete spectrum. The scattering matrix is then defined as

$$\mathbf{S} = (\mathbf{U}^+)^* (\mathbf{U}^-).$$

Faddeev's equations are complicated, nevertheless, due to the of the compactness of the kernel, the iterations converge efficiently and these equations can be solved numerically. This makes it an important tool for studying few body systems in quantum physics.

Another important work of Faddeev on scattering theory is the solution to the inverse scattering problem for the three-dimensional case. In [17], he gave necessary and sufficient conditions for reconstructing the potential from scattering data. In 1965 Faddeev wrote another influential paper [9] where he applied perturbation theory for continuous spectra to the Laplace–Beltrami operator on $\Gamma\backslash\mathbb{H}$ when the fundamental domain is non-compact but has finite hyperbolic area. In this case the space of $L^2(\Gamma\backslash\mathbb{H})$ decomposes naturally into two subspaces. The first subspace consists of functions having zero integrals over all horocycles in $\Gamma\backslash\mathbb{H}$. On this subspace the Laplace–Beltrami operator has a discrete spectrum [20]. Faddeev described spectral properties of this operator on the second, more complicated, subspace. This was a very influential work, with many important followup results, for example [10], [16], see [3] for the detailed description.

3. KdV equation as an integrable Hamiltonian system. In a paper published in 1967 Gardner, Green, Kruskal, and Miura (GGKM) discovered that the Cauchy problem for the famous Korteweg-de Vries (KdV) equation

$$u_t = 6uu_{xx} + u_{xxx} = 0, \quad u(x,t)|_{t=0} = u(x), \quad x \in \mathbb{R} \tag{4}$$

can be solved by the composition of direct and inverse spectral problems for (1). They found that if the initial data $u(x)$ is regarded as the potential in the differential operator (1), the evolving of the spectral data according to (4) is particularly simple:

$$r(k,t) = r(k)e^{8ik^3t}, \quad \kappa_l(t) = \kappa_l, \quad m_l(t) = m_l e^{8\kappa_l^3 t}. \tag{5}$$

In 1968 P. Lax gave an elegant explanation for this fact, observing that the KdV equation can be written as linear evolution of the operator $L = -\frac{\partial^2}{\partial x^2} + u(x,t)$:

$$\frac{\partial L}{\partial t} = [L, A]$$

where $A = 4\frac{\partial^3}{\partial x^3} - 6u\frac{\partial u(x,t)}{\partial x} - \frac{\partial u(x,t)}{\partial x}$.

In 1971 L. Faddeev and V. Zakharov [14] explained the evolution (5) of spectral data by showing that the KdV equation is an infinite dimensional Hamiltonian integrable system, and the spectral data are the action-angle variables.

They observed that the bracket

$$\{F, G\} = \int_{-\infty}^{\infty} \frac{d}{dx}\left(\frac{\delta F}{\delta u(x)}\right)\frac{\delta G}{\delta u(x)}dx \tag{6}$$

defines a Poisson structure on the space $\mathcal{M}$ of smooth rapidly decaying functions u. Functions satisfying condition $\int_{-\infty}^{\infty} u(x)dx = c$ form a symplectic leaf in the Poisson space $\mathcal{M}$.

The KdV evolution can be written as a Hamiltonian flow on $\mathcal{M}$:

$$u_t = \{H, u\}, \quad H = \int_{-\infty}^{\infty} \left(\frac{1}{2}u_x^2 + u^3\right) dx.$$

Faddeev and Zakharov computed the symplectic form

$$\Omega = \int_{-\infty}^{\infty} Du(x) \wedge \left(\int_{-\infty}^{x} Du(y)dy\right) dx \tag{7}$$

corresponding to the bracket (6) in terms of spectral data for (1):

$$\int_0^{\infty} dP(k) \wedge dQ(k)dk + \sum_{l=1}^{N} dp_l \wedge dq_l. \tag{8}$$

Here and in (7) D refers to the differential for $\mathcal{M}$ written either in terms of $u(x)$ or in terms of scattering data. Because of the bijection proved by Faddeev in his thesis, this is the same differential written in different "coordinates." Variables (8) are defined as

$$P(k) = \frac{4k}{\pi} \ln |a(k)|, \quad Q(k) = \arg b(k), \quad p_l = 2\kappa_l^2, \quad q_l = \ln c_l. \tag{9}$$

The function $a(k)$ is determined by scattering data through the dispersion relation (2), $b(k) = a(k)r(k)$. Formulae (7) and (8) show that variables $P(k)$, $Q(k)$, p_l, q_l are Darboux coordinates for the symplectic form Ω. Here the results obtained by Faddeev in his Ph.D. thesis were instrumental in identifying (8) and (7).

Infinitely many Poisson commuting Hamiltonians that are local differential polynomials in u can be derived by solving the Riccati equation

$$\sigma_x + \sigma^2 - u + 2ik\sigma = 0$$

in power series $\sigma(x, p) = \sum_{n\geq 1} \frac{\sigma_n(x)}{(2ik)^n}$. Faddeev and Zakharov showed that the functionals $I_n = \int_{-\infty}^{\infty} \sigma_n(x)dx$ Poisson commute and can be expressed only in terms of $P(k)$ and p_l. In particular

$$H = \int_{-\infty}^{\infty} \sigma_5(x)dx = 8\int_0^{\infty} k^3 P(k)dk - \frac{32}{5}\sum_{l=1}^{N} p_l^{5/2}.$$

Thus, they proved that (4) is a completely integrable infinite dimensional Hamiltonian system with the action variables $P(k), p_l$, and with the angle variables $Q(k), q_l$.

Thus, the KdV equation inspired the first example of an infinite-dimensional integrable system. Many others followed, such as Nonlinear Schrödinger and the Sine-Gordon equation. The Sine-Gordon equations, [15] became the first example of a two-dimensional relativistic classical field theory, which is an infinite-dimensional Hamiltonian integrable system.

4. Quantum field theory, integrable systems, and quantum groups. From the very beginning the study of soliton equations for L. Faddeev was a step towards constructing non-perturbative models of quantum field theory. The first significant progress in this direction was made in the semiclassical framework, the results were summarized in [18]. But the most important developments came from the "fusion" of inverse problems in soliton equations, Bethe ansatz, and from the realization that Baxter's work on the exact solution of 6- and 8-vertex models is intrinsically related to C.N. Yang's work on factorized scattering. Many important developments originated from this period; in particular, it led to the emergence of quantum groups, which, after Drinfeld, became one of the fundamental objects in representation theory.

A comprehensive overview of all Faddeev's work was done in [1], see also [2] [3], as well as the website http://faddeev.com/.

Nikolai Reshetikhin

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Andrei Alexandrovich Suslin (1950–2018)

Andrei Alexandrovich Suslin was born on December 27, 1950, in Leningrad. His father, Alexander Ivanovich, was a well-known engineer — a specialist on ships' power systems who worked at the Krylov State Research Center. His mother, Alexandra Nikolaevna, was an engineer-economist. Andrei's math skills were notable at a young age. Influenced by their father, all the children in Andrei's family enjoyed solving mathematical Olympiad problems and puzzles.

When he was only in the 6th grade, Andrei won the Leningrad City Olympiad for the 6th, 8th, and 10th (last) grade levels. In 8th grade he became the winner of the Russian National Mathematical Olympiad. In 1967 he won a gold medal at the International Mathematical Olympiad. In 1972 he graduated from the Department of Mathematics and Mechanics at Leningrad State University (LSU).

In 1974, Suslin got his Ph.D. degree (under the supervision of M.I. Bashmakov) and already in 1977 he defended his Habilitation for the confirmation of the Serre conjecture on projective modules over polynomial rings From 1969 to 1975, A.A. Suslin taught at the specialized boarding school №45 at LSU. From 1973 to 1977 he worked at LSU, and from 1977 to the end of his life he worked at the Leningrad Branch of the Steklov Institute. From 1994 onwards he was also a professor at Northwestern University (in Evanston, U.S.).

A.A. Suslin was invited to the International Congress of Mathematicians three times as a lecturer in 1978, 1986, and 1994 (including as a plenary lecturer in 1986).

In 1980, A.A. Suslin was awarded Lenin's Komsomol Prize for solving Serre's conjecture; in 2000, he received the Cole Prize in algebra for his work on motivic cohomology, particularly for his and V. Voevodsky's paper *Bloch–Kato conjecture and motivic cohomology with finite coefficients*, where they developed the basis of the theory of motivic cohomologies.

Only in the algebraic K-theory of fields and algebraic manifolds he proved the following:

• The Serre conjecture about projective modules over polynomial rings (1976);

• Quillen–Lichtenbaum conjecture about K-theory of algebraically closed fields (1983);

• A stable version of the Milnor conjecture about cohomologies with finite coefficients of a general linear group (1984);

• Bloch–Kato conjecture for $n = 2$ (the so-called Merkurjev–Suslin theorem) (1982);

• Positive solution (with M. Wodzicki) of the Karoubi conjecture which follows from an excision property for C^*-algebras in K-theory (1990, 1992);

• For every smooth algebraic variety X, a complex of free abelian groups was invented, which is now called the Suslin complex;

• The construction of the Suslin complex was a key part of the basis of the theory of motives, created by V. Voevodsky;

• In the case of a complex algebraic variety, the given complex, taken modulo n, calculates singular homologies of the given variety with coefficients $\mathbb{Z}/n\mathbb{Z}$.

A. Suslin jointly with E. Friedlander constructed a spectral sequence, starting with motivic cohomologies, and converging to Quillen's K-groups; this solves an A. Beilinson's conjecture.

Additionally, A.A. Suslin has a number of other remarkable results, including:

• B.B. Venkov's and D. Quillen's theorem (the detection theorem for finite groups) was extended to all finite group schemes over a field;

• B.B. Venkov's theorem on finite number of generators of a cohomology algebra of a finite group was extended (with E. Friedlander) to the case of finite group schemes over a field;

• He and E. Friedlander proved that for a finite group scheme over a field, its cohomology with coefficients in a finite-dimensional rational module is a finitely generated module over a ring of the cohomology of the given group scheme.

It's worth mentioning that the Merkurjev–Suslin theorem was unofficially regarded as the best result in algebraic K-theory for a long time. Another interesting fact is that French mathematicians in IHES gave a standing ovation to Suslin's proof of the Quillen–Lichtenbaum conjecture, breaking all the unwritten traditions.

A.A. Suslin almost always chose to attack iconic problems, a trait truly reflecting his personality. However, if you met him on the street or in a cafe, it would be hard for anybody to recognize him as a distinguished mathematician.

It's difficult to exaggerate A.A. Suslin's contribution to the development of modern algebra. In St. Petersburg, A.A. Suslin created a school of algebraic K-theory and motivic cohomology well-known around the world.

Some of my personal memories (a very tiny portion). Suslin was my advisor starting from my 3rd year at university and through postgraduate school (1981–1984) at the Steklov Institute in Leningrad. But what's more important to me is that he was my mentor in mathematics throughout my whole life. With these memories, I would like to remember some stories that strongly highlight Suslin's character as a mathematician and as a person.

In the '90s I would often go to Suslin (the Steklov Institute, room 306) and ask him what he was currently working on. Suslin would take a cup of tea and formulate what he was interested in. Then, he would start developing his approach to the problem on the blackboard, right in front of me.

If something went wrong, he would take a break for a cigarette or a cup of tea, and having thought some more, would propose a new approach. Every time, before doing any calculations, he would predict the passing result(s) or the general line of thought. Only after that would he start to check, let us say the first non-trivial case, through calculations.

And if this did not work (which was often the case), he would take another break, and repeat the process again and again. These conversations, which I would for the most part listen attentively to, lasted for one, two, or three hours. Several times they lasted up to four or five hours, with the mentioned breaks, during which we drank tea with bagels.[1] Thanks to those interactions, I adopted his method, which Suslin never explained to me, but which I witnessed him systematically use.

I will allow myself to repeat what Suslin told me, as I think this is the most important thing (technical things aside) that I learned from him. First, using all your prior experience, you have to formulate what it is that you want to prove. Then, you must firmly believe that the formulation is, in principle, true. Only after that should you start looking for an approach. As we all know, when trying to solve meaningful problems, such heavy technical (or essential) difficulties arise that we start to give up on and lose interest in the problem.

We must know a priori that the expected result is conceptually true. I learned this principle from Suslin. It is possibly the most important method that he taught me.

[1] The Russian version of the bagel is called a bublik. These bread rolls are similar to each other, but a bublik has a larger hole in the middle and the dough is more dense. It is eaten more as a snack than as a meal in Russia.

From a purely social perspective, it is thanks to Suslin that I met so many great mathematicians. Suslin's basic principle for interacting with others, as he told me, was: "I try not to offend anyone." And he was very successful at it.

Now I want to share a characteristic story that shows why Suslin would systematically attack specific, prominent problems. Once, we were sitting in his apartment in Evanston and drinking wine. "Vanya," he asked me, "what problem should we work on?" I thought about it for a moment and proposed something. After a short silence, Suslin said:

> No, that's boring. Let's prove the Hodge conjecture, or Grothendieck's standard conjectures.

That was Suslin. He was not interested in lifting a big rock or even a very big one, his true calling was to **move a cliff**. To me, this story clearly explains why he always took on specific prominent problems.

Having come back from the French institute IHES in 1983, Suslin joyfully told me: "Vanya, I proved the Quillen–Lichtenbaum conjecture the night before my lecture at IHES. Right after the lecture, Gabber generalized my result, and a few days later I used his generalization to prove that

$$K_i^{\mathbb{Q}}(\mathbb{C}; \mathbb{Z}/n) = K_i^{top}(pt; \mathbb{Z}/n) = \mathbb{Z}/n,$$

if i is even and it is the zero group if i is odd. As a result, one gets the stable Milnor–Friedlander conjecture, specifically that $H^*(GL(\mathbb{C}); \mathbb{Z}/n)$ is the ring of polynomials $\mathbb{Z}/n[c_1, c_2, c_3, \ldots]$." In other words, cohomologies with finite coefficients of the classified space of the group GL(C) seen as a discrete group, are the same as cohomologies of the classified space of the group $GL(C)$ seen as a Lie group. After this introduction, Suslin sketched out a proof of his famous rigidity theorem in just half an hour.

In 1999 at MPI in Bonn, Suslin asked me and Sergei Yagunov: "What are you working on?" We answered that we wanted to extend his rigidity theorem for K-theory to Voevodsky's cobordisms. For that, we know approximately how to build Gysin homomorphisms for Voevodsky's cobordisms. To our surprise, this inspired Suslin so much that he discussed this problem with us for three days. On the third day he came and said: "All you need to prove are the three properties of Gysin homomorphisms:

(1) The base change property,

(2) Covariant functoriality,

(3) Normalization ($\mathrm{id}^* = \mathrm{id}$).

Having these three properties, you can prove the rigidity." This story really illustrates Suslin's style in mathematics.

In January of 1994, Suslin came into his office at Steklov Institute (room 306) holding someone's preprint. "This is brilliant work," he said. It was the hand-written version of Voevodsky's famous preprint on presheaves with transfers. Immediately, Suslin started giving us a lecture about it and ended up reading four lectures, which were each 4 hours long. That preprint was fundamental for Voevodsky's construction of his triangulated category of motives, which ultimately led Voevodsky to proving the Milnor conjecture.

Once in 1995, I asked Suslin to explain Quillen's trick to me, the one created to prove Gersten's conjecture. Within 5 minutes, Suslin was able to explain the essence of the trick to me and formulate the principle that in such problems, you always have to start with the closed fibre. The same applies to Voevodsky's trick. For the following 20 years, I used this absolutely basic principle to solve many problems on my own and with co-authors. The next day, Suslin and I proved the Grothendieck–Serre conjecture for the group $SL_{1,A}$, where A is an Azumaya algebra. After that day, I started to work intensively on proving the Grothendieck–Serre conjecture, and Suslin returned to developing the theory of motivic cohomologies with Voevodsky.

Let me stop here. But I have to say: I will always fondly remember my great teacher and distinguished mathematician, Andrei Alexandrovich Suslin.

Ivan Panin

Remarkable Theorems of Andrei Suslin

Andrei Suslin (1950–2018) was deeply involved in both the formulation and the solution of many of the most important questions in algebraic K-theory. His own evolution from a "pure algebraist" led to a partnership with Vladimir Voevodsky in building the edifice of motivic cohomology. The interweaving of arithmetic algebraic geometry and algebraic K-theory, seen frequently in Andrei's work, has contributed much to the development of both fields. Later in his career Andrei made important contributions to the modular representation theory of finite group schemes.

Andrei was primarily a problem solver, a mathematician confident that clearly formulated questions could be answered by "direct, imaginative attack." Time and again, Andrei introduced new techniques and structures in order to solve challenging problems. Although he did not incline to "theory building," he has left us considerable theory with which to continue his efforts. For many years, Andrei's clear, precise, careful approach to fundamental questions placed him as the "final judge" of many current efforts at the interface of algebraic geometry and K-theory. Andrei freely shared his ideas, gave brilliantly clear lectures, encouraged the work of others.

In what follows, we mention a selection of Andrei's many fundamental results.

In 1976, Andrei and Daniel Quillen independently and essentially simultaneously proved the following theorem, known as "Serre's Conjecture"; prior to their proofs, the most famous problem of commutative algebra was to find a proof of this Quillen–Suslin Theorem.

Theorem 1 ([7])**.** *Let S be a finitely generated polynomial algebra over a field F, so that $S = F[x_1, \ldots, x_d]$ for some d. Then every finitely generated projective S-module P (i.e., any direct summand P of some free S-module M) is a free S-module.*

The key step in Andrei's proof is the following elementary algebraic fact designed for his goal: Let R be a commutative ring and set $f = (f_1, f_2) \in R[t]^{\oplus 2}$ (not necessarily unimodular). Let $c \in R \cap (f_1R[t] + f_2R[t])$. Then for any commutative R-algebra A and $a, a' \in A$ such that $a \equiv a'$ modulo cA, we have that $f(a)$ and $f(a')$ are conjugate by an element of $GL_2(A)$. This is a remarkable piece of ingenuity!

The following theorem of Andrei's (with A. Merkurjev) provided a quantum leap in the study of the Brauer group. Both the statement and proof served as a launching pad for explorations of the relationship between algebraic K-theory and motivic cohomology.

Theorem 2 ([5]). *Let F be a field, m an integer coprime to $\mathrm{ch}(F)$ and $\xi \in F$ a primitive m-th root of unity. Then the norm residue homomorphism*

$$h_2 = h_{F,2} : K_2(F)/mK_2(F) \;\to\; H^2(F, \mu_m), \quad \{a, b\} \mapsto C(a, b)_\xi$$

is an isomorphism, where $H^2(F, \mu_m) = \mathrm{Br}(F)[m] \subset \mathrm{Br}(F)$ consists of all elements whose exponent divides m. In particular, the subgroup $\mathrm{Br}(F)[m]$ of the Brauer group is generated by the classes of cyclic algebras $C(a, b)_\xi$ for $a, b \in F^\times$.

Theorem 2 quickly led to various new algebro-geometric results, typically guided by Andrei. For example, the following theorem of Andrei on "Serre's Conjecture II" paved the way for the work of many others, which eventually established that a simply connected semi-simple classical algebraic group over a field F of cohomological dimension ≤ 2 has no non-trivial principal homogeneous spaces over F.

Theorem 3 ([12]). *Let F be a field of cohomological dimension ≤ 2. Then the reduced norm homomorphism $Nrd\colon A^\times \to F^\times$ is surjective for every central simple algebra A over F of degree n.*

This implies that Serre's Conjecture II is valid for simply connected groups of inner type A_{n-1}.

In the past 50 years, many mathematicians have tried to understand the cohomology of $BG(F)$, the classifying space of the infinite discrete group $G(F)$ of F-valued points of an algebraic group G over an algebraically closed field F. Andrei proved the conjectured value for $G = GL_\infty$; we formulate Andrei's result in terms of algebraic K-theory.

Theorem 4 ([8, 9]). *Let F be an algebraically closed field and let n be a positive integer invertible in F. Then*

$$K_{2i}(F, \mathbb{Z}/n) \;=\; \mathbb{Z}/n, \quad K_{2i+1}(F, \mathbb{Z}/n) = 0, \quad i \geq 0.$$

Andrei's proof is very elegant, introducing the now fundamental notion of Suslin rigidity. When he presented the proof in Paris, there was a standing ovation.

The domain and range of the norm residue homomorphism in various forms were the subject of many of Andrei's impressive calculations involving Milnor K-theory on the one hand and norm varieties on the other. At its core, this is a study of quadratic forms and division algebras.

The algebraic K-groups $K_*(F)$ of a field F (or a more general ring) are defined in terms of homotopy groups which are notoriously difficult to compute. The Milnor K-groups $K_*^M(F)$ of a field are defined more simply in terms of generators and relations; only for $i \leq 2$ does $K_i^M(F) = K_i(F)$. The following theorem of Andrei nevertheless gives a close relationship between $K_*^M(F)$ and $K_*(F)$.

Theorem 5 ([10]). *If F is an infinite field, then the natural composition*

$$K_n^M(F) \to K_n(F) \to H_n(GL_\infty(F)) \simeq H_n(GL_n(F)) \to K_n^M(F)$$

is multiplication by $(n-1)!$.

The following theorem by Andrei and Yu. Nesterenko identifies Milnor K-theory with the diagonal summands of (bigraded) Bloch higher Chow groups for a field F. This theorem played an important role in the later development of motivic cohomology by Andrei and V. Voevodsky as in [15].

Theorem 6 ([6]). *For any field F, the Milnor K-theory $K_*^M(F)$ of F is naturally isomorphic to the diagonal part of Bloch's Higher Chow groups,*

$$K_n^M(F) \quad \simeq \quad CH^n(F, n).$$

When challenged to prove an excision property for C^*-algebras, Andrei in a paper written with M. Wodzicki proved the following theorem applicable to any ring.

Theorem 7 ([16]). *For any ring A, A satisfies excision in rational K-theory if and only if $A_{\mathbb{Q}}$ is homologically unital (in the sense of an explicit chain complex associated to A is acyclic).*

One achievement of etale cohomology as developed by M. Artin and A. Grothendieck was to provide a functorially algebraic method of computing the (singular) cohomology with finite coefficients of the underlying analytic space of a complex algebraic variety X. Using the Suslin complex $Sus_*(X)$ of an algebraic variety X over an algebraically closed field (involving maps from algebraic simplices to symmetric powers of X) and Suslin rigidity, Andrei and V. Voevodsky proved the following beautiful result giving a more "elementary" algebraic interpretation of $H_*(X(\mathbb{C})^{an}, \mathbb{Z}/n)$.

Theorem 8 ([14]). *If X is a quasi-projective variety over $\mathbb{C}$, then the natural map*

$$\pi_i(Sus_*(X), \mathbb{Z}/n) \to H_i(X(\mathbb{C})^{an}, \mathbb{Z}/n), \quad i \geq 0,$$

is an isomorphism.

Andrei worked closely with V. Voevodsky to prove fundamental properties of V. Voevodsky's motivic cohomology theory. The joint paper with Voevodsky [15] represented a major advance, connecting Milnor K-theory to motivic cohomology via Bloch's higher Chow groups. In particular, the following theorem by Andrei and V. Voevodsky played a central role in Voevodsky's proof of first the Milnor Conjecture and then the more general Bloch–Kato Conjecture.

Theorem 9 ([15]). *Let F be a field and ℓ a prime invertible in F. Then the norm residue homomorphsm*

$$h_n : K_n^M(F)/\ell K_n^M(F) \;\to\; H^n(F, \mu_\ell^{\otimes n})$$

is an isomorphism if and only if for all q with $q \le n$

$$H^p_M(X, \mathbb{Z}/\ell(q)) \;\simeq\; H^p_{et}(X, \mu_\ell^{\otimes q}),\; p \le q; \quad H^{q+1}_M(X, \mathbb{Z}/\ell(q)) \;\hookrightarrow\; H^{q+1}_{et}(X, \mu_\ell^{\otimes q}).$$

One justification of the formulation of Bloch's higher Chow groups (shown by Andrei and V. Voevodsky to agree with Voevodsky's motivic cohomology) is their relationship to algebraic K-theory as expressed in the motivic spectral sequence of the next theorem. Andrei was responsible for providing definitive proofs of this spectral sequence.

Theorem 10 ([3], [11]). *Let X be a smooth quasi-projective variety over a field. Then there is a strongly convergent spectral sequence*

$$E_2^{p,q} = H_M^{p-q}(X, \mathbb{Z}(-q)) \;\Rightarrow\; K_{-p-q}(X).$$

Finite group schemes arise in the study of the representation theory of finite groups and algebraic groups as well as occur in aspects of arithmetic algebraic geometry. Together with E. Friedlander, Andrei proved the most fundamental result about their cohomology. In doing so, they introduced a new form of functor cohomology developed expressly for this theorem and now of much independent interest; for an early example, see the paper by Andrei and others [1].

Theorem 11 ([2]). *Let G be a finite group scheme over a field k. Then $H^*(G, k)$ is a* finitely generated *algebra over k.*

Moreover, if M is a G-module (i.e., a $k[G]$-comodule) finite dimensional over k, then $H^*(G, M)$ is a finitely generated module over $H^*(G, k)$.

Among other results concerning the cohomology and representation theory of finite group schemes, we mention the following geometric description by Andrei (with C. Bendel and E. Friedlander) of the cohomology of infinitesimal group schemes over a field (i.e., group schemes whose coordinate algebras are finite-dimensional local rings). For the r-th Frobenius kernel $G = \mathbb{G}_{(r)}$ of one of the classical groups SL_n or Sp_{2n} or O_n (associated to a symmetric bilinear form on k^n), this theorem implies that $\operatorname{Spec} H^*(G, k)$ is homeomorphic to the variety of r-tuples of p-nilpotent, pairwise commuting elements of the Lie algebra of $\mathbb{G}$.

Theorem 12 ([13]). *Let G be an infinitesimal group scheme over a field k of height r. Then the morphisms $\mathbb{G}_{a(r)} \to G$ of group schemes over k from the r-th Frobenius kernel of the additive group $\mathbb{G}_a$ to G (i.e., the height r, 1-parameter subgroups of G) are the k-points of an affine scheme $V_r(G)$.*

There is a natural map of finitely generated commutative k-algebras

$$\psi : k[V_r(G)] \;\to\; H^*(G_{(r)}, k)$$

that induces a homeomorphism on prime ideal spectra.

The following theorem of Andrei (joint with E. Friedlander and J. Pevtsova) is the basis for new invariants of finite group schemes, the formulation of modules of constant Jordan type, and a novel construction of algebraic vector bundles. The key ingredient is an ingenious argument of Andrei's about the maximality of ranks occurring among linear combinations of commuting nilpotent square matrices.

The theorem concerns π-points of a finite group scheme G over a field k, namely flat maps $\alpha_K\colon K[t]/t^p \to KG$ factoring through an abelian subgroup scheme $C_K \hookrightarrow G_K$ with K/k an arbitrary field extension. Here, KG is the linear dual of the coordinate algebra $K[G_K]$. There is a natural equivalence relation on π-points of G defined in terms of G-modules, and the set of equivalence classes admits a scheme structure $\Pi(G)$ (also defined using representations of G) such that $\Pi(G) \simeq \operatorname{Proj} H^*(G, k)$.

Theorem 13 ([4]). *Let G be a finite group scheme, M a finite dimensional G-module, and $\alpha_K : K[t]/t^p \to KG$ a π-point of G which represents a generic point of $\Pi(G)$. Then the Jordan type of $\alpha_K(t) \in KG$ viewed as a nilpotent operator on M_K depends only upon the equivalence class of α_K as an element of $\Pi(G)$.*

Eric M. Friedlander[1] *and Alexander Merkurjev*[2]

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[1] Partially supported by the Simons Foundation.
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Appendix

Life and death of mathematicians in Petrograd 1918–1923 (The Committee for Improving the Wellfare of Petrograd Scientists)

H.G. Wells, who visited Petrograd[1] in September–October in 1920, wrote:

> For the rest of the arts, for literature generally and for the scientific worker, the catastrophe of 1917–18 was overwhelming. [...] For the scientific man at first the Soviet Government had as little regard as the first French revolution, which had "no need for chemists." These classes of worker, vitally important to every civilised system, were reduced, therefore, to a state of the utmost privation and misery. [1, p.34].

Wells further wrote in his book of essays entitled *Russia in the Shadows*:

> Nowhere in all Russia is the fact of that crash so completely evident as it is in Petersburg. [2, p. 316]

1919, 1920, and 1921 entered Russian history as years of unspeakable terror and hunger. A Civil War was under way. Railroad transportation almost ceased operating. The delivery of food to the city had practically come to a standstill. "Masses of people are dying," O.I. Vendrich, a resident of Petrograd, wrote in her diary during the winter of 1919, [3]. The academician Mikhail Ivanovich Rostovtzeff, in his article *Wake: In Memory of Murdered Friends and Colleagues* published in exile in early 1920, wrote:

> From time to time I receive letters from my teachers, colleagues and students who have fled the Bolshevik paradise. [...] And each letter contains, first and foremost, a Synodikon — dry lists of those who died, all featuring the same marginalia: died of starvation, execution by firing squad, committed suicide. [4, p. 431]

The Bolshevik government allocated almost no funds to support the scientific community, considering it politically hostile. In May of 1918, the

[1] Petrograd — that is how St. Petersburg was called during 1914–1924.

Petrograd Soviet[2] issued a decree on rationing by class[3]. Under this decree, the residents of Petrograd were divided into four categories according to the type of bread ration card they received. But even the ration given to workers (the first category) amounted to only 20% of the calories needed to sustain life. Scientists were allotted rations in the third category.

Historian N.I. Kareev, Corresponding Member of the Academy of Sciences, in his memoir *The Things I Lived Through and Experienced*, describes the ration as follows:

> Bread was allotted only by card in small quantities, sometimes amounting to as little as a quarter or even an eighth of a pound a day. [5, p. 272]

At that time, pretty much the only person in Petrograd to whom the intelligentsia could turn to for help was Maxim Gorky[4]. Thanks to his efforts, on December 13th, 1919, the board of Narkompros[5] decided to create a special committee with the goal of improving the welfare of scientists. On December 23rd, 1919, Sovnarkom[6] issued a decree that established a special increased food ration for scientists. In order to implement the decree of the Council of People's Commissars, the Executive Committee of Soviet Petrograd passed a resolution to establish the Petrograd Committee for the Improvement of the Welfare of Scientists — PetroKUBU. Maxim Gorky was appointed chairman. The decision to create the committee was published on January 13th, 1920, in the newspaper *Petrogradskaya Pravda*, the Petrograd edition of *Pravda*. Thus, the Petrograd KUBU was created, saving thousands of lives for the country and for science.

On January 15th, the Petrograd *Krasnaya Gazeta* responded to the creation of KUBU with an article by A. Bolotin entitled "How Scientists' Welfare Will Be Improved." The author informed the readers:

> The Committee for the Improvement of the Welfare of Scientists compiled lists of those involved in science, who will be allocated increased food rations. The list included only 1,800 scientists. Only scientists of exceptional merit were included in the list, as the number

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[2] Petrograd Soviet was an elected revolutionary committee.

[3] "The catastrophic food situation spawned the idea of a "class ration" in order to exclude some of the city's inhabitants from the rationed supply. The state took upon itself the obligation to feed only the workers: this included not only laborers, but also those who were registered as employed. Production and socio-political support for the new system depended on them." [24]

[4] Maxim Gorky (1868–1936) was a Russian Soviet writer, playwright, public figure, journalist, and publicist. He was nominated for the Nobel Prize for Literature in 1918, 1923, 1928, and 1933.

[5] Narkompros, People's Commissariat for Education RSFSR (Narkompros RSFSR, NKP), was a governmental committee of the RSFSR, which in the 1920s–1930s was in charge of nearly all cultural and humanitarian sectors: education, science and others.

[6] Sovnarkom (Council of People's Commissars of the RSFSR) was the government of Soviet Russia from 1917–1946.

> of individuals engaged in scientific activity in Petrograd is around 4,000 presently. [...]
>
> The committee then decided to organize a “house of scientists” in the former palace of Vladimir Alexandrovich[7]. In one of the wings overlooking the former Millionnaya Street, some of the halls will be lit and heated, where scientists will be able to conduct their research. These halls will thus become scientists’ heated shelters. [...] One can imagine, that thanks to these shelters, our scientists will become accustomed to collectivist forms of living.

KUBU’s main task at that time was to supply scientists with food rations, although KUBU did not have its own food supply. It was possible to get them only from the Petrokommuna[8]. For this reason, PetroKUBU had to submit a list of all scientists, indicating their position, scientific achievements, and where they worked, by February 1st. Scientists were required to write up their own biographies and append a list of their scientific works.

The KUBU archive contains handwritten documents compiled by Nadezhda Nikolayevna Gernet, Nikolai Maksimovich Günter, Ivan Matveevich Vinogradov, as well as a petition by Academician Markov requesting that the young scientist Vinogradov (born in 1891) be granted an academic ration, and a review of his research by Yakov Viktorovich Uspensky. The archive also contains the autobiographies of Alexander Vasilievich Vasiliev, Abram Samoilovitch Besicovitch (along with Vladimir Andreevich Steklov’s approval), and other mathematicians in Petrograd. Owing to the efforts of KUBU, food provisions for Petrograd scientists improved, but things were still difficult because the Petrokommuna could not provide all the food requested.

In May 1920, a delegation of English trade unions, arrived in Soviet Russia. Bertrand Russell, a famous mathematician and philosopher who later won the Nobel Prize, arrived with the delegation. Later on, in his autobiography, he reminisced:

> On one occasion in Petrograd (as it was called) four scarecrows came to see me, dressed in rags, with a fortnight’s beard, filthy nails, and tangled hair. [...] Equally ragged were the Mathematical Society of Petrograd. I went to a meeting of this society at which a man read a paper on non-Euclidean geometry. I could not understand anything of it except the formulae which he wrote on the blackboard, but these were quite the right sort of formulae, so that one may assume the paper to have been competent. Never, in England, have I seen tramps who looked so abject as the mathematicians of Petrograd. [6, p. 193]

---

[7] Grand Duke Vladimir Alexandrovich of Russia, son of Alexander II. The palace was one of the last imperial palaces built in Saint Petersburg.

[8] Petrokommuna was in charge of supplying the city with produce and manufactured goods, engaged in procurement, production and distribution of products during the Civil War and the *War Communism* policy.

A delegation of English labour party near the Labour Palace, May 1920, Petrograd. Bertrand Russell is the sixth from the right. Source [23].

Starting in 1920, PetroKUBU began publishing the journal *Science and Its Practitioners.*

Twelve issues were published in three years. Then the journal was closed down upon denunciation by the chairman of the Petrosoviet, Zinoviev. In the fourth issue of the 1921 journal, in the article "The Petrograd Society of Physics and Mathematics", the journal informed:

> During the second meeting on May 15th, in which Yakov Viktorovich Uspensky introduced the audience to the course he was giving on Non-Euclidean Geometry at the University, the famous English mathematician and philosopher Bertrand Russell was present. Russell, who was travelling with an English workers' delegation to Russia, introduced his views on the possibility of experimentally solving questions about the properties of space. [7, p. 39]

The same article reported:

> On May 14th, 1921, a new scientific society was finally organized in the mathematical office of the Pedagogical Institute at the University [...] The society was formed as a result of the activities of this seminar, which began to meet in the office of Prof. Alexander Vasilievitch Vasiliev, upon his initiative. [...] on May 14th, the board of the society was created and its main tasks were outlined [...] This was the 21st meeting of the seminar [...]

The cover of the journal *Science and its Practitioners.*

> The meetings of the seminar that founded the Society of Physics and Mathematics began on the 20th of March, 1920, on the day of Isaac Newton's death [...] [7, pp. 38–39]

Meanwhile the 12th meeting of the seminar (on November 23rd, 1920) was dedicated to the memory of Evgraf Stepanovich Fedorov.

> Prof. Boldyrev and Bogomolov highlighted the scientific achievements of the famous scientist in their presentations: the former — in the field of crystallography and crystal chemistry, the latter — in higher geometry, to which Fedorov contributed [...] broad, overarching ideas. [7, p.39]
>
> During a number of meetings [...] specialized mathematical papers were reviewed and published as part of the 42nd volume of the journal *Acta mathematica* (Yakov Davidovich Tamarkin, Abram Samoilovitch Besicovitch, Grigorii Mikhailovich Fichtenholz). Incidentally, Levi-Civita's extraordinarily important memoir on the three-body problem, is reviewed. [...]
>
> The organized society began its activity with two meetings on May 25th and 26th. During the first meeting, two presentations were given: A.V. Vasiliev on "Geometry of the World", and Academician Petr Petrovich Lazarev on "The Ion Theory of Excitation." Two presentations were given during the meeting on May 26th, the centennial of the memorial

> day of the famous Russian thinker Pafnuty Lvovich Chebyshev. A.V. Vasiliev reviewed his work on the history of mathematics in Russia, A.S. Besicovitch delivered a presentation on a topic from probability theory related to Chebyshev's works. [7, pp. 39–40]

Pafnuty Lvovich Chebyshev was honored by the Academy of Sciences in May of 1921. Sergey Fyodorovich Oldenburg, the permanent secretary of the Academy, wrote about this in the article "The Russian Academy of Sciences in 1921." The article was published as part of the first issue in 1922 of the journal *Science and its Practitioners*.

In the same article, S.F. Oldenburg reports that "over the past year, the Academy has established three new research institutes" [8, p. 8]. One of them is the Institute of Physics and Mathematics. The author further informs that: "The Institute of Physics and Mathematics participates in the international edition of Leonard Euler's work" and "the remarkable posthumous work of Alexander Mikhailovich Lyapunov." [8, p. 9]

Over time, the academic food ration provided more calories and became more varied. At Gorky's request, scientists who had families were given additional products.

During this time, the Soviet press featured caricatures of scientists, academic rations, and Gorky. Gorky annoyed Lenin and the Soviet government with his constant solicitude for the Russian creative and scientific intelligentsia. Without a doubt, the leaders of the state knew these caricatures were being published. The statements alleging the government's concern for scientists that were later made are largely hypocritical.

On August 13$^{\text{th}}$, 1922, a caricature of the academic ration appeared in the appendix to the *Krasnaya Gazeta*.

A pig and a donkey near a trough filled with academic rations.

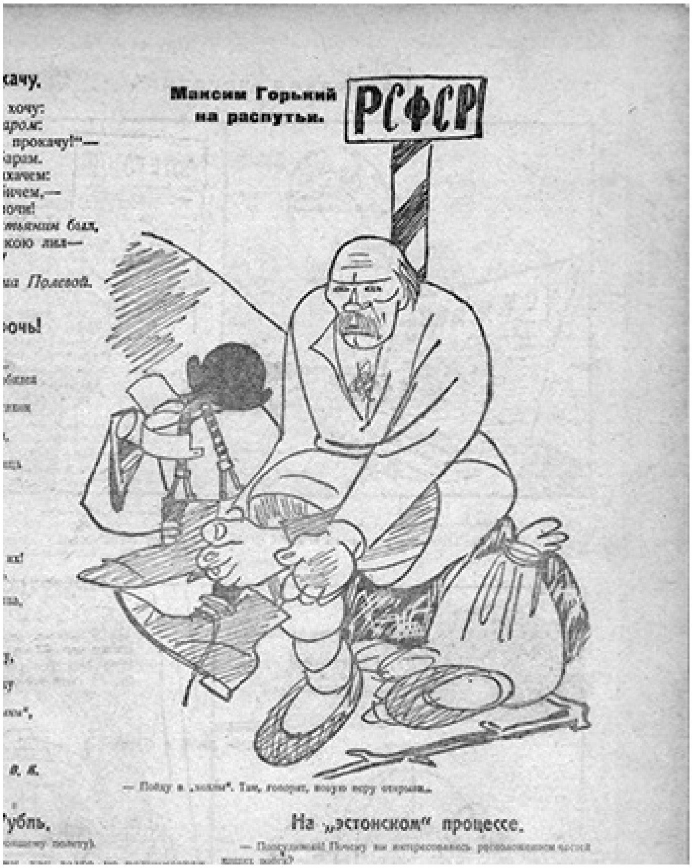
In 1923, the artist Manuel Andreev, famous during the beginning of the 20th century, was tasked by Soviet authorities to produce a series of satirical postcards "Everyday Life in Petrograd", [9]. One of the postcards with the title

"Ivan Ivanovich got his ration" caricatured an intellectual wearing glasses and homemade shoes on bare feet, carrying bulky sacks and a bag with groceries.

Ivan Ivanovich got his ration.

It's well-known that at the end of 1921 Lenin forced M. Gorky to go abroad under the pretext of getting treatment for his medical conditions. But a caricature of the writer that appeared in the media depicted Gorky leaving the Russian Soviet Federative Socialist Republic (RSFSR) of his own accord. At the border checkpoint, the writer swapped his Russian bast shoes for European shoes.

Scientists were having challenges with more than just obtaining food. At one of the meetings of the General Assembly of the Academy of Sciences, the



Caricature of Gorky.

application of Academician Andrei Andreyevich Markov was considered, who for lack of shoes could not attend the meetings of the Academy. During the meeting, the decision was made to ask Academician Alexander Evgenyevich Fersman to ask KUBU about the possibility of obtaining footwear. At KUBU's meeting on April 26, 1921, Fersman petitioned for Academician Markov to be provided shoes [10, p. 31]. Professor Nikolai Maksimovich Günther had to make a long and persistent effort to have his wife's only pair of shoes repaired. There were only a few craftsmen working at the KUBU shoe shop, and the waiting list for shoe mending extended for months. Korney Ivanovich Chukovsky[9] recorded in his diary how his then assistant Maria Benkendorf told him that

[9] Korney Ivanovich Chukovsky (1882–1969) was a Russian Soviet poet, publicist, literary critic, translator, literary scholar, children's writer, and journalist.

> ...when people kneel in church, it's very curious to behold the entire collection of holes on the soles of their shoes. Not a single sole without a hole! [11, p. 259]

The housing issue was also acute at that time. Random strangers were moved into scientists' apartments, making them more crowded. Other housing afflictions included forced relocations and evictions. The KUBU archives contain a petition submitted to PetroKUBU by Prof. Günther on February 9, 1920. The house where Günther lived was seized by the Society for the Disabled. The tenants were asked to immediately vacate their apartments but were not allowed to take their belongings. Prof. Günther's petition was supported by PetroKUBU and he was allowed to remain in his home, although he had to move to another wing of the house.

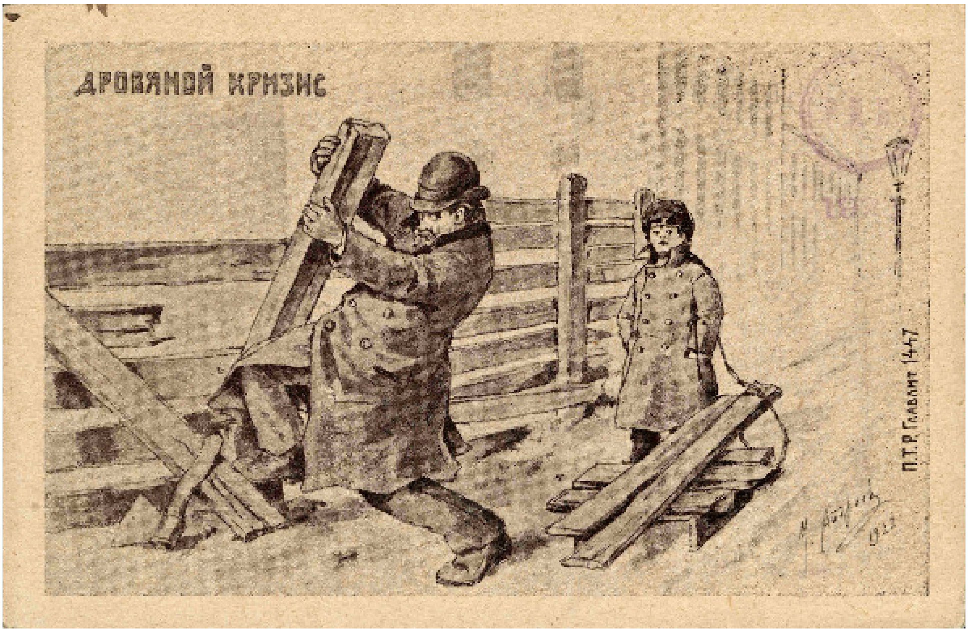
The ways in which the housing issue was solved were reflected in the printed propaganda of the Soviet authorities as well. One of the postcards in the "Children Playing the Revolution Game" series, was titled "Seizure of the Premises." The postcard shows children trying in every way possible to keep invaders out of their home.

Postcard "Seizure of the Premises".

KUBU protected its members from military and labor conscription. Hard physical labor, especially during the cold season, could mean sickness and death for the hungry and emaciated scientists. In the first issue of the journal *Science and its Practitioners* for 1921 in the section Personalia in the list of deceased scientists (178 names in total), we find the following record:

> Viliev, Mikhail Anatolievich, was a young, talented astronomer who already had about a hundred papers to his name. His exceptional abilities, highly recognized by specialists, guaranteed him a future as a scientist of the highest rank. He died after catching a cold while he was digging trenches. His death is one of the most profound modern tragedies of the life and death of a Russian scientist. [12, p. 35]

The fuel shortage was very acute during the early twenties. The winters were harsh and there was practically no firewood left in the city. Wooden houses, fences, and the ends of sidewalks were dismantled for firewood.



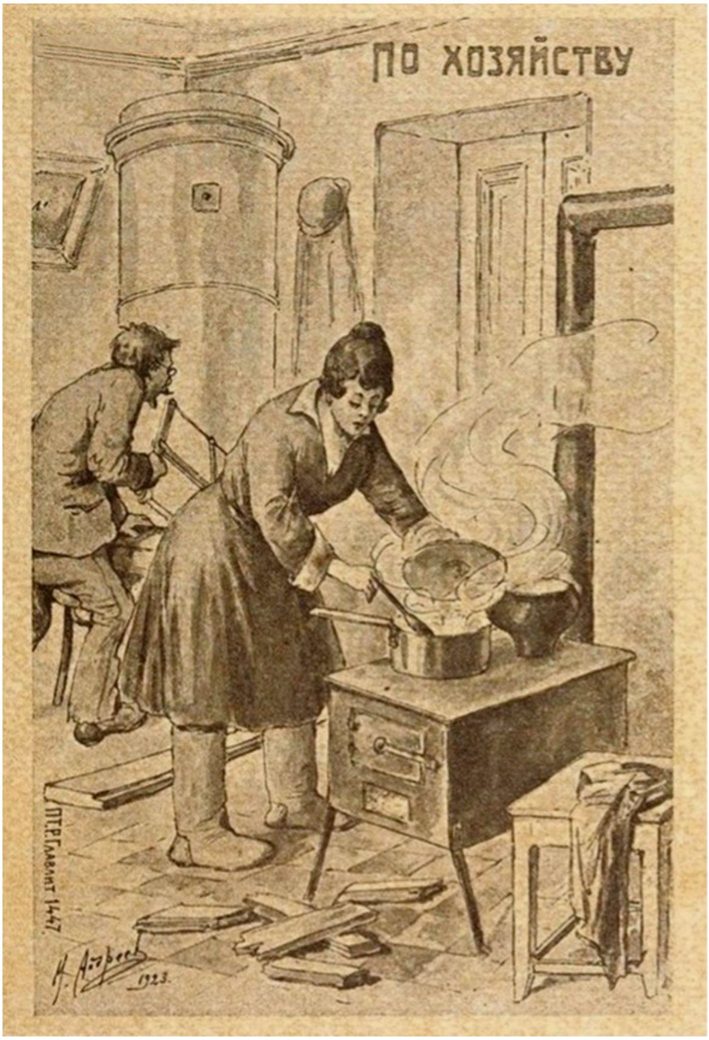
The Firewood Crisis.

They did not heat the furnaces in the apartments, but instead got "burzhuykas", which served both as a furnace and a kitchen stove.

In his diary, Sergei Grigorievich Eliseyev, a famous Japanese studies scholar, described Prof. Vladimir Fedorovich Matveev, from the Law Department of Petrograd University:

> It was so cold that when Professor Matveev came home and went to bed fully dressed, he froze [to death] in his apartment. [13, p. 23]

There were also difficulties in getting the works of Petrograd scientists published. In the Bibliography of the 3rd issue of *Science and Its Practitioners* for 1921, we find the following:



The burzhuyka stove.

> Of Faraday's testament "to work, to finish, to publish", only the first third, "to work", is feasible for the majority of scientific practitioners nowadays. It's the destiny of only a few to finish, and only the lucky few who manage to publish. After all, in Petrograd alone there are more than 12,000 pages of unpublished manuscripts, and this number continues to grow. [14, p. 29].

Nevertheless, some things were still published, in spite of the shortage of paper, as well as typographical difficulties. Books were published mostly in small formats.

In 1921, two works by Alexander Vasilyevich Vasiliev were published in Petrograd: *Mathematics* and *The Integer: A Historical Essay.* In the same year, Alexander Alexandrovich Ivanov's book *Error Theory and the Method of Least Squares* was published. In 1922, *An Introduction to the Non-Euclidean Geometry of Lobachevsky–Bolian* by Yakov Viktorovich Uspensky, and in 1923, his *Essay on the History of Logarithms*, were published. In 1923, the book *Lectures on Experimental Geometry* by the Ukrainian mathematician Alexander Matveyevich Astryab was published in Petrograd. Other works were planned for publication.

In the 1st 1920 issue of the journal *Science and its Practitioners*, readers were informed:

> During the fall semester of 1919, the Pedagogical Institute at Petrograd University established the Department of Physics and Mathematics... Professor A.V. Vasiliev is the dean of the Department of Physics and Mathematics. [15, p. 27–28]

KUBU performed another very important function. From 1919 to 1922, scientists were constantly arrested in Petrograd. Felix Edmundovich Dzerzhinsky[10] wrote:

> Our professionals are mostly people belonging to bourgeois circles and mentality. We usually arrest individuals belonging to such social groups, taking them hostage or placing them in concentration camps for communal labor. [16, p. 221]

One such arrest was described by S.G. Eliseev, a well-known Japanese studies scholar of Russian origin who emigrated to America, in his diary:

> ...When I came to the headquarters of the secret service, I saw many people from the university... Turns out, we were under arrest and were hostages. When I asked: "Why did you arrest us?", the investigating officer answered, "To shoot you, because you're hostages." [13, p. 17–18]

The prominent Russian philosopher Nikolay Onufriyevich Lossky wrote in his book Reminiscences: *Life and the Philosophical Journey*:

> In the summer of 1922, a new storm was closing in over the intelligentsia, of which none of us had an inkling of. Zinoviev, the chief of St. Petersburg and the Northwestern region, reported to Moscow that the intelligentsia was beginning to rear its head. He wrote that different factions of the intelligentsia are beginning to create professional journals and societies; they aren't working in concert, but will, in time, unite and become a force to be reckoned with. Therefore,

[10] Felix Edmundovich Dzerzhinsky (1877–1926), Head of several People's Commissariats, founder and head of the VChK. The VChK is a special security agency of the Soviet government.

> the Moscow government decided to arrest prominent scientists, writers, and public figures throughout Russia; and this was done on August 16$^{\text{th}}$, 1922. [17, p. 237]

Lossky remembered this arrest:

> [...] the government knew that we were not involved in political activity. [...] [But] it was already decided that we would be sentenced to exile abroad. At this time, the Bolshevik government was seeking *de jure* recognition by the Western European states. Individuals whose names and activities were known in Europe were arrested, and the Bolsheviks apparently wanted to show that their regime was not a barbaric despotism. It is said that Trotsky suggested this very measure of exile abroad. [17, p. 238–239]

Trotsky's statement on this subject is well-known:

> We expelled these people because there was no reason to shoot them, but it was impossible to tolerate them. [18, p. 266]

Lossky further recalls:

> After the interrogation I was taken to a large room where there were about fifty people under arrest [...] Among them were Karsavin, Lapshin, the professor of mathematics Selivanov and others from our group. Apparently, Selivanov was arrested for his "bourgeois" method of teaching mathematics to engineers. In his lectures, Selivanov not only provided the mathematical formulas used in the engineering profession, but also the mathematical proofs behind them. At this time, the Bolsheviks came to the conclusion that an engineer should know the formulas but is not required to know the mathematical reasoning behind them. [17]

Students, rising to the heights of their professions, are getting rid of "excessive ballast." Liberation from ballast probably implied liberation from professors like Selivanov, who taught mathematics "not in the red way." [19]

The FSB archives contain VChK-GPU documents where there is a note by F.E. Dzerzhinsky regarding the petition of party member Vladimir Ivanovich Nevsky: "Nevsky is petitioning for the release of Selivanov. We are against this." [20, p.328–329]. KUBU also petitioned for the 67-year-old mathematician Selivanov, but to no avail.

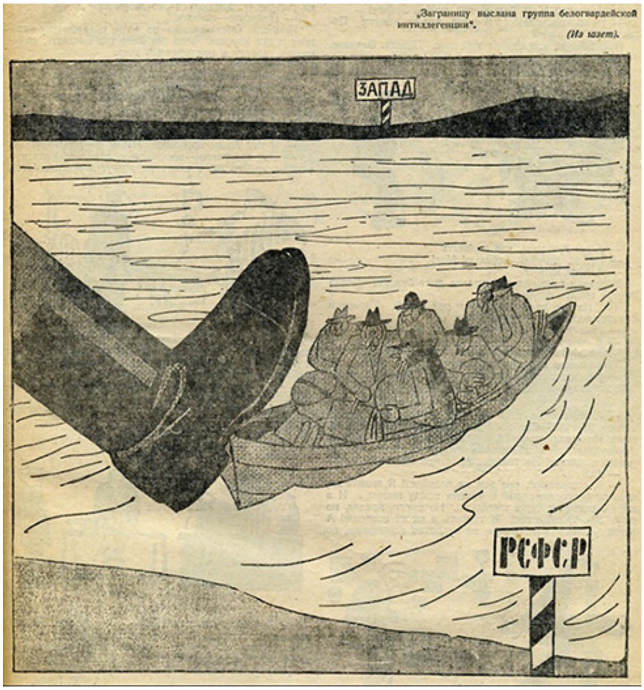
On October 10, 1922, Dmitri Fyodorovich Selivanov filed a letter to the GPU Presidium:

> I have now begun teaching at the University... Additionally, I am preparing two manuscripts for publication, one on an Advanced Course for Higher Algebra and the other on Projective Geometry. My wife, a

Lunacharsky with his students in a hot air balloon. During their meetings, the student body has spoken out in favor of relieving universities of unnecessary ballast. — Isn't that a lot, comrade? — It's okay, Anatoly Vasilyevich, the more ballast we throw out, the higher we will rise.

> mathematics teacher at the Rabfak[11] of the University, and I, would be terribly hindered in our work by being exiled abroad. This whole

[11] In 1919, for the first time in the history of the University of Petrograd (Leningrad), a new and unusual institution — the Workers' Faculty (Rabfak, or Rabochiy Fakultet) — was established. It was supported by the new Soviet authorities at the university, in their struggle against the "old" students and professors. Many people were educated through the Rabfak in 1922–1929, some of whom later became prominent scientists and statesmen. It



The magazine *Red Lights*, 1922. "A group of White Guard intelligentsia has been exiled abroad."

> hassle would be expensive and would take up a lot of our time. I don't know whether I should sell my last possessions in order to have enough money for this eventuality [...] I'm really afraid that I'll suddenly be told to board a steamship in two or three days, but that I won't have the money or the permission to bring my linen and clothes along with me. I would not want to come to Germany without the necessary things and ask German mathematicians for alms. [20, p. 329]

On October 14, 1922, Selivanov was ordered to leave Russia, and on November 16, 1922, he and a group of scientists sailed from Petrograd on the German steamship *Prussia*.

The Soviet press responded to the expulsion of scientists from Russia with yet another cartoon.

KUBU also cared about scientists' health. As early as June, 1920, KUBU scientists were first accepted as patients to the sanatorium in Detskoye Selo[12]. In 1921, Konstantin Alexandrovich Posse, professor of mathematics at Pet-

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is an example of the proletarianization of higher education as promulgated by the Soviet authorities after 1917. (Translated and adapted from an article in Russian, [25].)

[12] Detskoye Selo, known in pre-revolutionary times as Tsarskoye Selo, is a town 15 miles south of the center of St. Petersburg. Under Stalin, the town was renamed Pushkin and is still known to this day by that name.

rograd University, was treated there. In October 1922, the Internat for the Elderly was opened at KUBU. The mathematicians Julian Vasilyevich Sochocki and K.A. Posse, professors at Petrograd University, lived the last years of their lives in this residential home. Dmitri Ivanovich Mendeleev's wife spent her final years in this residential home as well [21, p. 13]. In 1921, Gorky arranged for Petrograd scientists to be treated at the CEKUBU Gaspra[13] sanatorium in Crimea. In May, 1926, Academician Vladimir Andreevich Steklov vacationed and was treated there. Thanks to the Petrograd Committee for Improving the Welfare of Scientists, scientists were able to survive through trying times, and the lives of many outstanding figures in Russian science were saved. According to archival data, 80 Petrograd mathematicians received food rations from KUBU. Many of them could not have survived those harsh and hungry post-revolutionary years without these rations. Here are the names of scientists who benefitted from KUBU:

1. Adamov, Aleksei Alekseevich, m.[14], Professor at the Polytechnical Institute.
2. Akimov, Pyotr Vasilyevich, Theory of Mechanics, Professor at Petrograd Mining Institute.
3. Angert, David Nikolaevich, m., and Education Studies, Professor.
4. Bashinsky, Viktor Vladimirovich, Methods of calculating statistical uncertainty of systems, Professor at Turkestan University.
5. Besicovitch, Abram Samoilovitch, m., Professor, 3rd Pedagogical Institute of the University of Perm.
6. Bentkovsky, Iosif Iosifovich, mechanics, Professor, strengths of materials.
7. Bilibin, Alexander Yakovlevich, m., Professor at the 2nd Polytechnical Institute.
8. Bogomolov, Stepan Alexandrovich. m., Professor at the 1st, 2nd, and 3rd Pedagogical Institute.
9. Borisov, Evgeny Vasilyevich, m., lecturer at the Technical Institute of Petrograd University.
10. Borisov, Alexander Alexandrovich, m., lecturer at the Institute of Agriculture.
11. Boyarchuk, Vladimir Prokofievich, m., stayed at the Department of the 3rd Pedagogical Institute.
12. Budaevsky, Sergei Andreevich, m., mechanics, instructor at the Engineering Academy.
13. Vasiliev, Alexander Vasilyevich, m., Professor at the University.
14. Vinogradov, Alexander Mikhailovich, m., instructor at the Institute of the National Economy.

[13] Gaspra is a region in Yalta City on the Black sea in Crimea.
[14] m. — mathematician.

15. Vinogradov, Ivan Matveevich, m., Professor at the University's 1st Polytechnical Institute.

16. Vulikh, Zakhar Zakharovich, m., Professor at the 1st Pedagogical Institute.

17. Gavrilov, Alexander Feliksovich, m., Professor at the 1st Polytechnical Institute.

18. Gelyvikh, Pyotr Avgustovich, m., instructor at the Artillery Academy.

19. Gernet, Nadezhda Nikolaevna, m., Professor at the 1st Pedagogical Institute and at the University.

20. Girman, Sergei Nikitich, m., independently.

21. Glagolev, Ivan Pavlovich, m., instructor at the 1st Pedagogical Institute.

22. Godytsky-Tsvirko, Alexander Mordarevich, Theory of mechanics, instructor at the Institute of Railway Engineers.

23. Gratsiansky, Ivan Ivanovich, Methodology of mathematics, Professor at the Institute of Early Childhood Education.

24. Grodsky, Georgy Dmitrievich, m., Professor at the Artillery Academy.

25. Günter, Nikolai Maksimovich, m., Professor at the University.

26. Davydov, Ivan Ivanovich, m., instructor at the Institute of Geography.

27. Di-Segni, Nikolai Konstantinovich, m., Dean of the Institute of Fire Engineering.

28. Egupov, Vladimir Andreevich, m. and theoretical mechanics, Professor at the Academy of Fine Arts.

29. Juravsky, Andrei Mitrofanovich, m., Professor at the Institute of Mining.

30. Ivanov, Ivan Ivanovich, Professor at the 1st Polytechnical Institute.

31. Ikornikov, Yuri Vasilievich, m. and geographer, instructor at the Technological Institute.

32. Kavun, Ivan Nikitich, Methodology of mathematics, instructor at the 3rd Pedagogical Institute.

33. Kargin, Dmitry Ivanovich, m., graphics, Professor at the Institute of Fire Engineering.

34. Kompaneits, Pyotr Andreevich, m., Professor at the 3rd Pedagogical Institute.

35. Kondratyev, Vladimir Andreevich, m., Head of Department at the National Pedagogical Museum.

36. Konyuchenko, Ivan Timofeevich, m., instructor at the Institute of Astronomy.

37. Koyalovich, Boris Mikhailovich, m., Professor at the University's Technological Institute.

38. Krechmer, Vasily Avgustovich, m., instructor at the Institute of Railways Engineers.

39. Kulisher, Alexander Ruvimovich, m., Professor at the 2nd Pedagogical Institute.

40. Lipin, Nikolai Vyacheslavovich, m., instructor at the 3 rd Pedagogical Institute, and at the University.

41. Lyush, Vasily Vladimirovich, m., instructor at the Technological Institute.

42. Malis, Leonid Germanovich, m., physics, instructor at the Technological Institute.

43. Markov, Andrei Andreyevich, m., Academician at the Academy of Sciences.

44. Melikov, Konstantin Venediktovich, m., instructor at the Institute of Railways Engineers.

45. Meshcherski, Ivan Vsevolodovich, mechanics, Professor at the 1st Polytechnical Institute.

46. Mitropol'sky, Aristarkh Konstantinovich, mechanics, applied mathematics, instructor at the Technological Institute.

47. Mikhel'son, Nikolai Semyonovich, m., Professor at the Technological Institute.

48. Naryshkina, Ekaterina Alekseevna, m., stayed at the Department of Mathematics at Petrograd University.

49. Penionzhekevich, Karl Boleslavovich, m., physics, instructor at the Institute of Photography.

50. Petrovich, Sergei Georgievich, m., Professor at the 1st Pedagogical Institute.

51. Pirozhkov, Mikhail Vasilyevich, m., instructor at the Worker's Department at the University.

52. Polosukhina, Olga Andreevna, m., Professor at the University.

53. Posse, Konstantin Alexandrovich, m., Professor at the University.

54. Radtsig, Alexander Alexandrovich, mechanics and mathematics, 1st Polytechnical Institute.

55. Selivanov, Dmitry Fedorovich, m., Professor at the University.

56. Sigov, Isaaky Alexandrovich, m., instructor at the 2nd Pedagogical Institute.

57. Sinakevich, Vladimir Ivanovich, m., instructor at the Academy of Agriculture.

58. Smirnov, Vladimir Ivanovich, m., Professor at the University.

59. Smirnova, Yulia Andreevna, m., Professor at the University.

60. Sochocki, Yulian Vasilyevich, m., Professor at the University.

61. Steklov, Vladimir Andreevich, m., Academician at the Academy of Sciences.

62. Sultan-Shakh, Ekaterina Semyonovna, m., physics, instructor at the 1st Pedagogical Institute.

63. Tamarkin, Yakov Davidovich, m., Professor at the University.

64. Tovstoles, Flavion Pavlovich, m., applied mathematics at the Military Engineering Academy.

65. Tokmachyov, Sergey Mikhailovich, m., meteorology instructor at the Forestry Institute.

66. Umnov, Boris Ivanovich, m., instructor at the 3rd Pedagogical Institute.

67. Uspensky, Yakov Viktorovich, m., Professor at the University.

68. Federman, Alexander Karlovich, m., Professor at the Institute of Agronomy.

69. Filippov, Vladimir Mikhailovich, m., instructor at the 1st Polytechnical Institute.

70. Fichtenholz, Grigorii Mikhailovich, m., Professor at the University.

71. Fridman, Alexander Alexandrovich, m., Mechanics, Head of the Department of the Physical Observatory.

72. Kharitonovich, Boris Georgievich, Mechanics, instructor at the Polytechnical Institute of the Naval Academy.

73. Tsinzerling, Dmitry Petrovich, m., Head of the Mathematics Collection of the Pedagogical Museum.

74. Tsytovich, Nikolai Platonovich, m., artillery, professor at the Artillery Academy.

75. Shatrov, Vladimir Dmitrievich, Applied mathematics, Professor at the Institute of Civil Engineering.

76. Shmulevich, Pyotr Kornivovich, m., instructor at the Institute of Photography and Photographic Technology.

77. Shokhat, Yakov Alexandrovich, m., Professor at the 2nd Pedagogical Institute.

78. Shokhor-Trotsky, Semyon Ilyich, m., instructor, researcher at the Institute of Agriculture and at the Institute of Optics.

79. Shchukin, Nikolai Leonidovich, Theoretical and applied mechanics, Professor at the 2nd Polytechnical Institute.

80. Krylov, Aleksey Nikolaevich, m., Professor, Academician at the Academy of Sciences.

In a letter to Academician S.F. Oldenburg, Gorky wrote:

> I observed how the creators of Russian science endured the excruciating days of hunger and freezing cold with such modest heroism and stoic courage, and I saw how they worked and saw how they died...
>
> I think the world was given a magnificent lesson on stoicism by Russian scientists with how they lived and worked during the years of the Intervention and the Blockade. Their story will teach the world about this time of suffering with the pride of a Russian man writing these simple words to you. [22, p. 260]

Afterword: Firstly, I would like to provide some historical background information. The text mentions the Entente blockade of Petrograd. The Entente, also known as the Triple Entente, was a military alliance between England, France and Russia that aimed to contain the Central Powers, which comprised Germany, Austria and Italy. During World War I, the Entente was a combatant force against the Central Powers bloc. The roots of this alliance can be traced back to an agreement between England, the United States and France, signed in 1897, known as "The Gentlemen's Agreement." The purpose of the agreement was to keep Germany in check. After the end of World War I, the Entente supported anti-Bolshevik governments and blockaded many major Russian ports. One important reason that the Entente was doing this was the hope of restoring the large loans Russia had made to England and France before and during the First World War.

The text also mentions the House of Scientists, the former palace of Prince Vladimir, on the Neva River embankment. Grand Duke Vladimir Alexandrovich, was born on April 22, 1847 in St. Petersburg and died on February 17, 1909 in St. Petersburg. He was the third son of Emperor Alexander II and Empress Maria Alexandrovna. He actively participated in government affairs and was a member of the State Council (1872). He was also a senator (1868), adjutant general (1872), and a general of infantry (1880). Prince Vladimir was the younger brother of Emperor Alexander III.

The Petrokommune, i.e. the Petrograd Workers' Commune, was an assembly of executive bodies of Soviet power in Petrograd, formed in March 1918 and lasting until February 1919. Due to the move of the RSFSR government from Petrograd to Moscow, the Petrosoviet's decree on March 10, 1918 established the Petrograd Council of Sovnarkom (SNK) and Commissariats for Food, Finance, Education, and others as local bodies of power.

Now I would like to add a few historical remarks about the scientists mentioned in the text. The text mentions Nadezhda Nikolaevna Gernet. She was born on April 18, 1877 in Simbirsk and died on June 24, 1943 in Leningrad (during the blockade). Nadezhda Nikolaevna was a Russian and Soviet mathematician and teacher. She was a student of D. Hilbert, and was the second woman mathematician in Russia, after S.V. Kovalevskaya, to earn a doctoral degree. In the lists of scientists who received rations from KUBU, four additional women scientists are listed: Ekaterina Alekseevna Naryshkina, Olga Andreevna Polosukhina, Yulia Andreevna Smirnova, and Ekaterina Andreevna Sultan-Shakh. It would be an interesting line of research to discover the fates of these women scientists.

A number of scientists supported by KUBU nevertheless didn't survive the conditions they found themselves in. One of them was Evgraf Stepanovich Fedorov, an outstanding crystallographer, mineralogist and mathematician, Academician at the Russian Academy of Sciences, and Director of the St. Petersburg Mining Institute (1905–1910). He was 64 years old at the time of the revolution (he was born on December 10, 1853 in Orenburg). He was an active scientist, and a man of liberal views. It is noteworthy that he was a member and active participant of the organization *Land and Liberty*, but then withdrew from it, because he did not share the terroristic ideology of the organization. Fedorov died of pneumonia at the age of 65, on May 21, 1919, during a hungry, cold time in Petrograd.

Another scientist who died during this period was Andrei Andreyevich Markov. The mathematician Andrei Andreyevich Markov (1856–1922) made major contributions to probability theory, mathematical analysis and number theory. Markov processes are still one of the main methods for financial modeling, and they are also foundational to search engines such as Google and many other applications. Markov died in Petrograd in 1922 at the age of 66, most likely from starvation.

Of the scientists mentioned in the text who survived those terrible times, many had brilliant careers abroad, or in the Soviet Union.

In the Soviet Union, the mathematician Ivan Matveyevich Vinogradov (1891–1983) made important contributions to science. He is recognized for his seminal results in analytic number theory. He was an academician in the USSR Academy of Sciences (1929) in the Department of Physics and Mathematics. In addition, he was also elected as a foreign member of the Royal Society of London (1942), and was also a member of many other academies. He was the director of the Steklov Mathematical Institute of the USSR Academy of Sciences for 45 years.

Alexander Vasilievich Vasiliev (1853–1929) was a mathematician, public figure, and professor emeritus. Vasiliev initiated the reestablishment of the Petrograd Mathematical Society in 1921, of which he was chairman until his move to Moscow in 1923.

Grigory Mikhailovich Fichtenholz (1888–1959) was an outstanding mathematician and educator. His three-volume classic *Differential and Integral Calculus*, was for many years a staple textbook for students and mathematicians not only in the Soviet Union, but internationally as well. After 1917, Fichtenholz was active in the Council of Experts at the RSFSR People's Commissariat for Education. He also contributed greatly as head of a committee for preparing school curricula, and actively supported the organization and running of mathematical olympiads. Fichtenholz created a special approach on the theory of functions of real variables and functional analysis at the Leningrad University's Departments of Mathematics and Mechanics. In addition, G.M. Fichtenholz founded the Department of Calculus at Leningrad University and was its chair until his forced resignation at the height of the campaign against cosmopolitanism in 1953.

Pyotr Petrovich Lazarev (1878–1942) was another scientist supported by KUBU, who became an outstanding physicist and an Academician at the USSR Academy of Sciences (starting from 1917). Lazarev founded the *Physics-Uspekhi Journal* (*Advances in Physical Sciences*) in 1918, and established the Physics and Biophysics Research Institute, the first of its kind in the Soviet Union. Lazarev was arrested by denunciation in 1931 and was subsequently exiled. During this time, his wife committed suicide. He died at the age of 64 during the Great Patriotic War (World War II) while being evacuated.

Jakob Viktorovich Uspensky (English name: James Victor Uspensky) (1883–1921), mathematician and academician. Uspensky specialized in number theory and probability theory. In 1929, Uspensky immigrated to the United States and secured a position at Stanford University. Soon after, he renounced his post as an Academician at the USSR Academy of Sciences, and the authorities accessioned his extensive home library. Uspensky died in 1947 at the age of 63 in San Francisco.

In conclusion, it is clear that the physical and mental hardships of these trying years had an effect on the life expectancy of these talented individuals. Out of the eight scientists spotlighted, five of them could have contributed much more to science if given the extra years. Gernet died at the age of 66, Fedorov at 65, Markov at 66, Lazarev at 64, and Uspensky at 63. Nevertheless, there is no doubt that if not for KUBU and the help it afforded scientists, these statistics could have been even more ominous.

Natalia Malysheva

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The story of mathematicians repressed by the NKVD in 1941–1942 in besieged Leningrad

The Soviet repressive machine affected the entire population. The community of mathematicians was not spared. While the Luzin case[1] and the dispersal of the Leningrad Mathematical Society can still be seen as an intra-academic squabble, in the case of the "Public Safety Committee," which will be discussed below, the NKVD dealt with the mathematicians in earnest.

In the winter of 1941–1942, people in the besieged city of Leningrad were starving to death — by the thousands — every day. The militia and the NKVD[2] caught deserters, saboteurs, and gangs of cannibals. The first secretary of the Leningrad provincial and city party committee, A.A. Zhdanov, was informed that, according to security services agents, some anti-Soviet individuals were waiting for the Germans and wanted to cooperate with them. Zhdanov instructed P.N. Kubatkin, the head of the NKVD Department for Leningrad and the Leningrad Region, to verify the intelligence and ensure the safety of the city during the enemy's blockade. The executors took the order as a signal to exterminate disloyal sectors of the population by "lawful" methods.

Much has been written about the case of "Public Safety Committee" and the "The Scientists' Case"[3] [4], [5], [6], [7], [8]. Especially noteworthy are the memoirs of G.R. Lorentz who knew the repressed scientists personally [17]. The purpose of the current note is to recreate for the reader the atmosphere of repressions, the zeitgeist, which was an integral, albeit not always visible, part of the worldview of Soviet citizens of that period.

The case of the "Public Safety Committee" began with the arrest of Vladimir S. Ignatowski[4], a mathematician, physicist, specialist in optics, and corre-

[1] In 1936, Nikolai Luzin, a Russian and Soviet mathematician, was accused of publishing his major results in foreign journals and being disloyal to Soviet authorities. His department at the Steklov Institute was closed and he lost all his official positions; however, he was neither arrested nor expelled from the Academy.

[2] The abbreviation stands for "People's Commissariat for Internal Affairs," i.e., the interior ministry.

[3] Case No. 3749 (P-29626) ("Public Safety Committee") and Case No. 555 (P-22163) ("The Scientists' Case") are kept in the archive of St. Petersburg and the Leningrad Region branch of the FSB of Russia.

[4] Also Waldemar Sergius von Ignatowski.

sponding member of the USSR Academy of Sciences. There were two NKVD agents in his inner circle, one of whom had had an affair with Ignatowski's wife since 1939, and the other, E.F. Merkulov, was a provocateur; in 1945 he was found guilty of fabricating reports to show his usefulness to the authorities.

Professor Ignatowski and his wife were arrested on 6 November 1941. Ignatowski (who was beaten up during interrogations) confirmed that he was a member of the fascist organization "Public Safety Committee"[5] and that he was waiting for the Germans and wished to join the government they would establish after taking the city. The testimony of Ignatowski and his wife was used as a reason to arrest Professors Sergey M. Chanyshev, Vladimir I. Milinsky, Konstantin I. Strakhovich, Nikolai A. Artemyev, and a senior engineer of the Institute of Precision Mechanics Konstantin A. Lyubov. Their biographies and achievements in mathematics can be found, for example, in [8].

They all incriminated themselves under torture and were sentenced to death. Vladimir I. Milinsky, Associate Professor of Geometry in the Faculty of Mathematics and Mechanics of the University, died in prison on 4 January 1942. Konstantin I. Strakhovich's execution was commuted to ten years in a penal camp in exchange for false testimony against some Leningrad scientists who had not yet been arrested. As for the others, the sentences were carried out.

The second case, Case No. 555 (the case of "The Union of the Old Intelligentsia") was created based on the testimony of Strakhovich, which led to the arrest of Andrey M. Zhuravsky, Director of LOMI[6], Nikolai V. Rose, Dean of the Faculty of Mathematics and Mechanics of the LSU, Boris I. Izvekov, Professor of the Faculty, Assistant Professors Natalia I. Postoyeva and Boris D. Verzhbitsky, Professor Nikolai S. Koshlyakov, and many others.

Rose, Izvekov, and Verzhbitsky died during the investigation as a result of beatings and starvation, at least in part.

Under pressure from investigator Kruzhkov, Nikolai Sergeevich Koshlyakov, corresponding member of the USSR Academy of Sciences, professor, chair of the Department of General Mathematics at Leningrad State University and chair of the Department of Higher Mathematics at LETI, and an employee of LOMI, testified that a counter-revolutionary organization consisting of eleven professors and associate professors existed at Leningrad State University. In 1954, Professor Koshlyakov reported the following to the Leningrad Military District Prosecutor's Office:

> At the very beginning of the investigation, I was warned by Kruzhkov and others that if I did not plead guilty to the charges brought against me, they would force me to confess to committing the crime, as they

[5] In all likelihood, the investigators named the "organization" after the infamous committee of the times of the French Revolution, which was the French government in effect, if not in name, during the Reign of Terror in 1793–94. The naming, it might be assumed, was not based on historical accuracy but rather on the connotations.

[6] Leningrad Branch of the Mathematical Institute of the USSR Academy of Sciences.

> had many means to do so. And when I attempted to refuse to sign the record made up by Kruzhkov, he beat me up. After that, I no longer resisted [...] I was in a state of complete exhaustion and no longer had any will. [2]

On 25 April 1942, all of the accused in the case of "The Union of the Old Intellectuals" were sentenced to death. On 28 May 1942, the Presidium of the Supreme Soviet of the USSR commuted the sentence to ten years of labor camps.

In 1944, Koshlyakov was transferred from labor camp to Moscow where he worked at the theoretical department of a *sharashka*[7] — a design engineering bureau SB-1, which was developing weapons for anti-aircraft defense. He was released half a year earlier, in 1951, after 9,5 years of imprisonment. Later he was fully rehabilitated, and in November 1953 he was restored to the rank of corresponding member of the Academy of Sciences of the USSR by the decree of the Presidium of the Academy of Sciences. While in detention, he published one paper, under the pseudonym Sergeev, with the help of I.M. Vinogradov, S.N. Bernstein, and Y.V. Linnik [10].

Natalya Ivanovna Postoyeva, associate professor of Mathematics and Mechanics Faculty of LSU, was arrested on 16 February 1942. In 1957, she described her tribulations as follows:

> ...under the pretext of checking passports, some people came to my apartment and presented me with a search and arrest warrant. The search was carried out hastily and, having put away the coffee, precious in those days, and the remains of sugar into a briefcase, they took me to prison. The investigation began, which turned into torture as soon as my case was assigned to investigators Ryabov and Kruzhkov. My testimony against myself and other persons was ripped out by the terrible cruelty of the investigators, exacerbated by the fact that these techniques were applied to a person already exhausted by the blockade, starvation, who was in a state of most severe dystrophy. The insults, beatings, standing all night on legs swollen from hunger, sleep deprivation, the cold in the cell, and, perhaps, the most horrible for a hungry maniac, deprivation of a bowl of soup forced me to sign the records and give testimony in which every word was perverted by the investigators, translated into their 'special language' which they replaced the ordinary, human language with. For six months I was being made into a state criminal, following a pre-prepared plan, and with no qualms they mocked me and rubbed their hands after every 'confession' torn out of my throat. The investigators were present at the trial and, fearing a repeating of the torture [by the investigators] more than death, I, having been warned by them in advance, repeated the memorized words about the crimes I had allegedly committed. (Note by Associate Professor Postoyeva, 1957, quoted in [1].)

[7] A secret research and development laboratory, a number of which used to operate within the Soviet Gulag labor camp system from 1930 to the 1950s.

Postoyeva served her ten-year sentence in full and then spent three more years in exile in Komi. She worked in a *sharashka*. The academician Vladimir I. Smirnov interceded on her behalf, "in his personal capacity" [14].

Andrey Zhuravsky, Professor at the Mining Institute and the head of the Leningrad Department of the Mathematical Institute of the Academy of Sciences of the USSR, was arrested on 17 February 1942 and was sentenced to 10 years in labor camps. In February 1943 he was transferred to Perm where he worked in a sharashka (OKB-172[8]) and in 1944 to prison #1 of Leningrad (known as Kresty) where he remained until the end of his 10-year sentence, engaged in the development of artillery weapons. Then, he was exiled to the city of Syktyvkar, where he managed to find a teaching position at the State Pedagogical Institute in 1952. He returned to Leningrad the next year, and in the spring of 1955 was reinstated to his former position as head of the Department of Higher Mathematics at the Mining Institute [11].

"He hoped that the involvement of such many scientists would make the very case absurd"

To give the reader a deeper insight into the atmosphere of "The Scientists' Case," here are some excerpts from the report describing the circumstances of the "investigation" into the case of Yevgeny I. Denisov, an associate professor at the Leningrad Polytechnic Institute:

> On the 18 of March 1942, I was unexpectedly arrested by the NKVD at my apartment. The arrest was preceded by a search with the seizure of the most valuable possessions and an inventory of property. As it turned out later, the reason for the arrest was as follows. One month before my unexpected arrest, my neighbor was arrested, Professor of Leningrad Polytechnic Institute Leonid Klimenko, whom I had known only very briefly. He himself told me that, being driven to despair by hunger and interrogation techniques used by the investigator demanding "sensational" exposures, he testified that everyone among his colleagues at the Institute, his acquaintances, and even neighbors were anti-Soviet persons. I also was included in that list of almost 60 people, all prominent scientists. As Klimenko further told me before his death (he died of severe exhaustion upon his arrival at the camp), he did not remember how he compiled the list and how he characterized particular persons, since he was nearly in a state of stupor due to the conditions created by the investigation. But the impulse towards making the list remained in his memory. He hoped that the involvement of such a high number of scientists 'would make the very case absurd' (Klimenko's own words) and thus ease his situation, as the investigators, in his opinion, would not dare to decapitate the leadership of the entire scientific workforce of Leningrad. Without a doubt, that action of Klimenko was also the result of a

[8] OKB is the Russian abbreviation for Experimental Design Bureau.

totally disturbed and clearly sick psyche that invariably accompanied the physical processes of acute exhaustion of the body, which, to a greater or lesser extent, applied to all of us who endured the first blockade winter.

Fortunately, a few days before Klimenko's testimony, an organized and complete evacuation of the teaching staff of the Institute was carried out and all the persons listed were already beyond the immediate reach of the investigative apparatus, which saved them from my fate. As for me, I could not evacuate with the others because my 75-year-old mother was ill, and had to wait for her to recover, and a few days later I was arrested.

After the very first interrogation, I realized how hopeless was the situation I had been put in. Despite my insistence, no specific charges were brought against me (I was familiarized with Klimenko's testimony at the end of the investigation), instead I was required to confess to some crimes unknown to me. The investigator Idashkin told me from the very beginning that he had incontrovertible evidence of my 'anti-Soviet' activity and that I should abandon all hope of breaking away from his grip, as in his investigative practice there had not yet been a case of acquittal of a person under investigation and I would be no exception.

My attempts to prove that due to my very essence, I could not be an anti-Soviet person were only met with ridicule by the investigators Idashkin and Mikhailov. Indeed, throughout the winter and up until the day of my arrest, I worked in my unheated laboratory, alone, voluntarily, and without payment, to manufacture medicines for Leningrad hospitals. Showing my swollen (because of burns and frostbite) hands, covered with wounds, which was inevitable when working in the indescribably difficult conditions of 1941–42, to investigators I tried to convince them how improbable it was for a hostile-minded person to voluntarily waste what remains of his strength to help his enemies. In response, I heard that the work was nothing but camouflage for my anti-Soviet activities.

Further, insisting on interviewing witnesses, I argued that one could interview many dozens of my acquaintances and colleagues who had known me for more than 15 years (i.e., my entire conscious life) and see that not a single one among them would speak unfavorably of me. The investigators answered that they were not going to interview any witnesses as positive responses would only mean that I had disguised myself well my whole life.

Finally, my explanation of the reason for not evacuating with the Institute staff was also categorically rejected by the investigators, who stated to me that I had remained in the city "awaiting the arrival of the Germans," despite that they had seen my sick mother lying in bed at home (she died after and as a result of my arrest) and had in their hands my evacuation documents, which I had received back in the month of February 1942, when I was going to evacuate as part of the so-called "Golden Pool" of Leningrad metallurgists, and could not do so only because I fell seriously ill with dysentery myself.

[...] The one who came, the deputy head of the Office, who later turned out to be Kruzhkov, in response to my statement about the need to charge me with specific crimes, indicating exactly what I had committed, where, and when, responded that his only advice to me was to admit to my anti-Soviet activities as soon as possible, as I would do it anyway sooner or later, but the later, the worse it would be for me.

At the end of the interrogation, seeing as I persisted in not admitting to any crimes, I was given an ultimatum threatening to involve my wife to "expose" me. The threat was doubtlessly real to me, for if I, knowing that I was not guilty, had been arrested, then my wife, too, could be arrested just as easily. Of course, my wife had nothing to expose me for, and the enormous power of the threat was not in that at all. It was in fact that after the evacuation of the Institute staff, there was no one we knew whose care for the rest of the family we could hope for in the event of my wife's arrest.

The only people left in the family in such a case would be my severely ill and elderly mother and my two children, a four-year-old daughter, and a ten-year-old son. To leave them on their own in those difficult conditions, without food, fuel, or help, not only for an unknown period of time but even for a day, would have been unthinkable, not to mention the possible prospect of the interrogation awaiting my wife, an example of which I had already experienced.

Thus, by the actions of the investigators, completely horrific conditions were created in which further defense of myself and attempts to restore the truth would prove to be not only pointless, leading only to an irreversible loss of what was left of my physical strength, but also immoral, for it threatened the doom of the entire family.

There was no way out of this situation (except the one left by the investigation). The only thing left to do was to take the most desperate step to at least temporarily avert the threat hanging over my own life and the existence of my entire family.

To achieve that, I had to assume the disgusting role of an anti-Soviet person and, having bought some time, try to find out the real reason for my arrest so that at the first opportunity (in court, as I believed) I could reveal all the circumstances that had forced me to make false statements and prove my innocence. This would make it possible to save my family because the investigation would no longer need to involve my wife in "exposing" me, and it would allow me to hope to save what was left of my strength as well, because the investigators had promised to improve my diet in case I "confessed," and, finally, it would make the conditions of the investigation more bearable. I did not hesitate long. A crazy fear for the welfare of my family and a sense of loss of my physical strength forced me to take the first step along the path that the investigators were pushing me to take.

[...] It can be said with absolute and unquestionable categorical certainty that the task assigned to themselves by the investigators was not at all to establish the truth but rather to create, deliberately and calculatingly, a spectacular and entirely bogus case. Therefore, they were not satisfied at all with the statement they tore out of me (i.e., that I "admitted" to being "anti-Soviet"). It was necessary to involve as many people as possible in the case by attributing to them all kinds of anti-Soviet talk, defeatist sentiments, etc. Investigator Idashkin himself gave me examples of such "conversations" and "rumors," apparently not having too much faith in my imagination. All I had to do was to vary their content and invent a setting and the "participants." As the investigation demanded more and more serious self-incriminating confessions, I completely stopped mentioning those who were still alive, because they could have been at risk of getting into the same situation I found myself in. Therefore, I began to connect all my subsequent confessions only to those among my acquaintances who, as of the day of my arrest, had already died, such as Prof. M.P. Slavinsky (who died in December 1941), M.G. Oknov (killed in February 1942) and laboratory assistant, I.I. Popinevsky (died in February 1942). It was clear to me that this would make it very difficult to expose the falsity of my testimony in the future, but I had no other choice because I could not stop giving testimony or limit it. By refusing to give further testimony, I would have found myself in the position of the first days of the investigation, but in immeasurably worse circumstances, created by the forced confession already made.

Attempts to limit my testimony provoked such anger from the investigator, who was no longer shy about his actions, that to resist him was tantamount to consciously and quickly destroying the rest of my physical strength. Thus, during one of the interrogations, Mikhailov together with Idashkin demanded from me a confession that I had not evacuated from Leningrad because I was "waiting for the Germans." While the matter was limited to making up the most stupid "conversations," which, rather than me, the investigation should have been more ashamed of, I could bear it, albeit with disgust, but to sign such a "confession" when in reality I was putting all my efforts into helping to produce medicine for the hospitals in the blockaded city, did not seem possible to me. After a heavy scuffle that lasted all night between me and both investigators, I ended up being beaten by Mikhailov, who hit me several times in the head with his fist. I had no strength to defend myself and could hardly stand on my feet, struggling to keep from moaning in unbearable pain all over my body and in my swollen feet after another many hours of "standing." However, the worst torture was sleep deprivation, which Idashkin and Mikhailov practiced for a long time, summoning me for interrogation shortly after lights out and releasing me to my cell shortly before waking up. As it was forbidden to sleep in the cell during the day, I sometimes went without sleep for a week. This meant that many of these nights of interrogation turned out to be spent as if in a state of delirium, and neither in the days following the interrogation nor later could I recall the content of the evidence taken from me, or even the successive course of the interrogation in general. Only vague fragments of disconnected memories remained in my memory. As a result of these conditions, I understood that I was powerless to resist the investigation in any measure and how futile my attempts in this respect were. From that time on, I decided to sign everything that was required of me without even bothering to read the evidence written by the investigators "in my own words." After this, the atmosphere of the investigation improved considerably. I was no longer on my feet for dozens of hours but sat and dozed off until awakened by an investigator to sign another testimony written by him. I was so tired of everything I had been through during the investigation that I was not interested in the contents of the dozens of pages of "my testimony" written out by the investigators, and I only waited for the end of the investigation and the beginning of the trial, where I was going to tell everything that had happened during the investigation and which had resulted in the emergence of this "testimony."

[...] On 21 May 1942, Mikhailov came unexpectedly to my solitary confinement cell, bringing me a packet of tobacco, a white bun, and a sausage, things one couldn't even dream of at the time. Unrecognizably courteous and amiable, he informed me that he had long intended to visit me, but the complexity of the case prevented him from coming sooner. He sat down on the bunk and had a friendly conversation with me about the state of affairs in the city and at the front, and not a single word concerning my case as if it had never existed and dozens of hours of abuse and beatings had never happened. Before he left, he told me in a trusting manner that according to his information, a trial was about to take place, and that he was quite convinced that I was an honest Soviet person, despite my testimony, and he felt obliged to inform me of the following: the investigation knows that my crimes were not as serious as they are described in my testimony, and in signing them I acted as a true Soviet person in whom the investigation now has every confidence. But on my part, I must also prove that I trust the investigation by fully corroborating my testimony in court. In such a case, I will be sent to the front, which has already been agreed upon with the chairman of the court at the request of the investigating authorities. If I refuse to testify, however, the re-investigation will be so hard that I am unlikely to be able to survive it.

[...] However, I had no more time to contemplate the situation, which had been precisely calculated by Mikhailov; a few minutes after he had left I was summoned to appear in court, and a short time later I had to answer the president of the Tribunal's question about my confession of guilt. I, like everyone else, answered in the affirmative.

Before entering the courtroom, I saw Mikhailov and Kruzhkov strolling down the corridor arm-in-arm with some military man who turned out to be the President of the Tribunal. Of course, this demonstration

> of the closeness of the investigators to the members of the court, calculated to have an appropriate effect on us defendants, was taken into account by all of us and was probably the main reason why all the accused confirmed their guilt, especially as the entire staff of the numerous investigative apparatus was fully present at this and all subsequent court hearings, closely following our every statement. Even during the breaks, the investigators did not leave us, apparently fearing that we might still conspire and lodge a united protest against the "case" in its entirety. My testimony was to take place the next morning, so Kruzhkov came to my cell in the evening (bearing the same "gift" Mikhailov had made in the morning) and repeated what Mikhailov already said, but the threat was more direct ("If you do not confirm your testimony, rest assured that you will be shot, and we, i.e., the investigation apparatus, will not protect you, although we can do everything including your release").
>
> The next day I confirmed my testimony against myself. On 24 May 1942, the court handed down its verdict, and from that moment none of us saw our investigators, whose moral character was indistinguishable from that of hardened criminals. (Denisov's note, 1957, quoted from [1].)

While incarcerated, Denisov was used as a specialist in an NKVD laboratory. Denisov was rehabilitated in February 1955, and in October 1955 he was reinstated as head of the Department of Analytical Chemistry at the Leningrad Polytechnic Institute [16].

Similar cases are described in numerous memoirs. The reader can reconstruct the general scheme from the story above. First, the authorities arrest several people who are linked to a community and who have "dubious" origins or backgrounds or who have studied or worked abroad. Agents' reports about those arrested may already exist, some more fictitious than others. As we shall see further on, in the eyes of some investigators, being skeptical of the Soviet authorities was enough to be guilty.

The arrestees are then isolated from one another and accused of participating in a counter-revolutionary organization. During interrogations, they may be threatened with execution and tempted with a pardon in exchange for a confession. Typically, one of the arrestees breaks and signs a "confession," stipulating several new suspects at the prompting of the investigator. Thus, a second, enlarged circle of suspects is created, which can also be arbitrarily expanded, as there is no real case, and therefore it has no objective boundaries.

For the convenience of paperwork and reporting, those who confessed were more or less arbitrarily included in groups with high-sounding goals such as "killing Stalin" or "overthrowing Soviet power." Following the peculiar aesthetics of the time, future "members of the government" and ministers were identified within the groups. From September 5, 1941, to October 1, 1942, 625 counter-revolutionary groups were uncovered and liquidated, according to Kubatkin's report to Zhdanov [3]. Inspections conducted in 1943 and later showed that many of these cases were falsified.

However, in the absence of a confession, it was not easy to "file" a new person for the case. On February 3, 1942, the first Dean of the Faculty of Optics at LITMO, Professor Vladimir N. Churilovsky, was arrested on charges of participation in the fascist and espionage organization set up by Ignatovsky (who had already been shot by that time).

The arrest warrant stated that Churilovsky was being incriminated by the testimony of the arrested professors Titov, Chanyshev, and Ignatovsky. No such testimony had been given by the professors [2]. Professor Churilovsky denied involvement in anti-Soviet activities, despite grueling interrogations and torture. Churilovsky's co-workers were questioned but could not testify to his anti-Soviet activities either. Then Ignatowskaya, who had already been sentenced to death, was interrogated as a witness. But she, too, testified only about Churilovsky's defeatist sentiments, which he allegedly expressed during a chance meeting.

All these testimonies were denied by Churilovsky during both the interrogations and the confrontation with Ignatovskaya on 6 March 1942 [2]. Having obtained no confession from Churilovsky and thus having no evidence of his involvement in the anti-Soviet organization of Rose, Koshlyakov, and others, the investigators put the materials on Churilovsky into a separate case and then terminated the criminal prosecution (see the Special Inspectorate report in [1]).

All the scientists convicted in these two cases were rehabilitated in 1954–1955, and the investigators who fabricated the case (Ogoltsov, Zanin, Altshuller, Podchasov, Kozhemyakin) were expelled from the Communist Party (CPSU).

> A review by the Prosecutor's Office established that the "Public Safety Committee" did not exist and that it had been artificially created by employees of the former NKVD Department of the Leningrad Region [...] The review also established that the criminal practice of interrogating prisoners already sentenced to capital punishment had been widespread in the NKVD of the Leningrad Region. At those interrogations, by promising to keep the convicts alive, testimonies against other people that the investigation needed were extorted [...] 25 February 1958 [2].

In 1955, investigator Kruzhkov was expelled from the CPSU and sentenced by the military tribunal of the 8th Naval Fleet to 20 years of penal labor camps "for falsification of criminal cases against a group of prominent Leningrad scientists." After serving seven years out of twenty, he was released in 1962. We should mention that Kruzhkov studied at the Mechanical and Mathematical Department of the Lomonosov Moscow State University from 1934 to 1939. You can read about this period of his life in the memoirs of his classmate [15]. It is possible that Kruzhkov continued to study by correspondence during the war, as he asked Postoyeva, who was then under investigation, to solve problems in complex analysis [13]. Investigator Kruzhkov fabricated 12 criminal cases against 65 people [4]. Kruzhkov's son became a well-known mathematician.

"... convicted scientists should not be shot, and their execution should be commuted to imprisonment in a [labor] camp, with them being utilized according to their specializations"

It is interesting to learn the views of the investigators convicted of falsification in this case. Investigator Altshuller saw the scientists and teachers primarily as former Socialist Revolutionaries, cadets, representatives of another culture, another class.

> I am a soldier of the revolution who has devoted his entire conscious life to carrying out the Party's orders in the sharp[9] section of the class struggle and, as such, has committed mistakes arising from the flawed system of work of the state security organs of the Yezhov and Beria period. (From the appeal of the former investigator Altshuller, 1970, quoted in [1].)

Kruzhkov also considered those under investigation to be enemies of the people and saw nothing reprehensible in torture:

> The defendant Kruzhkov partially admitted his guilt at the hearing, i.e., the use of unlawful methods of investigation against certain individuals, such as the use of standings, nighttime interrogations, swearing to break the will of the interrogated, and, in some cases, beatings. He also admitted the fact that on a number of occasions he had failed to put negative [i.e., denying] testimony in the interrogation records and obtained testimony from the person sentenced to capital punishment while stating that he had used illegal methods of investigation and obtained confessions, believed them at the time and thought that he was fighting the enemies of the people. (Quoted from [6].)

Ryabov obeyed instructions from his superiors and preferred not to think about what was happening.

> Both Zanin and Podchasov, as well as Kruzhkov and, in some cases, Artemov, came to me during the interrogations of the arrestees. The very fact of their entering the interrogation rooms, their demanding attitude towards the arrested, their "logical" and "convincing" — as it seemed to me at the time — use of circumstances taken from the testimonies of other arrested persons, and, on the part of Kruzhkov, a harsher treatment played a decisive role in the confessions of the arrested, whom I had been ordered to interrogate [...].
>
> In accordance with the orders of my former superiors, I also carried out other investigative actions without being aware of the actual merits of

[9] I.e., "intensive" in the political lingo of the time.

> the cases at the time. (Testimony of Lieutenant Colonel Ryabov, 1957, quoted in [1].)

Artemov believed the arrested to be guilty even before the investigation had begun simply because of their disloyalty to the Soviet authorities:

> [...] because the accused indeed did not shoot, did not kill, but were engaged in anti-Soviet conversations about the need, also through anti-Soviet conversations, to identify like-minded people and connect with them, about the need to "save" the civilian population of the city and for this purpose to seek the surrender of Leningrad to the Germans, about the need to offer their services to the Nazi command in an organized manner to establish a "new order" in Leningrad [...] (Testimony of Colonel Artemov, 1957, cited in [1].)

Shevelev failed to notice the tortures by Kruzhkov but was principled in his unwillingness to falsify the case and dictate to those under investigation what they should themselves confess to. For this, he was accused of stalling the investigation, being dismissed from the agencies, expelled from the CPSU, and sent to the army (from the conclusion of the Special Inspectorate, the testimony of former investigator Shevelev, quoted from [1]).

> After the persons under investigation tell their interrogator all about their counter-revolutionary activities, they enter a period of complete indifference when they may sign whatever the interrogator tells them to sign, without any objection. Both Podchasov and Altshuller saw this, and when it became clear to them that I could not be persuaded to create "a phony," naturally it became necessary to remove me from the case as someone standing in the way of their glory [...] Thus, after a day or two, almost the entire apparatus of the Leningrad Directorate was made aware that the Counterintelligence Department's investigation team was trying to puff up a big case against the scientists, and that I, for standing in the way, obstructing them, would obviously be arrested and imprisoned, as had been done with many others before that. (Testimony of former investigator Shevelev, quoted from [1].)

Podchasov, being a mid-ranking boss, "did not notice" the falsifications by his subordinates, but when he refused to arrest more scientists on the testimony of those already sentenced to death, they ... simply stopped calling him to meetings (testimony of Colonel Podchasov, quoted from [1]).

Zanin was following instructions from above "not to delay the development." He might not have given direct orders to falsify materials, seeing that Altshuller, Kruzhkov, and Artemov were doing fine on their own. It was enough for Zanin to turn a blind eye to something every now and then.

> The leadership of the MGB and the Leningrad Directorate demanded to ensure internal security in the city. There were instructions by the MGB and f[ormer] head of the UKGB Kubatkin not to drag

> out developments and to close them quickly. With the same goal in mind, instructions were given to review all archival materials in the departments and, if there were indications of c-r[counter-revolutionary] activity, to realize them as well by investigative means [...]. The investigations that started were over in a short period of time. There was a large number of people arrested. In such an environment and under such conditions, big mistakes and shortcomings could and did occur in the agency's investigative work. Materials were not always fully checked and documented. At the time, the primary evidence in an investigation was the confession of the arrested, and this was the main and big mistake stemming from the wrong policy of the ministry. In addition, the situation in Leningrad did not quite allow for prolonged development and investigation. The whole process of the investigation into the cases took place under the supervision and with the participation of representatives of the prosecutor's office, who, like us, were in the office day and night with no breaks [...] The arrested were indeed a-s [anti-Soviet] minded. This they do not seem to deny even now. (Testimony of Colonel Zanin, 1958, quoted in [1].)

Ogoltsov simply did not have the time (or inclination) to delve into whether these scientists should have been arrested, let alone tried in the first place, or what his subordinates were doing:

> [The case] required a thorough investigation, I would have needed to spend time comparing the facts and checking individual discrepancies and data, but the situation in the city besieged by the enemy did not allow this to be done. Based on these considerations, on my initiative, the UNKVD made a proposition: convicted scientists should not be shot, and their execution should be commuted to imprisonment in a [labor] camp, with them being utilized according to their specializations [...].
>
> My fault is that I did not sufficiently supervise either the progress of the undercover development or the investigation. In particular, I should have met personally with the agents who had reported information about the existence of a group of scientists carrying out anti-Soviet work and made sure that this information was correct. But I did not and could not do this because in addition to my primary duties as the deputy chief of the UNKVD I was appointed head of the operational staff to fight against enemy paratroopers in the Leningrad region. I formed 136 fighter battalions, and trained, and prepared these battalions. Besides this tremendous and important work in the conditions of war, I was charged with commanding subversive activities behind enemy lines, forming and sending two Chekist partisan detachments to the field of operations, selecting Chekists for the people's militia, forming a division out of the UNKVD and police members, and, finally, overseeing the front line defenses on the city outskirts, which was assigned to the 5th and 20th divisions of the NKVD. (Testimony of Lieutenant General Ogoltsov, quoted in [1].)

In 1948, as Deputy Minister of State Security, Ogoltsov directed the murder of Solomon Mikhoels, head of the Jewish Anti-Fascist Committee. In 1959, Ogoltsov was stripped of his military rank of lieutenant general and government decorations "as having discredited himself during his work in the agencies." Colonels Zanin, Artemov, Podchasov, Kozhemyakin, and Altshuller had already been dismissed in 1957, and their discharge was re-issued as "due to facts discrediting the rank of officer." Lieutenant-Colonel Ryabov was reprimanded. Kruzhkov, as we have already noted, was sentenced to 20 years of imprisonment but was released after seven years in a camp.

The scientists' rehabilitation case appears to have started as follows. Professor Strakhovich sent a complaint to Prosecutor General Rudenko in 1954 that his testimony had been obtained by investigators Kruzhkov and Artemov through long night interrogations and threats against relatives. An intra-departmental struggle resulted in the complaint reaching Khrushchev who decreed that "All those responsible are to be found and severely punished" (From the appeal of former investigator Altshuller, quoted in [1]).

Such was the zeitgeist of the time. Of course, teachers and students knew that some had died in the war, some had gone missing, and some had been taken by the security agencies for interrogation: the community was not that numerous. Many Leningrad mathematicians died on the frontlines and in the rear during the war. Detailed biographies of fifty of them are given in the two-volume book [9]. One can only guess what role the repressions, passing so close to everyone, played in the worldview of Soviet mathematicians.

Nikita Kalinin

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Index